

# Part VII: Sensitivity Functions and Frequency Response Analysis

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# Outline

- 1 One-degree and Two-degree of Freedom Control Systems
- 2 Sensitivity Functions
- 3 Nyquist Stability Criterion
- 4 Example for Two Degree of Freedom Control System
- 5 Design Examples

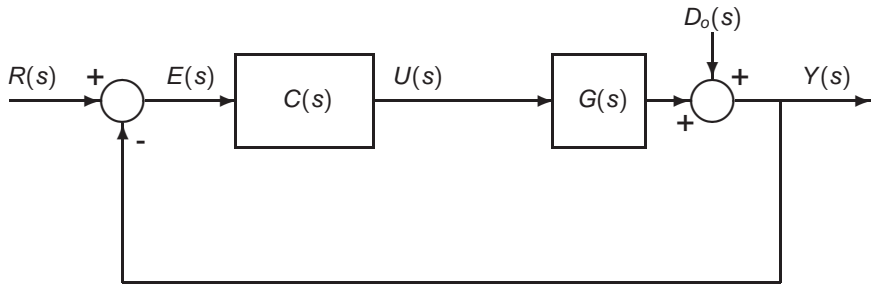
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# Learning Objectives

- One degree of freedom and two degrees of freedom control systems.
- The sensitivity functions and the relationship between the complementary sensitivity function and sensitivity function.
- Understand how sensitivity functions are used to describe closed-loop performance in terms of reference following, disturbance rejection and measurement noise attenuation.
- Understand how closed-loop bandwidth is used in a trade-off relationship between disturbance rejection and measurement noise attenuation.

# One Degree of Freedom Control System Structure



**Figure:** One-degree of freedom control system structure. Only one-degree of freedom is available in the controller structure to influence the output response  $Y(s)$  to the reference signal  $R(s)$  and to the disturbance  $D_o(s)$ .

# Closed-loop Signals

- The Laplace transform of the error signal  $E(s)$  is expressed as

$$\begin{aligned} E(s) &= R(s) - Y(s) = R(s) - G(s)U(s) - D_o(s) \\ &= R(s) - G(s)C(s)E(s) - D_o(s) \end{aligned} \quad (1)$$

Therefore, the error signal is expressed as

$$E(s) = \frac{R(s)}{1 + G(s)C(s)} - \frac{D_o(s)}{1 + G(s)C(s)} \quad (2)$$

- Then, the output of the control system is

$$\begin{aligned} Y(s) &= R(s) - E(s) = \left(1 - \frac{1}{1 + G(s)C(s)}\right)R(s) + \frac{D_o(s)}{1 + G(s)C(s)} \\ &= \frac{G(s)C(s)}{1 + G(s)C(s)}R(s) + \frac{D_o(s)}{1 + G(s)C(s)} \end{aligned} \quad (3)$$

- The control signal is

$$U(s) = C(s)E(s) = \frac{C(s)}{1 + G(s)C(s)}R(s) - \frac{C(s)}{1 + G(s)C(s)}D_o(s) \quad (4)$$

# The Closed-loop Transfer Functions

- The transfer function between the set-point signal and the plant output is

$$\frac{Y(s)}{R(s)} = \frac{G(s)C(s)}{1 + G(s)C(s)} \quad (5)$$

- The set-point signal and the control signal is

$$\frac{U(s)}{R(s)} = \frac{C(s)}{1 + G(s)C(s)} \quad (6)$$

- The transfer functions between the output disturbance and the output, and the output disturbance and the control signal are

$$\frac{Y(s)}{D_o(s)} = \frac{1}{1 + G(s)C(s)} \quad (7)$$

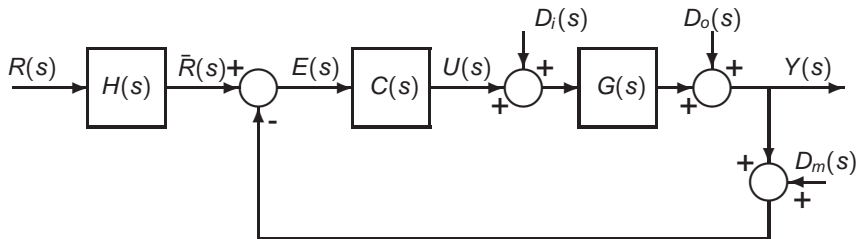
$$\frac{U(s)}{D_o(s)} = -\frac{C(s)}{1 + G(s)C(s)} \quad (8)$$

# One-degree of Freedom Design: Summary

In this controller structure, once the controller  $C(s)$  is selected, all four closed-loop transfer functions are fixed, only one-degree of freedom is available to influence the output response  $Y(s)$  to the reference signal  $R(s)$  and to the disturbance  $D_o(s)$ . This is called one-degree of freedom design.



## Two Degrees of Freedom Control System Structure



**Figure:** Two-degrees of freedom control system structure. An extra component  $H(s)$  is placed after the reference signal  $R(s)$ , which will be used in the design.

## Closed-loop Transfer Functions

- With the assumption that  $D_i(s) = 0$  and  $D_m(s) = 0$ , we calculate the output response  $Y(s)$  in relation to the reference signal  $R(s)$  and the output disturbance  $D_o(s)$ ,

$$Y(s) = \frac{G(s)C(s)H(s)}{1 + G(s)C(s)}R(s) + \frac{D_o(s)}{1 + G(s)C(s)} \quad (9)$$

- From this, we have the two transfer functions

$$\frac{Y(s)}{R(s)} = \frac{G(s)C(s)H(s)}{1 + G(s)C(s)} \quad (10)$$

$$\frac{Y(s)}{D_o(s)} = \frac{1}{1 + G(s)C(s)} \quad (11)$$

(12)

- Here, in comparison to one degree of freedom control system, we have the transfer function  $H(s)$  act on the reference signal  $R(s)$ .

# Two Degrees of Freedom Control System: Summary

- Transfer function  $H(s)$  provides one more degree of freedom to shape the output response to the reference signal  $R(s)$ .
- This extra degree of freedom plus the original one degree of freedom gives the two degrees of freedom in the design.
- If the control system is configured as a two degrees of freedom, then we can shape, independently, the output response to the reference signal and to the disturbance.

## Two Degrees of Freedom Design of PI Controllers (i)

- The Laplace transform of the control signal,  $U(s)$ , is expressed as the function of feedback error signal  $E(s)$  using the relation

$$U(s) = C(s)E(s) = \frac{c_1 s + c_0}{s} E(s) \quad (13)$$

where  $E(s) = R(s) - Y(s)$ .

- The closed-loop transfer function between the set-point signal  $R(s)$  and the output signal  $Y(s)$  is then

$$\frac{Y(s)}{R(s)} = \frac{G(s)C(s)}{1 + G(s)C(s)} = \frac{b(c_1 s + c_0)}{s(s + a) + b(c_1 s + c_0)} \quad (14)$$

where  $G(s)$  is the first order transfer function  $\frac{b}{s+a}$ .

- The closed-loop transfer function becomes:

$$\frac{Y(s)}{R(s)} = \frac{(2\xi w_n - a)s + w_n^2}{s^2 + 2\xi w_n s + w_n^2} = \frac{w_n^2 \left( \frac{2\xi w_n - a}{w_n^2} s + 1 \right)}{s^2 + 2\xi w_n s + w_n^2}$$

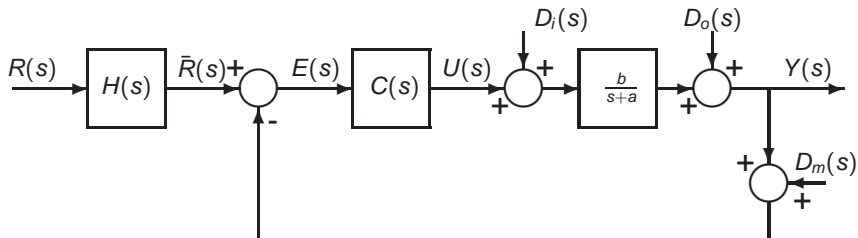
## Two Degrees of Freedom Design of PI Controllers (ii)

- By choosing the reference filter  $H(s) = \frac{1}{\frac{2\xi\omega_n - a}{\omega_n^2}s + 1}$ , with the two degrees of freedom structure, the relationship between the reference signal and the output response is

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad (15)$$

- Note that the parameter  $\frac{2\xi\omega_n - a}{\omega_n^2} = \tau_I$ , thus, the set-point filter is  $H(s) = \frac{1}{\tau_I s + 1}$ .
- This closed-loop transfer function is equivalent to what we obtained from the alternative PI controller structure.
- However, in the two degree of freedom design, we can choose any reference filter  $H(s)$ , as long as it is stable, to reflect the requirement of desired reference following.

## Two Degrees of Freedom Design of PI Controllers (iii)



**Figure:** Two-degrees of freedom PI control system structure:  $H(s) = \frac{1}{\tau_I s + 1}$ ,  
 $C(s) = K_C(1 + \frac{1}{\tau_I s})$ .

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# Sensitivity Functions (i)

- We calculate the feedback error of the closed-loop system firstly as

$$\begin{aligned}
 E(s) &= H(s)R(s) - (Y(s) + D_m(s)) \\
 &= H(s)R(s) - [G(s)(U(s) + D_i(s)) + D_o(s) + D_m(s)] \\
 &= H(s)R(s) - G(s)C(s)E(s) - G(s)D_i(s) - D_o(s) - D_m(s) \quad (16)
 \end{aligned}$$

- The closed-loop feedback error by re-arranging (16) is

$$E(s) = \frac{H(s)}{1 + G(s)C(s)}R(s) - \frac{G(s)}{1 + G(s)C(s)}D_i(s) - \frac{D_o(s)}{1 + G(s)C(s)} - \frac{D_m(s)}{1 + G(s)C(s)}$$

- The expression of the closed-loop output  $Y(s)$  is

$$\begin{aligned}
 Y(s) &= \frac{G(s)C(s)H(s)}{1 + G(s)C(s)}R(s) + \frac{D_o(s)}{1 + G(s)C(s)} + \frac{G(s)}{1 + G(s)C(s)}D_i(s) \\
 &\quad - \frac{G(s)C(s)}{1 + G(s)C(s)}D_m(s) \quad (17)
 \end{aligned}$$



## Sensitivity Functions (ii)

Also, from the feedback error (16), we calculate the closed-loop control signal as

$$\begin{aligned} U(s) &= C(s)E(s) = \frac{C(s)H(s)}{1 + G(s)C(s)}R(s) - \frac{C(s)G(s)}{1 + G(s)C(s)}D_i(s) \\ &\quad - \frac{C(s)}{1 + G(s)C(s)}D_o(s) - \frac{C(s)}{1 + G(s)C(s)}D_m(s) \end{aligned} \quad (18)$$

# Definitions of Sensitivity Functions

Based on these relationships, the following sensitivity functions are defined:

- **Sensitivity function.**

$$S(s) = \frac{1}{1 + G(s)C(s)}$$

- **Complementary sensitivity function.**

$$T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}$$

- **Input disturbance sensitivity.**

$$S_i(s) = \frac{G(s)}{1 + G(s)C(s)}$$

- **Control sensitivity.**

$$S_u(s) = \frac{C(s)}{1 + G(s)C(s)}$$

# Relationships between Sensitivity Functions

- The sensitivity plus complementary sensitivity equals to one:

$$S(s) + T(s) = \frac{1}{1 + G(s)C(s)} + \frac{G(s)C(s)}{1 + G(s)C(s)} = 1 \quad (19)$$

- The input disturbance sensitivity is related to sensitivity:

$$S_i(s) = \frac{G(s)}{1 + G(s)C(s)} = S(s)G(s) \quad (20)$$

- The control sensitivity is related to sensitivity:

$$S_u(s) = \frac{C(s)}{1 + G(s)C(s)} = S(s)C(s) \quad (21)$$

# Expressions of Input and Output

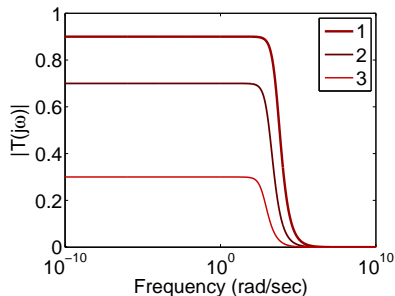
With the sensitivity functions, we re-write the output of the closed-loop system (17) as

$$Y(s) = H(s)T(s)R(s) + S(s)D_o(s) + S_i(s)D_i(s) - T(s)D_m(s) \quad (22)$$

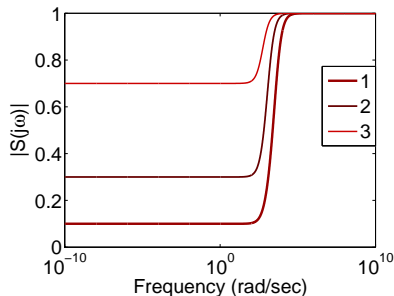
and the control signal (18) as

$$U(s) = H(s)S_u(s)R(s) - S_u(s)D_o(s) - S_u(s)G(s)D_i(s) - D_m(s) \quad (23)$$

# Example of Sensitivity Functions (Proportional Control)



(a) Complementary sensitivity



(b) Sensitivity

**Figure:** Magnitude of complementary sensitivity and sensitivity functions using proportional control. Key: line(1) steady-state gain  $\alpha = 0.9$ , line(2)  $\alpha = 0.7$ , line(3)  $\alpha = 0.3$ .

# Implications of the Sensitivity Functions

- The complementary sensitivity function  $T(s)$  represents the effect of both reference signal and measurement noise on the output. If we want a fast response speed to a reference signal, then the closed-loop bandwidth will be wider (larger  $w_n$ ). As a consequence, the closed-loop control system will amplify the measurement noise.
- The sensitivity  $S(s)$  represents the effect of output disturbance on the output.
- The input sensitivity  $S_i(s)$  represents the effect of input disturbance on the output.

# Disturbance Rejection and Noise Attenuation (i)

- There are both noise and disturbance existed in a physical system. A good closed-loop performance requires minimization of the effects of both disturbance rejection and noise. Or we call it disturbance rejection and noise attenuation.
- For minimization of the effects of both input and output disturbances, we will make the magnitude of the output in frequency response

$$|Y_d(j\omega)| = |S(j\omega)(D_o(j\omega) + G(j\omega)D_i(j\omega))| \quad (24)$$

as small as possible.

- For minimization of the measurement noise, we will make the magnitude of the output in frequency response

$$|Y_m(j\omega)| = |T(j\omega)D_m(j\omega)| \quad (25)$$

as small as possible.

## Disturbance Rejection and Noise Attenuation (ii)

We can not alter the disturbances and noise, because they already existed in the system. What we will do is to make

- the magnitude of sensitivity  $S(j\omega)$  ( $|S(j\omega)|$ ) small for disturbance rejection;
- the magnitude of complementary sensitivity  $T(j\omega)$  ( $|T(j\omega)|$ ) small for noise attenuation.

These are the basic design principles for control systems.



## Disturbance Rejection and Noise Attenuation (iii)

- Noting that the relationship between the sensitivity and complementary sensitivity is constrained by

$$S(j\omega) + T(j\omega) = 1 \quad (26)$$

which says that we can not make both  $|S(j\omega)|$  and  $|T(j\omega)|$  small over the same frequency bands.

- In other words, if the disturbance is minimized in a given frequency region where  $|S(j\omega)|$  is small, then inevitably the measurement noise is not attenuated in the same frequency region where  $|T(j\omega)|$  is large.

# Disturbance Rejection and Noise Attenuation (iv)

- So how are we going to design a closed-loop control system that will minimize the effects of disturbance and the measurement noise?
- Note that the disturbances existed in the system correspond to slow movement of the variables or slow changes, therefore, the frequency contents of the disturbance term  $|D_o(j\omega) + G(j\omega)D_i(j\omega)|$  are concentrated in the low frequency region.
- In contrast, the measurement noise corresponds to fast movement of the variables or fast and frequent changes of the variables, therefore, the frequency contents of the measurement noise  $|D_m(j\omega)|$  are concentrated in the higher frequency region.

# Disturbance Rejection and Noise Attenuation (v)

The strategies for disturbance rejection and noise attenuation are

- to achieve disturbance rejection by choosing the sensitivity function  $S(j\omega) \approx 0$  at the low frequency region, implying  $T(j\omega) \approx 1$  at the low frequency region, because  $S(j\omega) + T(j\omega) = 1$ .
- This is not too bad for noise attenuation because  $|D_m(j\omega)|$  is small in the low frequency region.
- At the high frequency region, to avoid the amplification of measurement noise, we choose  $|T(j\omega)| \approx 0$ , which implies  $|S(j\omega)| \approx 1$ .
- This is not too bad for disturbance rejection because  $|D_o(j\omega) + G(j\omega)D_i(j\omega)|$  is small in the high frequency region.
- Essentially, we will adjust the closed-loop bandwidth to compromise disturbance rejection with measurement noise attenuation.

# Summary of Disturbance Rejection and Noise Attenuation

- The parameter  $w_n$  corresponds to the desired closed-loop bandwidth.
- Larger  $w_n$  leads to a wider bandwidth for  $|T(j\omega)|$ , which implies a faster disturbance rejection and higher noise amplification.
- On the other hand, smaller  $w_n$  leads to a more narrow bandwidth for  $|T(j\omega)|$ , which implies a slower disturbance rejection and lower noise amplification.
- Larger  $w_n$  will also lead to faster reference response.

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# Nyquist Stability Criterion (i)

## Nyquist diagram

When using the Nyquist stability criterion, we examine the frequency response of the loop transfer function ( $M(j\omega) = C(j\omega)G(j\omega)$ ) that contains the transfer functions for the plant and the controller. We plot this frequency response in a complex plane.

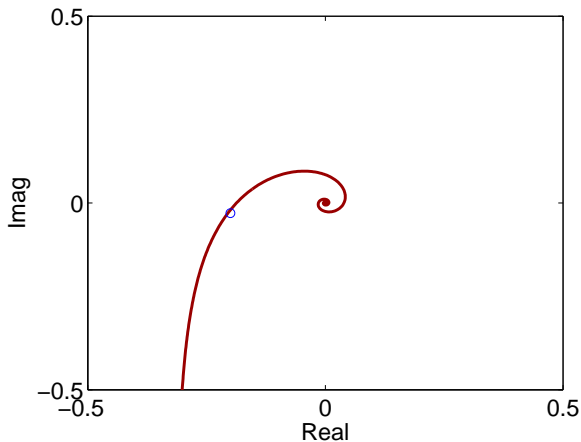
## Stability Criterion

The criterion states that a feedback control system with single input and single output is stable *if and only if*, for the frequency response of the loop transfer function, number of counter clockwise encirclements of the  $(-1, 0)$  point is equal to the number of poles of this loop transfer function with positive real parts.

## Nyquist Stability Criterion (ii)

- For majority of the control problems, this loop transfer function  $M(s)$  does not contain any poles that have positive real parts.
- For this type of systems, the closed-loop system will be stable if and only if the frequency response  $M(j\omega)$  is not encircle  $(-1, 0)$  point on the complex plane.

# Example



**Figure:** Nyquist loci with with  $K_{T1} = 0.04$ . on which  $o$  is the estimated value at  $\omega_1 = \frac{2\pi}{T}$ .



# Gain Margin

## Definition

Gain margin is defined as  $GM = \frac{1}{k_g}$ , where  $k_g$  is the distance between the origin of the complex plane and the point that  $M(j\omega)$  intersects the real axis.

## What does it mean

It means that if the loop gain were to exceed the reciprocal of  $k_g$ , then the closed-loop system would become unstable.

# Phase Margin

## Definition

It is the angle between the negative real axis and the line that intersects the circle  $|M(j\omega)| = 1$  (see Figure 6).

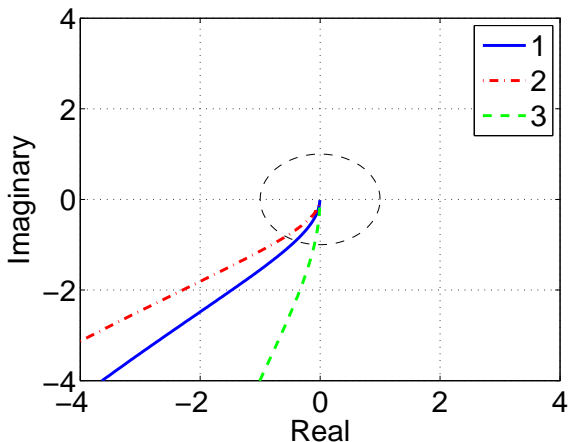
## What does it mean

Phase margin indicates the additional phase lag that could be associated with  $M(j\omega)$  before the closed-loop system became unstable.

## Good design

A good design should have reasonable gain and phase margins to ensure that the closed-loop system is robustly stable in the presence of factors known and unknown.

# Example of Phase Margin



**Figure:** Nyquist plots with a unit circle. Key: line (1) with  $J_m = J_m^0$ , line (2)  $J_m = 0.5J_m^0$ , line (3)  $J_m = 3J_m^0$  where  $J_m^0 = 0.00214 \text{ kg} \cdot \text{m}^2$ .

# Summary

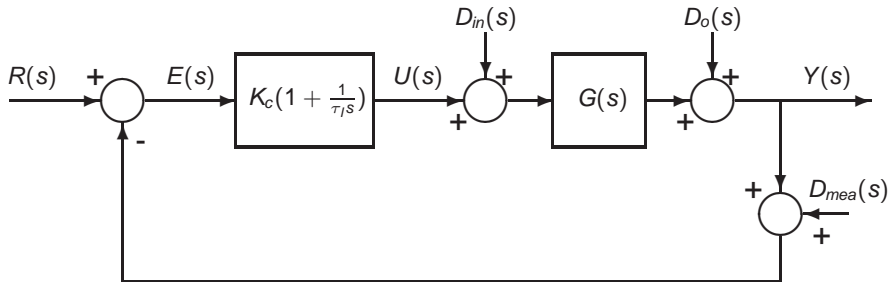
- Configuration of control systems in terms of one degree of freedom and two degrees of freedom design;
- Sensitivity functions, the relationship between them and their roles in control system analysis;
- The trade-off relationship between disturbance rejection and noise attenuation.
- Nyquist stability criterion with gain margin and phase margin.

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## Example (i)

Derive closed-loop transfer functions for the following PI control system.



$R(s)$  is the reference signal

$D_{in}(s)$  is the input disturbance

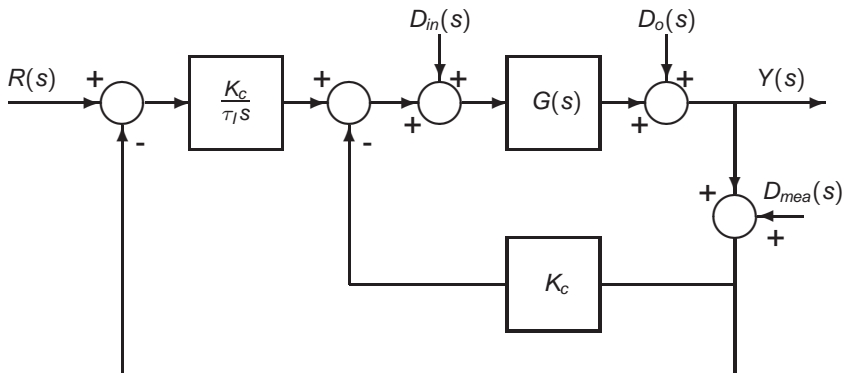
$D_o(s)$  is the output disturbance

$D_{mea}(s)$  is the measurement noise

$Y(s)$  is the output signal

# Example (ii)

Second configuration of the PI controller



## Example (iii)

Given  $K_c = 0.56$ ,  $\tau_I = 8$

and the plant transfer function  $G(s) = \frac{1}{s(s+3)^3}$ .

Find the following transfer functions:

$$\frac{Y(s)}{R(s)}, \quad \frac{Y(s)}{D_{in}(s)}, \quad \frac{Y(s)}{D_{out}(s)}, \quad \frac{Y(s)}{D_{mea}(s)}$$

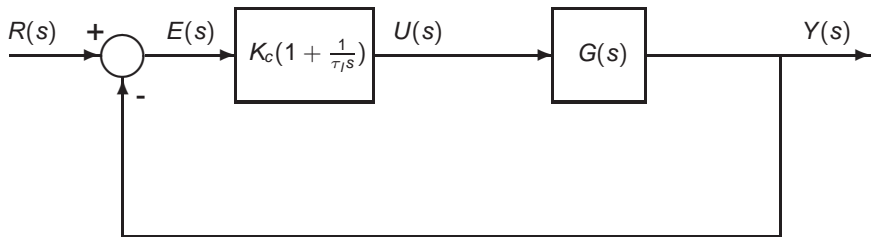
for both controller configurations.

Discuss the differences.



## Solution I

System diagram for reference and output signals in first configuration.

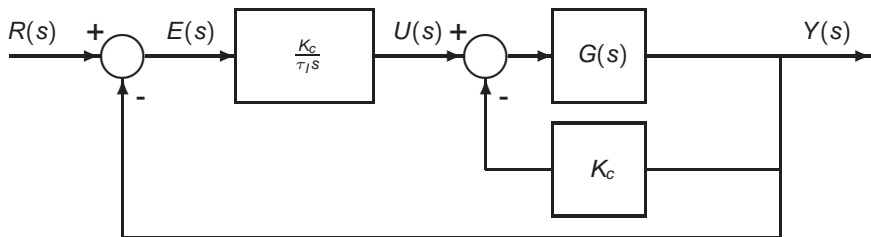


The transfer function

$$\begin{aligned}
 \frac{Y(s)}{R(s)} &= \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{\frac{K_C \tau_I s + K_C}{\tau_I s} G(s)}{1 + \frac{K_C \tau_I s + K_C}{\tau_I s} G(s)} = \frac{K_C \tau_I s G(s) + K_C G(s)}{\tau_I s + K_C \tau_I s G(s) + K_C G(s)} \\
 &= \frac{K_C \tau_I s \frac{1}{s(s+1)^3} + K_C \frac{1}{s(s+1)^3}}{\tau_I s + K_C \tau_I s \frac{1}{s(s+1)^3} + K_C \frac{1}{s(s+1)^3}} = \frac{K_C \tau_I s + K_C}{\tau_I s^2 (s+1)^3 + K_C \tau_I s + K_C} \\
 &= \frac{4.48s + 0.56}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}
 \end{aligned}$$

## Solution II

Block Diagram of  $Y(s)$  and  $R(s)$  in the second configuration of the PI controller

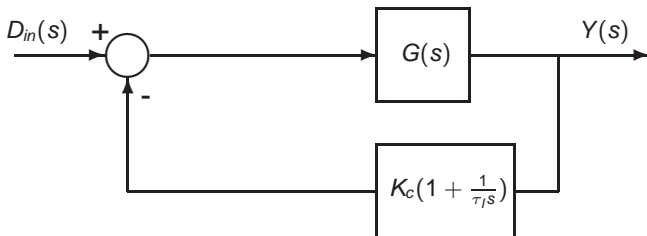


The transfer function is shown as following

$$\begin{aligned}
 \frac{Y(s)}{R(s)} &= \frac{\frac{K_c}{\tau_I s} \frac{G(s)}{1+K_c G(s)}}{1 + \frac{K_c}{\tau_I s} \frac{G(s)}{1+K_c G(s)}} = \frac{K_c G(s)}{\tau_I s + K_c \tau_I s G(s) + K_c G(s)} \\
 &= \frac{K_c \frac{1}{s(s+1)^3}}{\tau_I s + K_c \tau_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}} = \frac{K_c}{\tau_I s^2 (s+1)^3 + K_c \tau_I s + K_c} \\
 &= \frac{0.56}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}
 \end{aligned}$$

## Solution III

System diagram for input disturbance and output signals in first configuration.

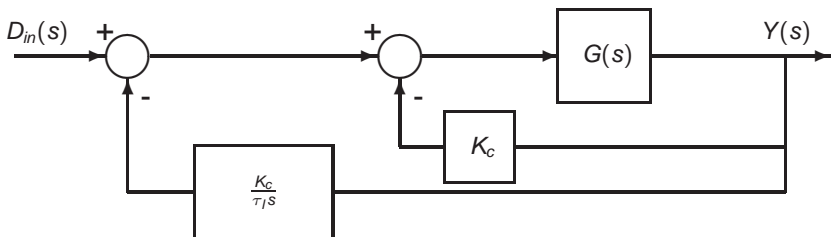


The transfer function of input disturbance is

$$\begin{aligned}
 \frac{Y(s)}{D_{in}(s)} &= \frac{G(s)}{1 + \frac{K_c \tau_I s + K_c}{\tau_I s} G(s)} = \frac{\tau_I s G(s)}{\tau_I s + K_c \tau_I s G(s) + K_c G(s)} \\
 &= \frac{\tau_I s \frac{1}{s(s+1)^3}}{\tau_I s + K_c \tau_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}} = \frac{\tau_I s}{\tau_I s^2 (s+1)^3 + K_c \tau_I s + K_c} \\
 &= \frac{8s}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}
 \end{aligned}$$

## Solution IV

Block Diagram of  $Y(s)$  and  $D_{in}(s)$  in the second configuration of the PI controller

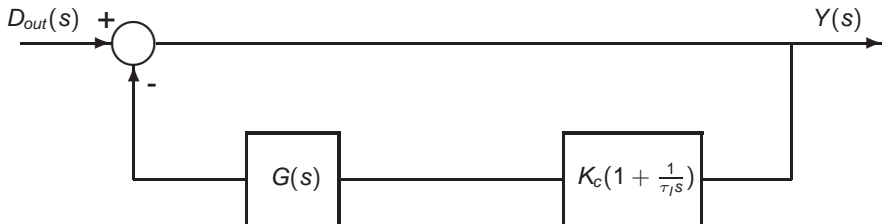


The transfer function of input disturbance in second configuration is

$$\begin{aligned}
 \frac{Y(s)}{D_{in}(s)} &= \frac{\frac{G(s)}{1+G(s)K_c}}{1 + \frac{G(s)}{1+G(s)K_c} \frac{K_c}{\tau_I s}} = \frac{\tau_I s G(s)}{\tau_I s + K_c \tau_I s G(s) + K_c G(s)} \\
 &= \frac{\tau_I s \frac{1}{s(s+1)^3}}{\tau_I s + K_c \tau_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}} = \frac{\tau_I s}{\tau_I s^2 (s+1)^3 + K_c \tau_I s + K_c} \\
 &= \frac{8s}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}
 \end{aligned}$$

## Solution V

System block diagram for output disturbance and output signals in first configuration.

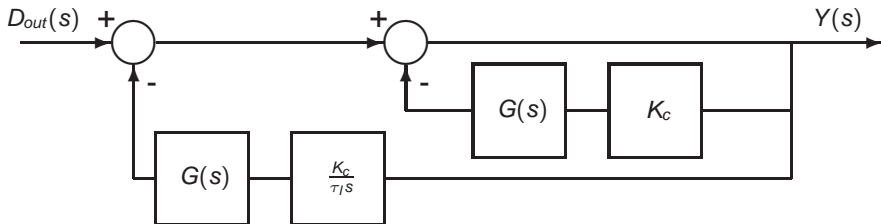


The transfer function in this case is

$$\begin{aligned}
 \frac{Y(s)}{D_{out}(s)} &= \frac{1}{1 + G(s)C(s)} = \frac{1}{1 + \frac{K_c \tau_I s + K_c}{\tau_I s} G(s)} = \frac{\tau_I s}{\tau_I s + K_c \tau_I s G(s) + K_c G(s)} \\
 &= \frac{\tau_I s}{\tau_I s + K_c \tau_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}} = \frac{\tau_I s^2 (s+1)^3}{\tau_I s^2 (s+1)^3 + K_c \tau_I s + K_c} \\
 &= \frac{8s^5 + 24s^4 + 24s^3 + 8s^2}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}
 \end{aligned}$$

## Solution VI

After simplifying, the system block diagram for  $D_{out}(s)$  and  $Y(s)$  in second configuration is shown below

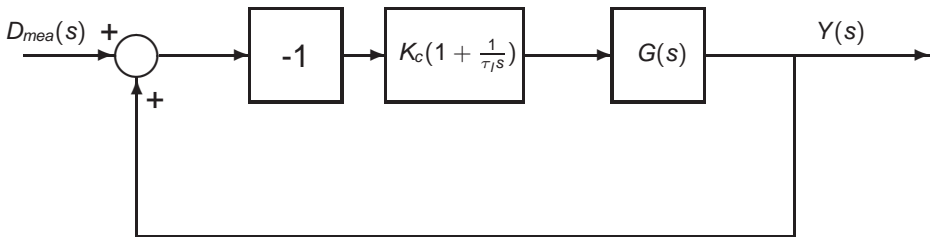


The transfer function in this case is

$$\begin{aligned}
 \frac{Y(s)}{D_{out}(s)} &= \frac{\frac{1}{1+K_c G(s)}}{1 + \frac{1}{1+K_c G(s)} G(s) \frac{K_c}{\tau s}} = \frac{\tau s}{\tau s + K_c \tau s G(s) + K_c G(s)} \\
 &= \frac{\tau s}{\tau s + K_c \tau s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}} = \frac{\tau s^2 (s+1)^3}{\tau s^2 (s+1)^3 + K_c \tau s + K_c} \\
 &= \frac{8s^5 + 24s^4 + 24s^3 + 8s^2}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}
 \end{aligned}$$

## Solution VII

System block diagram for measurement noise and output signals in first configuration.

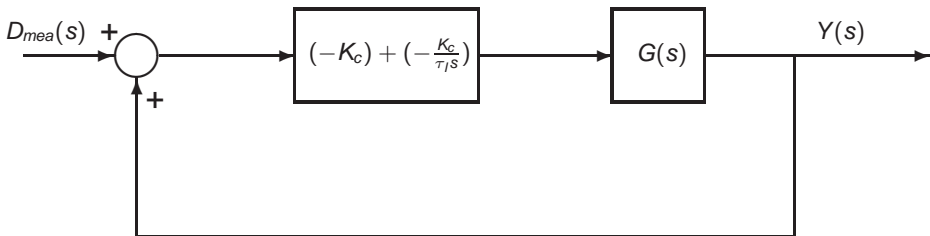


The transfer function in this case is

$$\begin{aligned}
 \frac{Y(s)}{D_{mea}(s)} &= \frac{-C(s)G(s)}{1 - (-C(s)G(s))} = \frac{-\frac{K_c\tau_I s + K_c}{\tau_I s} G(s)}{1 + \frac{K_c\tau_I s + K_c}{\tau_I s} G(s)} = -\frac{K_c\tau_I s G(s) + K_c G(s)}{\tau_I s + K_c\tau_I s G(s) + K_c G(s)} \\
 &= -\frac{K_c\tau_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}}{\tau_I s + K_c\tau_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}} = -\frac{K_c\tau_I s + K_c}{\tau_I s^2 (s+1)^3 + K_c\tau_I s + K_c} \\
 &= -\frac{4.48s + 0.56}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}
 \end{aligned}$$

## Solution VIII

After the simplifying rule of the block diagram, the system for measurement noise and output signals in second configuration become



The transfer function is identical to the previous case

$$\begin{aligned}
 \frac{Y(s)}{D_{mea}(s)} &= \frac{-\frac{K_c T_I s + K_c}{T_I s} G(s)}{1 + \frac{K_c T_I s + K_c}{T_I s} G(s)} = -\frac{K_c T_I s G(s) + K_c G(s)}{T_I s + K_c T_I s G(s) + K_c G(s)} \\
 &= -\frac{K_c T_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}}{T_I s + K_c T_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}} = -\frac{K_c T_I s + K_c}{T_I s^2 (s+1)^3 + K_c T_I s + K_c} \\
 &= -\frac{4.48s + 0.56}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}
 \end{aligned}$$



# Conclusions

- From the transfer functions before, we can obtain that the transfer functions for both controller configurations are different only for the Reference signal, other disturbance cases have identical transfer function.
- That's because the second configuration of PI controller is designed to reduce the overshoot of the set-point change during the transient.
- However, the second configuration does NOT affect the transfer function of the disturbances, which means the configuration will only change the response of the reference signal change.

# Exercise

Given  $K_c = -4.5$ ,  $\tau_I = 1.68$

and the plant transfer function  $G(s) = \frac{s-2}{(s+1)(s+2)(s+3)}$ .

Find the following transfer functions:

$$\frac{Y(s)}{R(s)}, \frac{Y(s)}{D_{in}(s)}, \frac{Y(s)}{D_{out}(s)}, \frac{Y(s)}{D_{mea}(s)}$$

for both controller configurations.

Discuss the differences.

# Outline

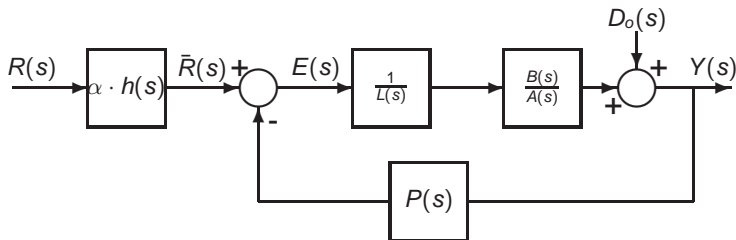
- 1 One-degree and Two-degree of Freedom Control Systems
- 2 Sensitivity Functions
- 3 Nyquist Stability Criterion
- 4 Example for Two Degree of Freedom Control System
- 5 Design Examples**

## Two degree of freedom Design Example

Two degrees of freedom pole-assignment controller design.  
Suppose that the plant transfer function is given by:

$$G(s) = \frac{3}{s^2 - s} = \frac{3}{s(s-1)}$$

The block diagram of the control system is shown as following:



## Two degree of freedom Design Example

we wish to obtain the output response to a set-point input as

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where we choose that  $\omega_n^2 = 2.44$  and  $\xi = 0.768$ .

Therefore,

$$\begin{aligned} T(s) &= \frac{2.44}{s^2 + 2.4s + 2.44} \\ &= \frac{2.44}{(s + 1.2 - j1)(s + 1.2 + j1)} \end{aligned}$$

The closed-loop transfer function is derived from the

$$\frac{Y(s)}{R(s)} = \frac{\alpha B(s)h(s)}{L(s)A(s) + B(s)P(s)} = \frac{\alpha B(s)}{\hat{A}_{cl}(s)}$$

Here  $\hat{A}_{cl}(s) = s^2 + 2\xi\omega_n s + \omega_n^2$ .

## Two degree of freedom Design Example

By applying the Final Value Theorem,

$$\alpha \times \frac{B(0)}{\hat{A}_{cl}(0)} = 1$$

Therefore,

$$\alpha = \frac{\hat{A}_{cl}(0)}{B(0)} = \frac{\omega_n^2}{3} = \frac{2.44}{3} = 0.813$$

$h(s)$  does not affect set-point response, but affects disturbance rejection. Let

$$h(s) = s + h_0.$$

**Question:** How  $h_0$  affects the controller  $L(s)$  and  $P(s)$  for  $h_0 = 0.2, 2$  and  $20$ ? (where

$$C(s) = \frac{p_1 s + p_0}{s + l_0}$$

# Two degree of freedom Design Example

**Case A:**  $h_0 = 0.2$

The Diophantine Equation:

$$\begin{aligned} L(s)A(s) + B(s)P(s) &= \hat{A}_{cl}(s)h(s) \\ (s + l_0)(s(s - 1)) + 3(p_1s + p_0) &= (s^2 + 2.4s + 2.44)(s + 0.2) \\ s^3 + (l_0 - 1)s^2 - l_0s + 3p_1s + 3p_0 &= s^3 + 2.6s^2 + 2.92s + 0.488 \end{aligned}$$

By comparing the coefficients, we have

$$\begin{aligned} l_0 - 1 &= 2.6 \\ -l_0 + 3p_1 &= 2.92 \\ 3p_0 &= 0.488 \end{aligned}$$

Therefore, the solution of the controller design is:

$$\begin{cases} l_0 = 3.6 \\ p_1 = 2.17 \\ p_0 = 0.163 \end{cases}$$

# Two degree of freedom Design Example

**Case B:**  $h_0 = 2$

The Diophantine Equation:

$$\begin{aligned} L(s)A(s) + B(s)P(s) &= \hat{A}_{cl}(s)h(s) \\ (s + l_0)(s(s - 1)) + 3(p_1s + p_0) &= (s^2 + 2.4s + 2.44)(s + 2) \end{aligned}$$

By comparing the coefficients, the solution of the controller design is derived:

$$\begin{cases} l_0 = 5.4 \\ p_1 = 4.21 \\ p_0 = 1.63 \end{cases}$$

**Case C:**  $h_0 = 20$

The Diophantine Equation:

$$\begin{aligned} L(s)A(s) + B(s)P(s) &= \hat{A}_{cl}(s)h(s) \\ (s + l_0)(s(s - 1)) + 3(p_1s + p_0) &= (s^2 + 2.4s + 2.44)(s + 20) \end{aligned}$$



## Two degree of freedom Design Example

By comparing the coefficients, the solution of the controller design is derived:

$$\begin{cases} l_0 = 23.4 \\ p_1 = 24.61 \\ p_0 = 16.26 \end{cases}$$

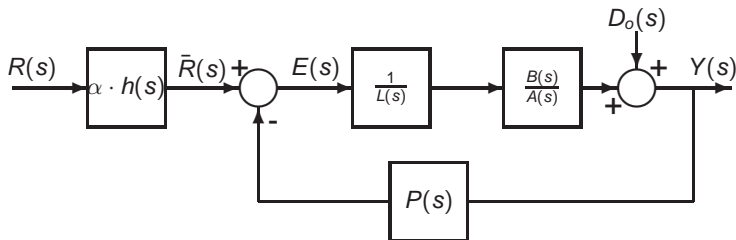
By comparing the controller parameters solution from case A, B and C, we see that as  $h_0$  increases,  $l_0$ ,  $p_1$  and  $p_0$  increases.

In other words, as we move the closed-loop poles  $h_0$  further away from the origin of the complex plane, the gain of the controller increases. Higher controller gain will result in the faster disturbance rejection response.

## Two degree of freedom Design Example

Implementation of two-degree of freedom control system.

Previously, we proposed the design based on the diagram as following:

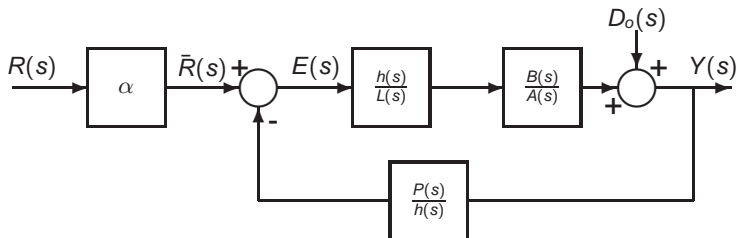


However, this structure of the feedback control system is not good for implementation. That's because the sectors of  $h(s) = s + h_0$  and  $P(s) = p_n s^n + p_{n-1} s^{n-1} + \dots + p_0$  involve the differentiation operation.

**Question:** How can we modify the configuration so that the control system is realizable?

## Two degree of freedom Design Example

**Solution:** The modified configuration is displayed as following:



**Proof** This is equivalent because the closed-loop transfer functions for output from both reference signal and output disturbance signal are the same as before.

$$\begin{aligned}
 \frac{Y(s)}{R(s)} &= \frac{\alpha \frac{h(s)}{L(s)} \frac{B(s)}{A(s)}}{1 + \frac{h(s)}{L(s)} \frac{P(s)}{h(s)} \frac{B(s)}{A(s)}} \\
 &= \frac{\alpha h(s) B(s)}{L(s) A(s) + P(s) B(s)} = \frac{\alpha h(s) B(s)}{\hat{A}_{cl}(s) h(s)} \\
 &= \frac{\alpha B(s)}{\hat{A}_{cl}(s)}
 \end{aligned}$$

# Two degree of freedom Design Example

$$\begin{aligned}
 \frac{Y(s)}{D_o(s)} &= \frac{1}{1 + \frac{h(s)}{L(s)} \frac{P(s)}{h(s)} \frac{B(s)}{A(s)}} \\
 &= \frac{L(s)A(s)}{L(s)A(s) + P(s)B(s)} \\
 &= \frac{L(s)A(s)}{\hat{A}_{cl}(s)h(s)}
 \end{aligned}$$

# Use of Sensitivity Function in Design

Suppose that a model of plant has Laplace transfer function:

$$G(s) = \frac{4}{(s+1)(s+2)^2}$$

the plant has output disturbance

$$d_o(t) = k + d_v(t)$$

where  $d_v(t)$  is a zero mean signal with energy in the band  $B_d : [0, 4]$  rad/s,  $k$  is a constant.

If we choose the complementary sensitivity

$$T(s) = \frac{\alpha}{(s^2 + 1.2\omega_n s + \omega_n^2)(\tau s + 1)^2}$$

**Question:** How should we design a feedback controller to reject the disturbance by choosing  $\alpha$ ,  $\omega_n$  and  $\tau$ ?

# Use of Sensitivity Function in Design

## Solution:

(1) The constant disturbance with unknown amplitude  $k$  has frequency content at  $\omega = 0$ . In order to reject a constant disturbance we want to choose  $S(0) = 0$ , which makes

$$Y_d(0) = S(0)D_o(0) = 0$$

at zero frequency.

This choice implies  $T(0) = 1$ , because of  $S(j\omega) + T(j\omega) = 1$ .

Therefore,  $\alpha = \omega_n^2$ .

(2) How do we reject  $d_v(t)$ ?

We want  $S(j\omega)$  to be small in  $[0, 4]$  rad/s to reject  $d_v(t)$ .

For a second order system, this means we need to choose  $\omega_n \gg 4$  rad/s.

Let us select  $\omega_n = 10$  rad/s.

# Use of Sensitivity Function in Design

(3) How to choose  $\tau$ ?

We select  $\tau$  value to be small, so that, it's not to interfere with  $\omega_n$ .

So we have  $\tau = 0.01$ .

Therefore, the complementary sensitivity transfer function is:

$$T(s) = \frac{100}{(s^2 + 12s + 100)(0.01s + 1)^2}$$

# Exercise

If the complementary sensitivity is given as:

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

$$C(s)G(s) = \frac{T(s)}{1 - T(s)}$$

$$C(s) = \frac{T(s)}{1 - T(s)} G^{-1}(s)$$

By using the information given in the part A, find the controller structure and parameters based on the last equation above.

**Question:**

- (1) Does  $C(s)$  have an integrator? If yes, where was it generated?
- (2) What are the orders of the numerator and denominator of the controller? Do you think the orders are reasonable choices?