Part IV: Linearization

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Outline

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Introduction

- One of the approaches to obtain the models for the control system design is based on analysis of the system dynamics using first principles, such as mass balance, Newton's laws, current law and voltage law. The majority of these types of models are nonlinear in nature.
- Thus, in order to use them for the PID controller design or other linear time invariant controller design, these nonlinear models need to be linearized around the operating conditions of the system.

The General Principle

- Assume that the nonlinear models have the general form:

$$\dot{x}(t) = f[x(t), u(t), t] \quad (1)$$

  where $f[.]$ is a nonlinear function. The purpose of linearization is to find a linear function (a set of linear functions) to describe the dynamics of the nonliear model at a given operating condition.
- Note that this linear model is obtained at a given operating condition.

Linearization of Nonlinear Functions (i)

- We will use Taylor series expansion to approximate a nonlinear function.
- A single variable case. A function with variable x, $f(x)$ can be expressed in terms of Taylor series expansion as

$$f(x) = f(x^0) + \frac{df(x)}{dx}|_{x=x^0}(x - x^0) + \frac{1}{2}\frac{d^2f(x)}{dx^2}(x - x^0)^2 + \ldots \qquad (2)$$

  if the function $f(x)$ is smooth and its derivatives exist for all the orders.
- Using first two terms in the Taylor series expansion leads to the approximation of the original function $f(x)$ at the specific point x^0,

$$f(x) \approx f(x^0) + \frac{df(x)}{dx}|_{x=x^0}(x - x^0) \qquad (3)$$

- This first order Taylor series approximates the original nonlinear function $f(x)$ using the function evaluated at x^0 and its first derivative at $x = x^0$.
- The approximation holds well in the vicinity of $x = x^0$.

Illustration of Linearization

Figure 1 illustrates an example of linear approximation of a nonlinear function where $x^0 = 5.3$, $f(x^0) = 140$ and $\frac{df(x)}{dx}|_{x=x^0} = 85$. It is seen that within the region where x is close to x^0, $f(x)$ is closely approximated by the first order Taylor series expansion (3).

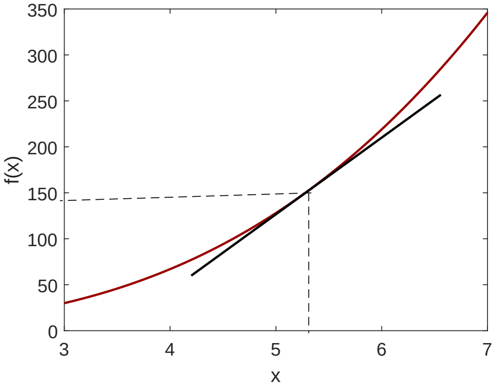


Figure 1: Approximation of a nonlinear function at $x^0 = 5.3$.

Linearization of Nonlinear Functions (ii)

- Intuitively, we can think of the original variable x as a 'large' variable because it covers a large region, and the perturbed variable $x - x^0$ as a 'small' variable because it covers a small region around x^0.
- The term of 'linear' comes from the second term of the right-hand side of the equation for its linear relationship between $f(x)$ and $x - x^0$.
- The first term is a constant, $f(x_0)$. If it is not zero, then it is not truly linear because it violates the homogeneity and additivity conditions required for linearity. In this case, on the $\dot{(x - x^0)}$ and $x - x^0$ plane, the function is a straight line between $\dot{(x - x^0)}$ and $x - x^0$, but it will not pass through the origin.

Linearization of Nonlinear Functions with Multiple Variables (i)

If the nonlinear function $f(x)$ contains n variables, meaning that $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ is a vector with dimension n, then the function is approximated using the first $n+1$ terms in the multivariable Taylor series expansion as

$$\begin{aligned} f(x_1, x_2, x_3, \dots, x_n) &\approx f(x_1^0, x_2^0, x_3^0, \dots, x_n^0) + \frac{\partial f(x)}{\partial x_1}|_{x_1=x_1^0, x_2=x_2^0, \dots}(x_1 - x_1^0) \\ &+ \frac{\partial f(x)}{\partial x_2}|_{x_1=x_1^0, x_2=x_2^0, \dots}(x_2 - x_2^0) + \dots + \frac{\partial f(x)}{\partial x_n}|_{x_1=x_1^0, x_2=x_2^0, \dots}(x_n - x_n^0) \end{aligned} \quad (4)$$

Linearization of Nonlinear Functions with Multiple Variables (ii)

- (4) is not linear, because it has a constant offset, which violates the homogeneity and additivity conditions required for linearity.
- In some applications, by appropriate selection of operating conditions, the constant is equal to zero.
- If this constant is not zero, it is regarded as a constant disturbance.
- This is one of the important reasons why integrator is often required in a feedback control system, which will overcome the effect of the offset in the system.

Linearization of Nonlinear Model(i)

- The nonlinear models obtained from using first principles of the physical laws are differential equations.
- We assume that the nonlinear differential equation used to describe a physical system takes the general form:

$$\dot{x}(t) = f(x(t), u(t), t) \tag{5}$$

where $x(t)$ is a vector that represents the state variables of dimension n and $u(t)$ is a vector for the control signals of dimension m.

Linearization of Nonlinear Model(ii)

- In the linearization of a nonlinear dynamic system, we will firstly choose the constant vectors $x^0 = \begin{bmatrix} x_1^0 & x_2^0 & \dots & x_n^0 \end{bmatrix}^T$, and $u^0 = \begin{bmatrix} u_1^0 & u_2^0 & \dots & u_m^0 \end{bmatrix}^T$, and apply the linearization procedure of the nonlinear functions as outlined in the previous section.
- The linearization of differential equations is basically to apply the linearization of functions as outlined in the previous section to each term in the differential equation.

Steady-state Solutions

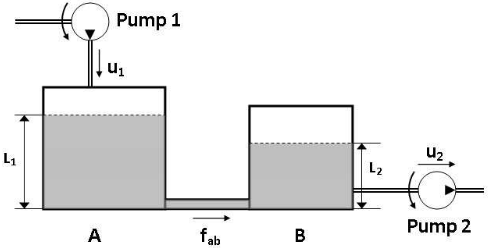
- The constant vectors x^0 and u^0 play an important role in the linearized model.
- To make the linearized system truly linear, these vectors need to be selected carefully. The point of interest is called an equilibrium point These equilibrium points in control system design and implementation are often referred to as stationary points, which represent a steady-state solution to the dynamic equation (5).
- The equilibrium points satisfy the following steady-state solution of the nonlinear differential equation (5):

$$\dot{x}(t) = f(x^0, u^0) = 0 \tag{6}$$

Outline

Case Study: Nonlinear Water Tank Model (i)

Two cubic water tanks are connected in series. Water flows into the first tank and flows out from the second tank. A pump controls the water in-flow rate $u_1(t)$ to the first tank; and another pump controls the water out-flow rate $u_2(t)$ from the second tank. Water flows from tank A to tank B, with a flow rate $f_{ab}(t)$. The units for the flow rate is m/sec and the units for the water level is m.



Case Study: Nonlinear Water Tank Model (ii)

- Using mass balance, the rate of change of water volume $V_1(t)$ in tank A is

$$\frac{dV_1(t)}{dt} = u_1(t) - f_{ab}(t) \tag{7}$$

- The water volume can also be expressed as $V_1(t) = S_1 L_1(t)$, where S_1 is the cross-sectional area of the tank A, and $L_1(t)$ is the water level in tank A.
- The dynamic equation to describe the rate of change in the water level $L_1(t)$ (tank A) is

$$S_1 \frac{dL_1(t)}{dt} = u_1(t) - f_{ab}(t) \tag{8}$$

- Likewise, the rate of change in the water level $L_2(t)$ is

$$S_2 \frac{dL_2(t)}{dt} = f_{ab}(t) - u_2(t) \tag{9}$$

where S_2 is the cross-sectional area for tank B.

Case Study: Nonlinear of Water Tank Model (iii)

- Applying Bernoulli's principle for small orifice, the flow rate f_{ab} is related to the difference between the two water tank levels by

$$g(L_1(t) - L_2(t)) = \frac{1}{2} f_{ab}(t)^2 \tag{10}$$

where g is acceleration due to gravity ($= 9.81 m/sec^2$); f_{ab} is the flow rate (m/sec), leading to

$$f_{ab}(t) = \sqrt{2g(L_1(t) - L_2(t))} \tag{11}$$

Case Study: Nonliner Water Tank Model (iii)

By substituting (11) into (8) and (9), we obtain

$$\frac{dL_1(t)}{dt} = -\frac{1}{S_1}\sqrt{2g(L_1(t) - L_2(t))} + \frac{1}{S_1}u_1(t) \tag{12}$$

$$\frac{dL_2(t)}{dt} = \frac{1}{S_2}\sqrt{2g(L_1(t) - L_2(t))} - \frac{1}{S_2}u_2(t) \tag{13}$$

Both of these models are nonlinear.

Solution: Linearization of Water Tank Model (i)

- In the linearization, the independent variables are $L_1(t)$, $L_2(t)$, $u_1(t)$ and $u_2(t)$. We will linearize the two equations (12) and (13) separately in terms of those independent variables.
- We let L_1^0 and L_2^0 denote the operating points for the tanks.
- The coefficients $\gamma_1 = \frac{\sqrt{2g}}{S_1}$ and $\gamma_2 = \frac{\sqrt{2g}}{S_2}$ are used to simplify the notation in both (12) and (13).

Solution: Linearization of Water Tank Model (ii)

- The first term in (12) is approximated by the first order Taylor series expansion as

$$\begin{aligned}\gamma_1\sqrt{L_1(t)-L_2(t)} &\approx \gamma_1\sqrt{L_1^0-L_2^0}\\ &+ \gamma_1\frac{\partial(\sqrt{L_1(t)-L_2(t)})}{\partial L_1}|_{L_1^0,L_2^0}(L_1(t)-L_1^0)\\ &+ \gamma_1\frac{\partial(\sqrt{L_1(t)-L_2(t)})}{\partial L_2}|_{L_1^0,L_2^0}(L_2(t)-L_2^0)\end{aligned} \quad (14)$$

- Note that

$$\frac{\partial(\sqrt{L_1(t)-L_2(t)})}{\partial L_1}|_{L_1^0,L_2^0} = \frac{1}{2}\frac{1}{\sqrt{L_1^0-L_2^0}} \quad (15)$$

$$\frac{\partial(\sqrt{L_1(t)-L_2(t)})}{\partial L_2}|_{L_1^0,L_2^0} = -\frac{1}{2}\frac{1}{\sqrt{L_1^0-L_2^0}} \quad (16)$$

Solution: Linearization of Water Tank Model (iii)

- Therefore, (14) is written as

$$\gamma_1\sqrt{L_1(t)-L_2(t)} = \gamma_1\sqrt{L_1^0-L_2^0} + \frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0-L_2^0}}(L_1(t)-L_1^0) - \frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0-L_2^0}}(L_2(t)-L_2^0) \quad (17)$$

- The second term in the differential equation (12) is already linear in relation to $u_1(t)$, therefore, we keep it unchanged.

Solution: Linearization of Water Tank Model (iv)

- By substituting the Taylor series approximation (17) into the differential equation (12), we obtain the linearized model for water tank A (do not forget that there is a negative sign):

$$\begin{aligned}\frac{dL_1(t)}{dt} &= -\gamma_1\sqrt{L_1^0 - L_2^0} - \frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}(L_1(t) - L_1^0) \\ &+ \frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}(L_2(t) - L_2^0) + \frac{1}{S_1}u_1(t) \qquad (18)\end{aligned}$$

- Firstly, we notice that in order for the linearization to be valid, the operating points $L_1^0 > L_2^0$.
- Secondly, the first term is a constant that is not zero because $L_1^0 \neq L_2^0$.
- We can choose the steady-state value of $u_1(t)$ according to this constant.

Linearized Model for Water Tank

- For this purpose, we re-write (18) as

$$\begin{aligned}\frac{dL_1(t)}{dt} &= -\frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}(L_1(t) - L_1^0) \\ &+ \frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}(L_2(t) - L_2^0) + \frac{1}{S_1}(u_1(t) - S_1\gamma_1\sqrt{L_1^0 - L_2^0}) \quad (19)\end{aligned}$$

- To find the small signal model for the Tank A, we define the deviation variables as

$$\tilde{L}_1(t) = L_1(t) - L_1^0; \;\; \tilde{L}_2(t) = L_2(t) - L_2^0; \;\; \tilde{u}_1(t) = u_1(t) - S_1\gamma_1\sqrt{L_1^0 - L_2^0}$$

- This leads to the linearized model for the Tank A as

$$\frac{d\tilde{L}_1(t)}{dt} = -\frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}\tilde{L}_1(t) + \frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}\tilde{L}_2(t) + \frac{1}{S_1}\tilde{u}_1(t) \quad (20)$$

- Linearization of Tank B is left as an exercise

Discussions: Linearization of Water Tank Model

- The coefficients to represent the operating conditions of the two tanks must be positive and $L_1^0 > L_2^0$ in order for the linear models to be valid.
- Note that the steady-state value of the control signal $S_1\gamma_1\sqrt{L_1^0 - L_2^0}$ is a function of the system parameters S_1, γ_1. If there are errors in these parameters, then there is an error in the steady-state value of the control signal. This error could be modelled as an input disturbance. This is one of the important reasons why integrator is needed in control system.

Steps in Linearization of Nonlinear Plant Model

- Choose the operating conditions for the plant model.
- Use Taylor series to approximate each nonlinear term in the plant model by taking the derivative of the nonlinear function and calculate its value at the operating points.
- Collecting all the approximated linear terms to form the linearized model.

Exercise: Linearization of PMS Motor

A Permanent Magnetic Synchronous Motor (PMSM) is described by the differential equations in the d-q rotating reference frame

$$\frac{di_d(t)}{dt} = \frac{1}{L_d}(v_d(t) - Ri_d(t) + \omega_e(t)L_q i_q(t)) \quad (21)$$

$$\frac{di_q(t)}{dt} = \frac{1}{L_q}(v_q(t) - Ri_q(t) - \omega_e(t)L_d i_d(t) - \omega_e(t)\phi_{mg}) \quad (22)$$

$$\frac{d\omega_e(t)}{dt} = \frac{p}{J}(T_e - \frac{B}{p}\omega_e(t) - T_L) \quad (23)$$

$$T_e = \frac{3}{2}p\phi_{mg}i_q \quad (24)$$

where ω_e is the electrical speed and is related to the rotor speed by $\omega_e = p\omega_m$ with p denoting the number of pole pairs, v_d and v_q represent the stator voltages in the d-q frame, i_d and i_q represent the stator currents in this frame, and T_L is load torque that is assumed to be zero if no load is attached to the motor.