## *Robotics 1*

# Differential kinematics

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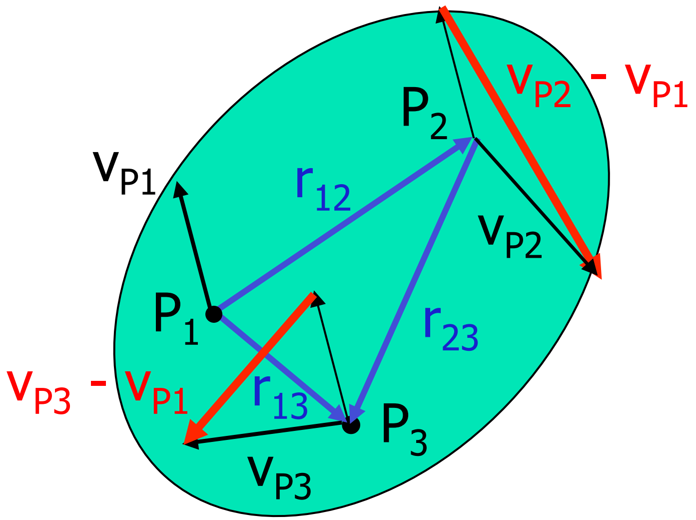
# Differential kinematics

- “relations between motion (velocity) in joint space and motion (linear/angular velocity) in task space (e.g., Cartesian space)”
- instantaneous velocity mappings can be obtained through time derivation of the direct kinematics or in a geometric way, directly at the differential level
  - different treatments arise for rotational quantities
  - establish the link between angular velocity and
    - time derivative of a rotation matrix
    - time derivative of the angles in a minimal representation of orientation

# Angular velocity of a rigid body



"rigidity" constraint on distances among points:

$\|r_{ij}\| = \text{constant}$

→ $v_{Pi} - v_{Pj}$ orthogonal to $r_{ij}$

1 $\quad v_{P2} - v_{P1} = \omega_1 \times r_{12}$

2 $\quad v_{P3} - v_{P1} = \omega_1 \times r_{13}$

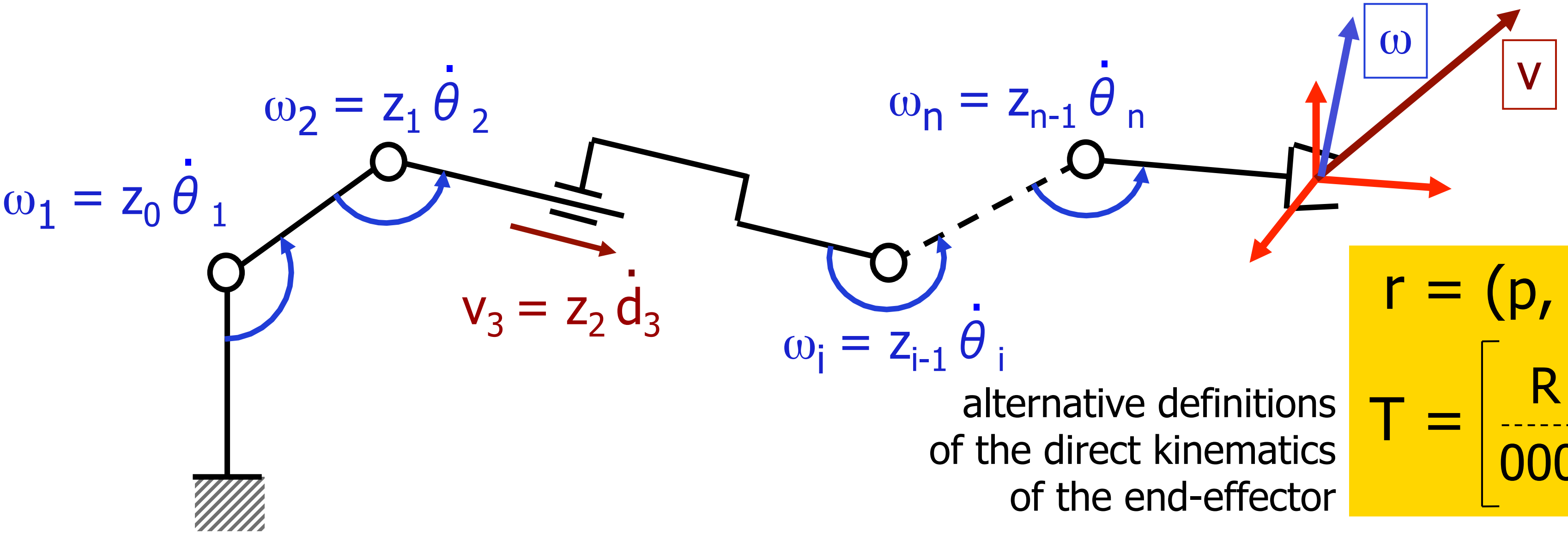
3 $\quad v_{P3} - v_{P2} = \omega_2 \times r_{23}$

$\forall\ P_1, P_2, P_3$ $\quad$ 2 - 1 = 3 → $\omega_1 = \omega_2 = \omega$

$$v_{Pj} = v_{Pi} + \omega \times r_{ij} = v_{Pi} + S(\omega)\, r_{ij} \quad \Leftrightarrow \quad \dot{r}_{ij} = \omega \times r_{ij}$$

- the angular velocity $\omega$ is associated to the whole body (**not** to a point)
- if $\exists\ P_1, P_2$ with $v_{P1} = v_{P2} = 0$: pure rotation (circular motion of all $P_j \notin$ line $P_1P_2$)
- $\omega = 0$: pure translation (**all** points have the same velocity $v_P$)

# Linear and angular velocity of the robot end-effector



alternative definitions
of the direct kinematics
of the end-effector

$$r = (p, \phi)$$

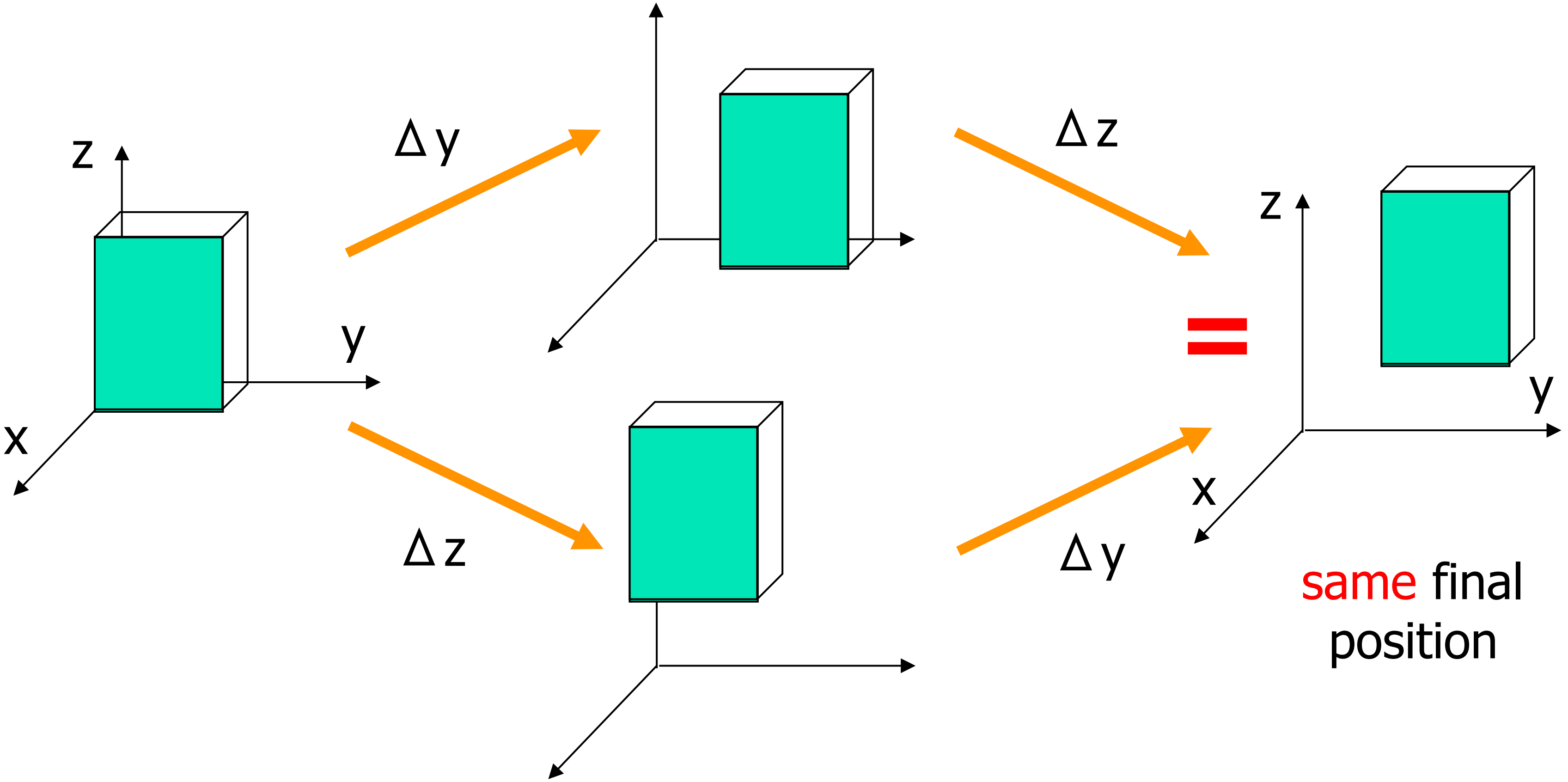
$$T = \left[\begin{array}{c:c} R & p \\ \hdashline 000 & 1 \end{array}\right]$$

- v and $\omega$ are "vectors", namely are elements of vector spaces
  - they can be obtained as the sum of single contributions (in any order)
  - these contributions will be those of the single the joint velocities
- on the other hand, $\phi$ (and $d\phi/dt$) is not an element of a vector space
  - a minimal representation of a sequence of two rotations is not obtained summing the corresponding minimal representations (accordingly, for their time derivatives)

in general, $\omega \neq d\phi/dt$

# Finite and infinitesimal translations

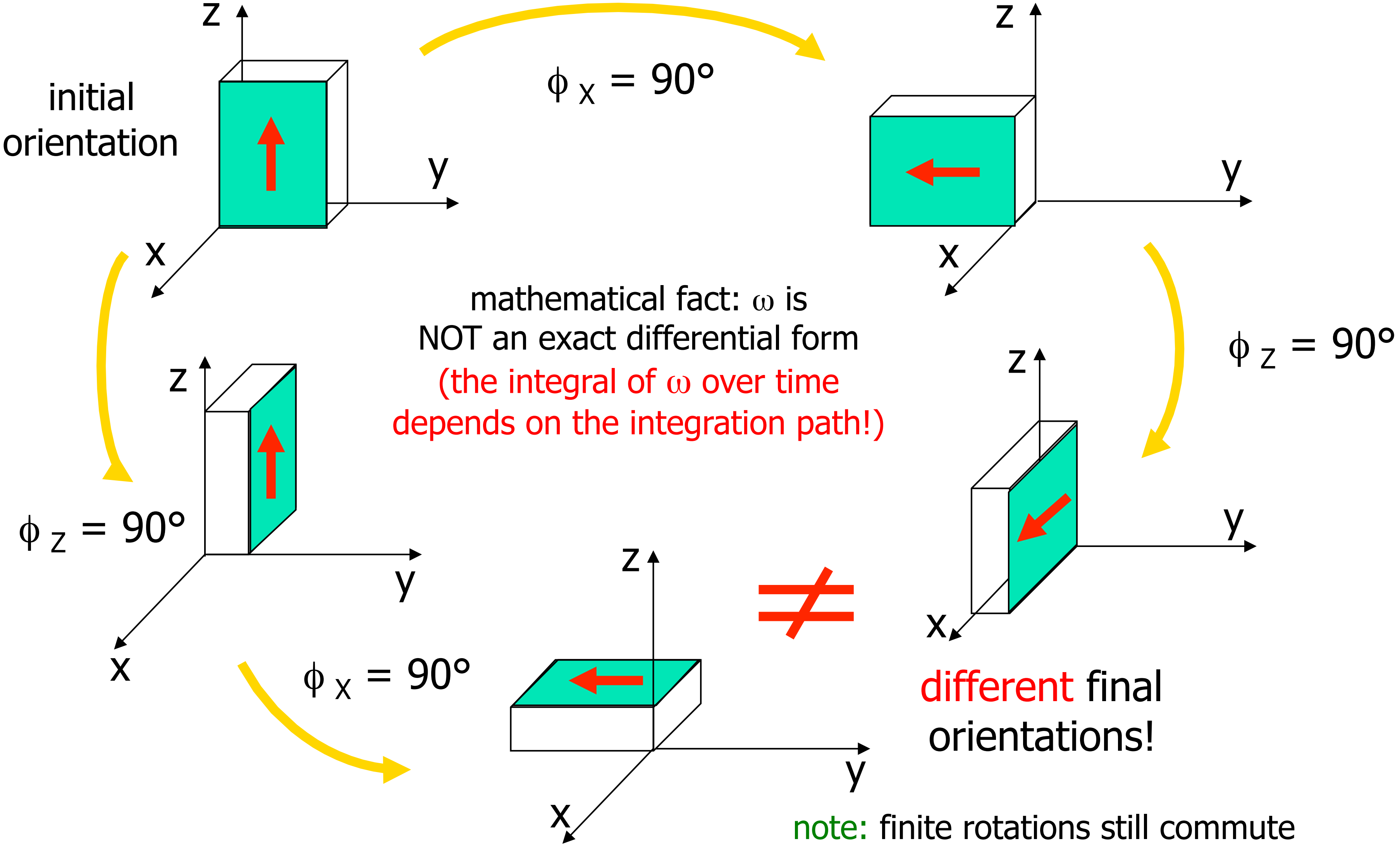
- finite $\Delta x, \Delta y, \Delta z$ or infinitesimal $dx, dy, dz$ translations (linear displacements) always commute



same final position

# Finite rotations do not commute

example



initial orientation

mathematical fact: $\omega$ is NOT an exact differential form (the integral of $\omega$ over time depends on the integration path!)

different final orientations!

note: finite rotations still commute when made around the same fixed axis

# Infinitesimal rotations commute!

- infinitesimal rotations $d\phi_X, d\phi_Y, d\phi_Z$ around x, y, z axes

$$R_X(\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_X & -\sin\phi_X \\ 0 & \sin\phi_X & \cos\phi_X \end{bmatrix} \Rightarrow R_X(d\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_X \\ 0 & d\phi_X & 1 \end{bmatrix}$$

$$R_Y(\phi_Y) = \begin{bmatrix} \cos\phi_Y & 0 & \sin\phi_Y \\ 0 & 1 & 0 \\ -\sin\phi_Y & 0 & \cos\phi_Y \end{bmatrix} \Rightarrow R_Y(d\phi_Y) = \begin{bmatrix} 1 & 0 & d\phi_Y \\ 0 & 1 & 0 \\ -d\phi_Y & 0 & 1 \end{bmatrix}$$

$$R_Z(\phi_Z) = \begin{bmatrix} \cos\phi_Z & -\sin\phi_Z & 0 \\ \sin\phi_Z & \cos\phi_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow R_Z(d\phi_Z) = \begin{bmatrix} 1 & -d\phi_Z & 0 \\ d\phi_Z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $R(d\phi) = \underset{\substack{\uparrow \\ \text{in any order}}}{R(d\phi_X, d\phi_Y, d\phi_Z)} = \begin{bmatrix} 1 & -d\phi_Z & d\phi_Y \\ d\phi_Z & 1 & -d\phi_X \\ -d\phi_Y & d\phi_X & 1 \end{bmatrix} \leftarrow$ neglecting second- and third-order (infinitesimal) terms

$$= I + S(d\phi)$$

# Time derivative of a rotation matrix

- let $R = R(t)$ be a rotation matrix, given as a function of time
- since $I = R(t)R^T(t)$, taking the time derivative of both sides yields

$$0 = d[R(t)R^T(t)]/dt = dR(t)/dt\ R^T(t) + R(t)\ dR^T(t)/dt$$

$$= dR(t)/dt\ R^T(t) + [dR(t)/dt\ R^T(t)]^T$$

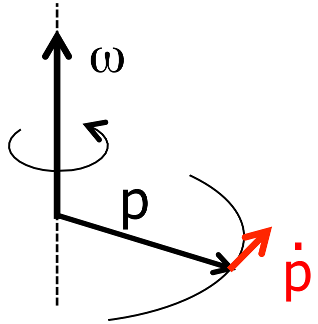
thus $dR(t)/dt\ R^T(t) = S(t)$ is a skew-symmetric matrix

- let $p(t) = R(t)p'$ a vector (with constant norm) rotated over time
- comparing

$$dp(t)/dt = dR(t)/dt\ p' = S(t)R(t)\ p' = S(t)\ p(t)$$

$$dp(t)/dt = \omega(t) \times p(t) = S(\omega(t))\ p(t)$$

we get $S = S(\omega)$



$$\boxed{\dot{R} = S(\omega)\ R} \iff \boxed{S(\omega) = \dot{R}\ R^T}$$

# Example

## Time derivative of an elementary rotation matrix

$$R_X(\phi(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi(t) & -\sin\phi(t) \\ 0 & \sin\phi(t) & \cos\phi(t) \end{bmatrix}$$

$$\dot{R}_X(\phi)\, R^T{}_X(\phi) = \dot{\phi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & -\cos\phi \\ 0 & \cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi} \\ 0 & \dot{\phi} & 0 \end{bmatrix} = S(\omega)$$

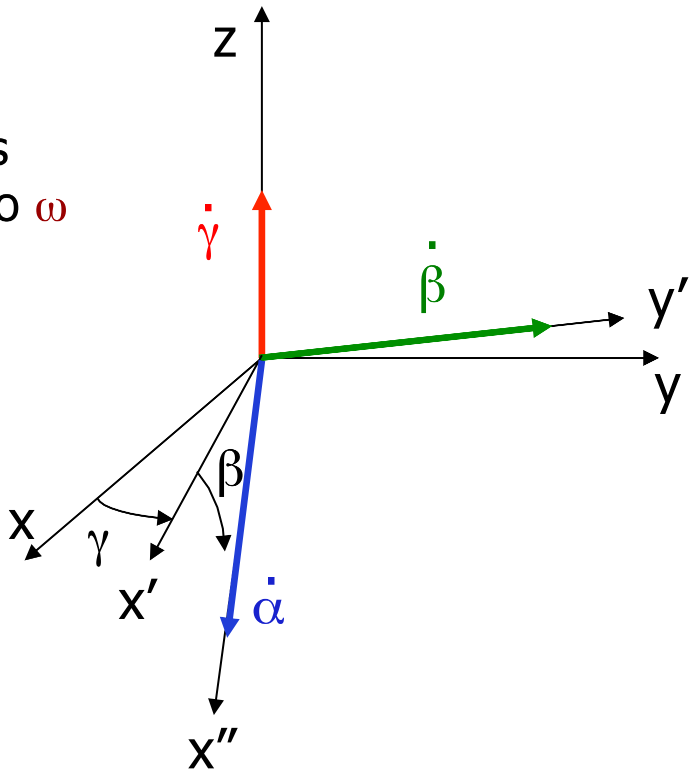
$$\Downarrow$$

$$\omega = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

# Time derivative of RPY angles and $\omega$

$$R_{RPY}(\alpha_x, \beta_y, \gamma_z) = R_{ZY'X''}(\gamma_z, \beta_y, \alpha_x)$$

the three contributions $\dot{\gamma}z$, $\dot{\beta}y'$, $\dot{\alpha}x''$ to $\omega$ are simply summed as vectors



$$\omega = \overbrace{\begin{array}{ccc} c\beta\, c\gamma & -s\gamma & 0 \\ c\beta\, s\gamma & c\gamma & 0 \\ -s\beta & 0 & 1 \\ x'' & y' & z \end{array}}^{T_{RPY}(\beta,\gamma)} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

1st col in $R_Z(\gamma_z)R_{Y'}(\beta_y)$

2nd col in $R_Z(\gamma_z)$

$$\det T_{RPY}(\beta,\gamma) = c\beta = 0$$

for $\beta = \pm\pi/2$
(singularity of the RPY representation)

similar treatment for the other 11 minimal representations...

# Robot Jacobian matrices

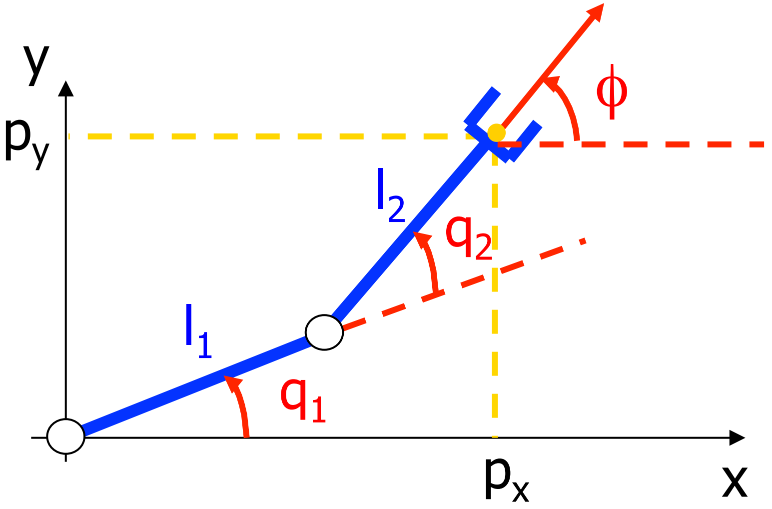
- analytical Jacobian (obtained by time differentiation)

$$r = \begin{pmatrix} p \\ \phi \end{pmatrix} = f_r(q) \quad \Rightarrow \quad \dot{r} = \begin{pmatrix} \dot{p} \\ \dot{\phi} \end{pmatrix} = \frac{\partial f_r(q)}{\partial q} \dot{q} = J_r(q)\,\dot{q}$$

- geometric Jacobian (no derivatives)

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} \dot{p} \\ \omega \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = J(q)\,\dot{q}$$

# Analytical Jacobian of planar 2R arm



direct kinematics

$$r \left\{ \begin{array}{l} p_x = l_1\, c_1 + l_2\, c_{12} \\ p_y = l_1\, s_1 + l_2\, s_{12} \\ \hdashline \phi \;= q_1 + q_2 \end{array} \right.$$

$$\dot{p}_x = -\, l_1\, s_1\, \dot{q}_1 - l_2\, s_{12}\, (\dot{q}_1 + \dot{q}_2)$$

$$\dot{p}_y = l_1\, c_1\, \dot{q}_1 + l_2\, c_{12}\, (\dot{q}_1 + \dot{q}_2)$$

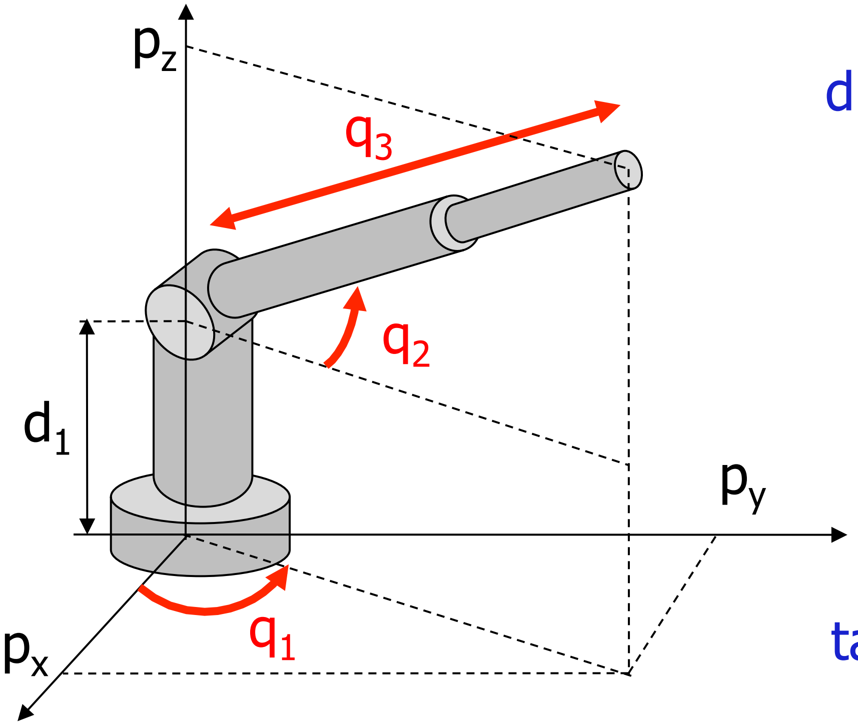
$$\dot{\phi} \;= \omega_z = \dot{q}_1 + \dot{q}_2$$

$$\Rightarrow \quad J_r(q) = \left( \begin{array}{cc} -\, l_1\, s_1 - l_2\, s_{12} & -\, l_2\, s_{12} \\ l_1\, c_1 + l_2\, c_{12} & l_2\, c_{12} \\ \hdashline 1 & 1 \end{array} \right)$$

here, all rotations occur around the same fixed axis z (normal to the plane of motion)

given r, this is a 3 x 2 matrix

# Analytical Jacobian of polar robot



direct kinematics (here, r = p)

$$\left.\begin{aligned} p_x &= q_3\, c_2\, c_1 \\ p_y &= q_3\, c_2\, s_1 \\ p_z &= d_1 + q_3\, s_2 \end{aligned}\right\} f_r(q)$$

taking the time derivative

$$v = \dot{p} = \underbrace{\begin{pmatrix} -q_3c_2s_1 & -q_3s_2c_1 & c_2c_1 \\ q_3c_2c_1 & -q_3s_2s_1 & c_2s_1 \\ 0 & q_3c_2 & s_2 \end{pmatrix}}_{\frac{\partial f_r(q)}{\partial q}} \dot{q} = J_r(q)\,\dot{q}$$

# Geometric Jacobian

always a 6 x n matrix

end-effector
instantaneous
velocity

$$\begin{pmatrix} v_E \\ \omega_E \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = \begin{pmatrix} J_{L1}(q) & \cdots & J_{Ln}(q) \\ J_{A1}(q) & \cdots & J_{An}(q) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

superposition of effects

$$v_E = \underbrace{J_{L1}(q)\,\dot{q}_1} + \ldots + J_{Ln}(q)\,\dot{q}_n \qquad \omega_E = \underbrace{J_{A1}(q)\,\dot{q}_1} + \ldots + J_{An}(q)\,\dot{q}_n$$

contribution to the linear e-e velocity due to $\dot{q}_1$

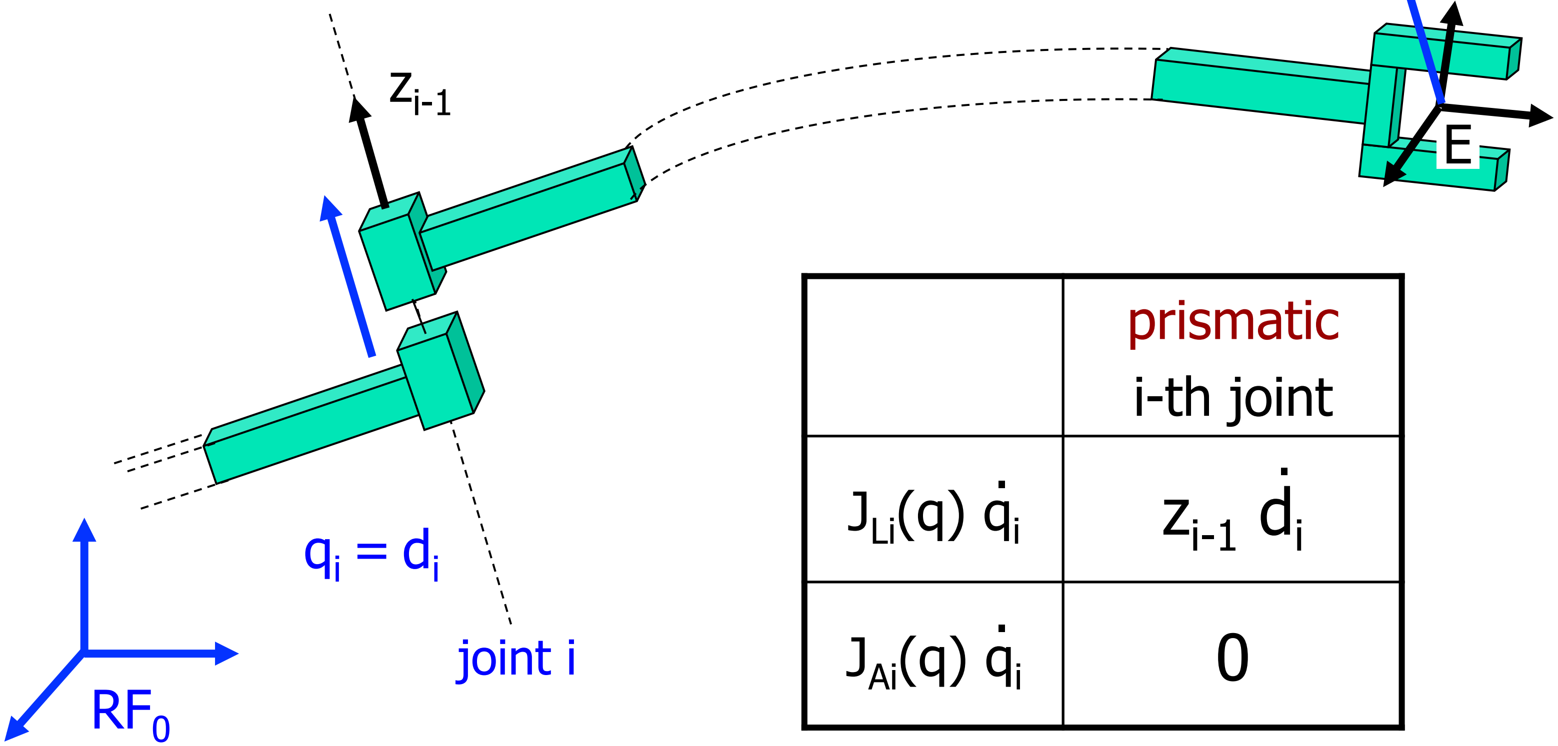
contribution to the angular e-e velocity due to $\dot{q}_1$

linear and angular velocity belong to (linear) vector spaces in $R^3$

# Contribution of a prismatic joint

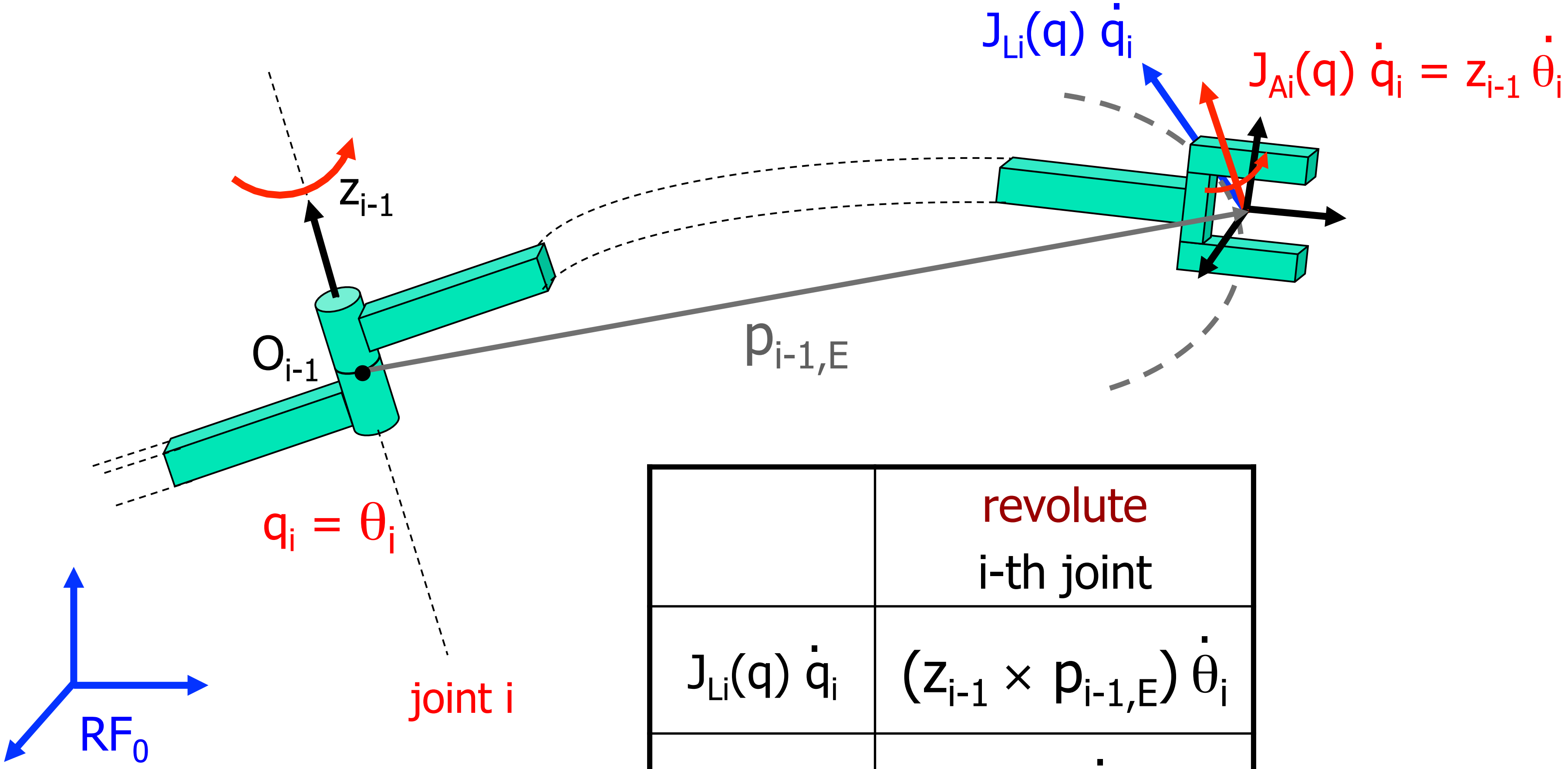
note: joints beyond the i-th one are considered to be "frozen", so that the distal part of the robot is a single rigid body

$J_{Li}(q)\,\dot{q}_i = z_{i-1}\,\dot{d}_i$



| | prismatic<br>i-th joint |
|---|---|
| $J_{Li}(q)\,\dot{q}_i$ | $z_{i-1}\,\dot{d}_i$ |
| $J_{Ai}(q)\,\dot{q}_i$ | 0 |

# Contribution of a revolute joint



| | revolute<br>i-th joint |
|---|---|
| $J_{Li}(q)\,\dot{q}_i$ | $(z_{i-1} \times p_{i-1,E})\,\dot{\theta}_i$ |
| $J_{Ai}(q)\,\dot{q}_i$ | $z_{i-1}\,\dot{\theta}_i$ |

# Expression of geometric Jacobian

$$\left(\begin{pmatrix} \dot{p}_{0,E} \\ \omega_E \end{pmatrix} =\right) \begin{pmatrix} v_E \\ \omega_E \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = \begin{pmatrix} J_{L1}(q) & \cdots & J_{Ln}(q) \\ J_{A1}(q) & \cdots & J_{An}(q) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

| | prismatic i-th joint | revolute i-th joint |
|---|---|---|
| $J_{Li}(q)$ | $z_{i-1}$ | $z_{i-1} \times p_{i-1,E}$ |
| $J_{Ai}(q)$ | $0$ | $z_{i-1}$ |

this can be also computed as

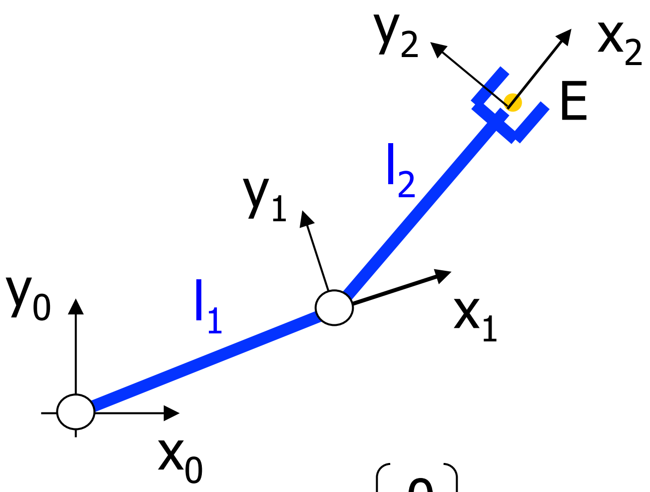
$$= \frac{\partial p_{0,E}}{\partial q_i}$$

$$z_{i-1} = {}^{0}R_1(q_1)\ldots{}^{i-2}R_{i-1}(q_{i-1}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$p_{i-1,E} = p_{0,E}(q_1,\ldots,q_n) - p_{0,i-1}(q_1,\ldots,q_{i-1})$$

all vectors should be expressed in the same reference frame (here, the base frame $RF_0$)

# Example: planar 2R arm



DENAVIT-HARTENBERG table

| joint | $\alpha_i$ | $d_i$ | $a_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $l_1$ | $q_1$ |
| 2 | 0 | 0 | $l_2$ | $q_2$ |

$$z_0 = z_1 = z_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

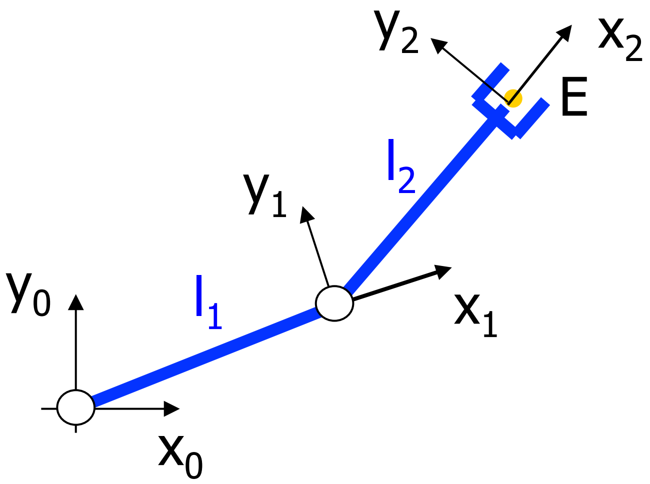
$${}^0A_1 = \begin{pmatrix} c_1 & -s_1 & 0 & l_1c_1 \\ s_1 & c_1 & 0 & l_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow p_{0,1}$$

$$p_{1,E} = p_{0,E} - p_{0,1}$$

$$J = \begin{pmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{pmatrix}$$

$${}^0A_2 = \begin{pmatrix} c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_{12} \\ s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow p_{0,E}$$

# Geometric Jacobian of planar 2R arm



$$J = \begin{pmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{pmatrix}$$

$$= \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

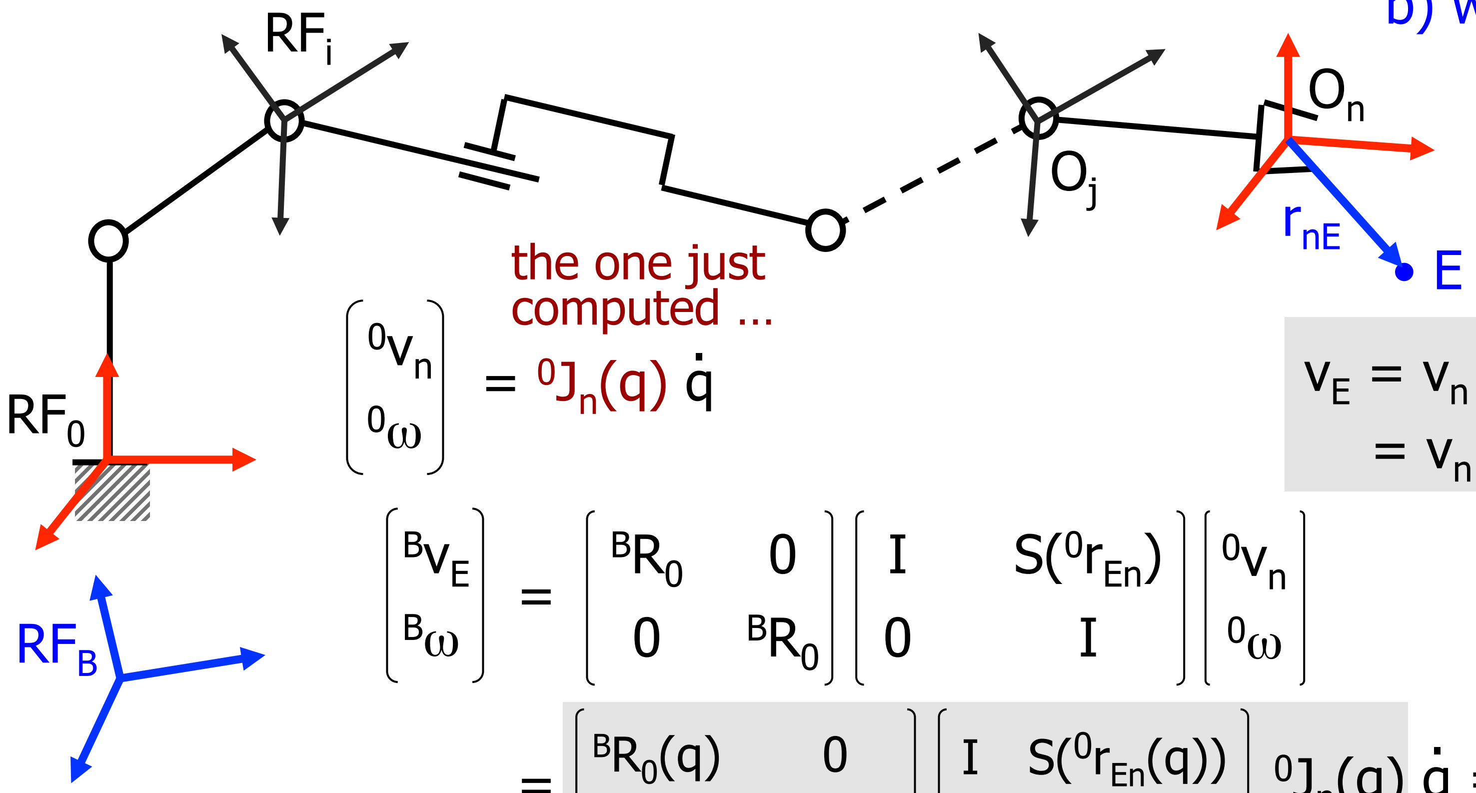
compare rows 1, 2, and 6 with the analytical Jacobian in slide #12!

**note:** the Jacobian is here a 6×2 matrix, thus its maximum rank is 2

⇓

at most 2 components of the linear/angular end-effector velocity can be independently assigned

# Transformations of the Jacobian matrix



b) we may choose
E ⇒ $O_j(q)$

the one just computed …

$$\begin{pmatrix} {}^0v_n \\ {}^0\omega \end{pmatrix} = {}^0J_n(q)\,\dot{q}$$

$$\begin{aligned} v_E &= v_n + \omega \times r_{nE} \\ &= v_n + S(r_{En})\,\omega \end{aligned}$$

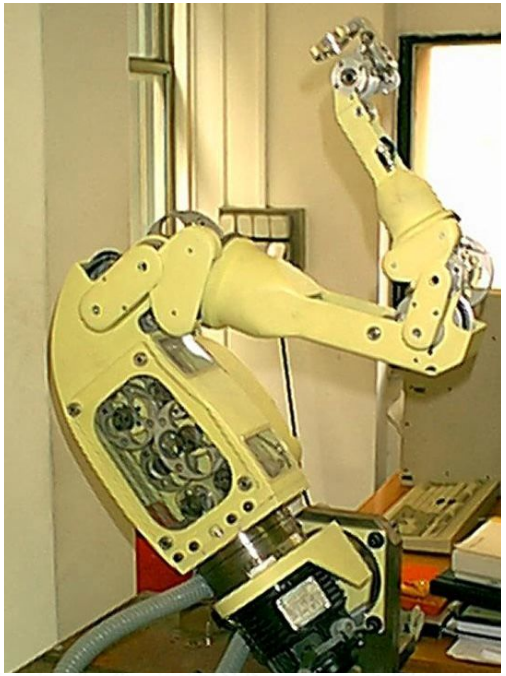
$$\begin{pmatrix} {}^Bv_E \\ {}^B\omega \end{pmatrix} = \begin{pmatrix} {}^BR_0 & 0 \\ 0 & {}^BR_0 \end{pmatrix} \begin{pmatrix} I & S({}^0r_{En}) \\ 0 & I \end{pmatrix} \begin{pmatrix} {}^0v_n \\ {}^0\omega \end{pmatrix}$$

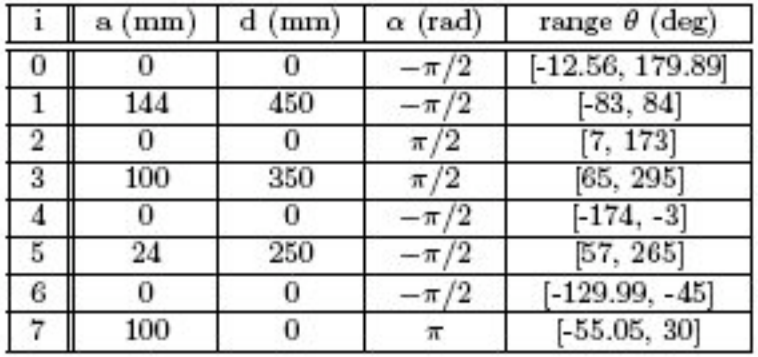
$$= \underbrace{\begin{pmatrix} {}^BR_0(q) & 0 \\ 0 & {}^BR_0(q) \end{pmatrix} \begin{pmatrix} I & S({}^0r_{En}(q)) \\ 0 & I \end{pmatrix}}_{\text{never singular!}} {}^0J_n(q)\,\dot{q} = {}^BJ_E(q)\,\dot{q}$$

a) we may choose
RF$_B$ ⇒ RF$_i(q)$

# Example: Dexter robot

- 8R robot manipulator with transmissions by pulleys and steel cables (joints 3 to 8)
  - lightweight: only 15 kg in motion
  - motors located in second link
  - incremental encoders (homing)
  - redundancy degree for e-e pose task: n-m=2
  - compliant in the interaction with environment





| i | a (mm) | d (mm) | $\alpha$ (rad) | range $\theta$ (deg) |
|---|---|---|---|---|
| 0 | 0 | 0 | $-\pi/2$ | [-12.56, 179.89] |
| 1 | 144 | 450 | $-\pi/2$ | [-83, 84] |
| 2 | 0 | 0 | $\pi/2$ | [7, 173] |
| 3 | 100 | 350 | $\pi/2$ | [65, 295] |
| 4 | 0 | 0 | $-\pi/2$ | [-174, -3] |
| 5 | 24 | 250 | $-\pi/2$ | [57, 265] |
| 6 | 0 | 0 | $-\pi/2$ | [-129.99, -45] |
| 7 | 100 | 0 | $\pi$ | [-55.05, 30] |

# Mid-frame Jacobian of Dexter robot

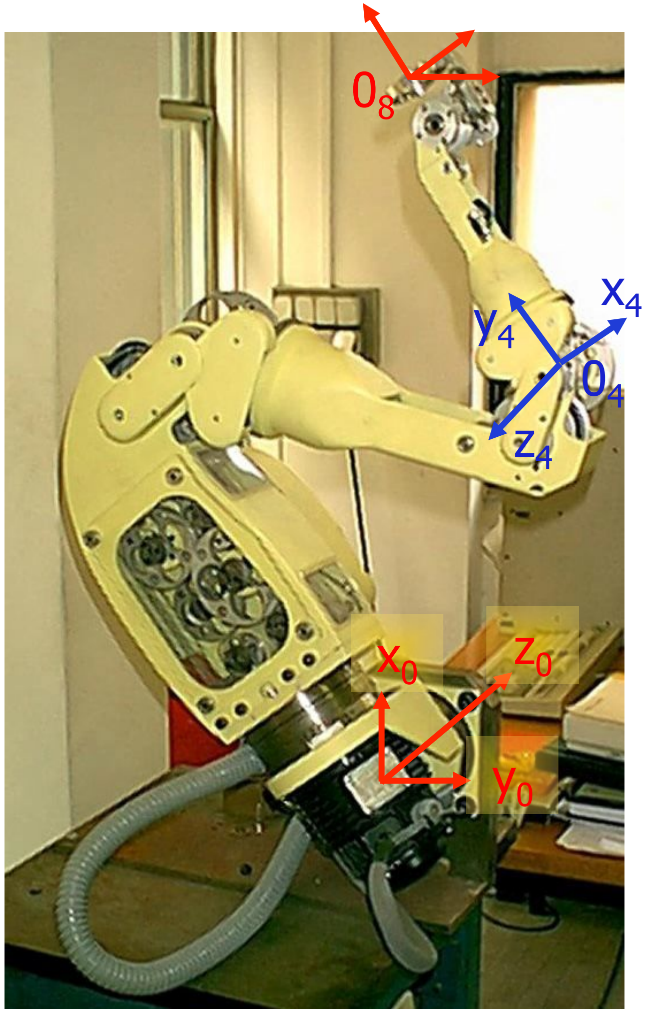
- geometric Jacobian ${}^{0}J_{8}(q)$ is very complex
- "mid-frame" Jacobian ${}^{4}J_{4}(q)$ is relatively simple!

$$
{}^{4}\hat{J}_4 = \left[ \begin{array}{c}
d_1 s_1 s_3 + d_3 s_3 c_2 s_1 - a_1 c_3 c_1 s_2 - d_1 c_3 c_1 c_2 - d_3 c_1 c_3 \\
-a_3 s_3 c_2 s_1 + a_3 c_3 c_1 + a_1 c_1 c_2 - d_1 c_1 s_2 \\
-d_3 c_3 c_2 s_1 - a_1 s_3 c_1 s_2 - d_1 s_3 c_1 c_2 - d_3 s_3 c_1 - d_1 s_1 c_3 + a_3 s_2 s_1 \\
-c_3 c_2 s_1 - s_3 c_1 \\
-s_2 s_1 \\
-s_3 c_2 s_1 + c_3 c_1
\end{array} \right.
$$

6 rows,
8 columns

$$
\begin{array}{ccccc}
a_1 s_3 + d_3 s_3 s_2 & d_3 c_3 & 0 & 0 & 0 \\
-a_3 s_3 s_2 & -a_3 c_3 & 0 & 0 & 0 \\
-a_1 c_3 - d_3 c_3 s_2 - a_3 c_2 & d_3 s_3 & -a_3 & 0 & 0 \\
-c_3 s_2 & s_3 & 0 & 0 & -s_4 \\
c_2 & 0 & 1 & 0 & c_4 \\
-s_3 s_2 & -c_3 & 0 & 1 & 0
\end{array}
$$

$$
\left. \begin{array}{cc}
-a_5 s_4 - d_5 c_5 c_4 & -a_5 s_5 c_4 c_6 + d_5 s_5 s_6 c_4 \\
-d_5 c_5 s_4 + a_5 c_4 & d_5 s_5 s_6 s_4 - a_5 s_5 s_4 c_6 \\
d_5 s_5 & -a_5 c_6 c_5 + d_5 c_5 s_6 \\
-c_4 s_5 & -c_4 c_5 s_6 + s_4 c_6 \\
-s_4 s_5 & -s_4 c_5 s_6 - c_4 c_6 \\
-c_5 & s_5 s_6
\end{array} \right]
$$



# Summary of differential relations

**$\dot{p} \rightleftarrows v$** $\quad \dot{p} = v$

**$\dot{\phi} \rightleftarrows \omega$**

$$\omega = \omega_{\dot{\phi}_1} + \omega_{\dot{\phi}_2} + \omega_{\dot{\phi}_3} = a_1\,\dot{\phi}_1 + a_2(\phi_1)\,\dot{\phi}_2 + a_3(\phi_1, \phi_2)\,\dot{\phi}_3 = T(\phi)\dot{\phi}$$

(moving) axes of definition for the sequence of rotations $\phi_i$

$$r = \begin{pmatrix} p \\ \phi \end{pmatrix} \Rightarrow \quad J(q) = \begin{pmatrix} I & 0 \\ 0 & T(\phi) \end{pmatrix} J_r(q) \qquad J_r(q) = \begin{pmatrix} I & 0 \\ 0 & T^{-1}(\phi) \end{pmatrix} J(q)$$

$T(\phi)$ has always a singularity ⇔ singularity of the specific minimal representation of orientation

**$\dot{R} \rightleftarrows \omega$**

$$\dot{R} = S(\omega)\,R \quad \Longleftrightarrow$$

for each column $r_i$ of R (unit vector of a frame), we have $\dot{r}_i = \omega \times r_i$

# Acceleration relations (and beyond…)

## Higher-order differential kinematics

- differential relations between motion in the joint space and motion in the task space can be established at the second order, third order, ...
- the analytical Jacobian always "weights" the highest-order derivative

velocity $\dot{r} = J_r(q)\,\dot{q}$

matrix function $N_2(q,\dot{q})$

acceleration $\ddot{r} = J_r(q)\,\ddot{q} + \dot{J}_r(q)\,\dot{q}$

jerk $\dddot{r} = J_r(q)\,\dddot{q} + 2\,\dot{J}_r(q)\,\ddot{q} + \ddot{J}_r(q)\,\dot{q}$

snap $\ddddot{r} = J_r(q)\,\ddddot{q} + \dots$

matrix function $N_3(q,\dot{q},\ddot{q})$

- the same holds true also for the geometric Jacobian J(q)

# Primer on linear algebra

given a matrix J: $m \times n$ (m rows, n columns)

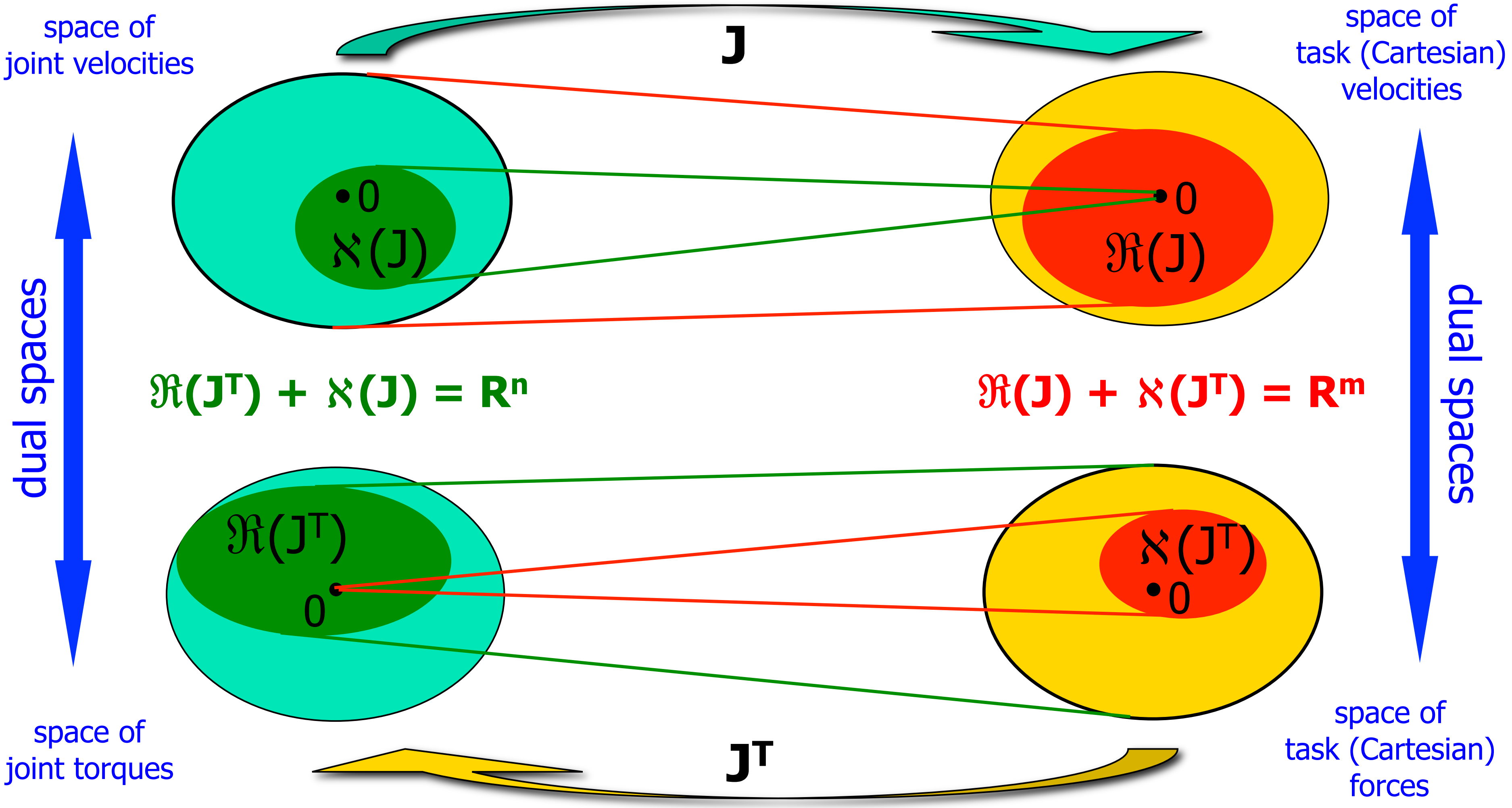
- rank $\rho(J)$ = max # of rows or columns that are linearly independent
  - $\rho(J) \leq \min(m,n)$ (if equality holds, J has "full rank")
  - if $m = n$ and J has full rank, J is "non singular" and the inverse $J^{-1}$ exists
  - $\rho(J)$ = dimension of the largest non singular square submatrix of J
- range $\Re(J)$ = vector subspace generated by all possible linear combinations of the columns of J ← also called "image" of J

  $$\Re(J)=\{v \in R^m : \exists\, \xi \in R^n,\ v = J\,\xi\}$$

  - $\dim(\Re(J)) = \rho(J)$
- kernel $\aleph(J)$ = vector subspace of all vectors $\xi \in R^n$ such that $J \cdot \xi = 0$ ← also called "null space" of J
  - $\dim(\aleph(J)) = n - \rho(J)$
- $\boxed{\Re(J) + \aleph(J^T) = R^m}$ e $\boxed{\Re(J^T) + \aleph(J) = R^n}$
  - sum of vector subspaces $V_1 + V_2$ = vector space where any element v can be written as $v = v_1 + v_2$, with $v_1 \in V_1$ , $v_2 \in V_2$
- all the above quantities/subspaces can be computed using, e.g., Matlab

# Robot Jacobian

## decomposition in linear subspaces and duality



$$\Re(J^T) + \aleph(J) = R^n$$

$$\Re(J) + \aleph(J^T) = R^m$$

(in a given configuration q)

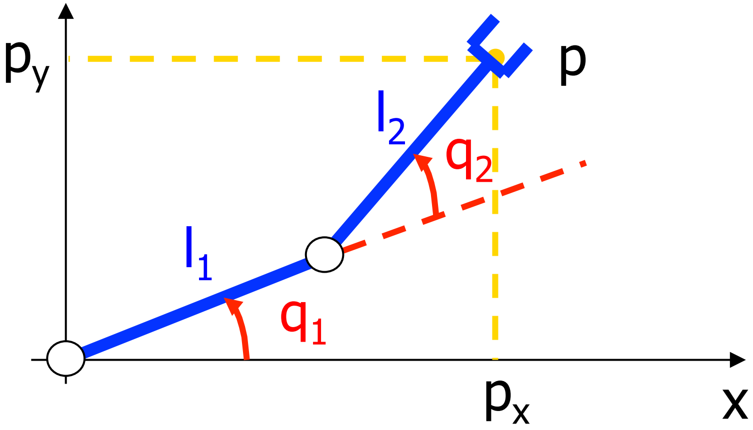
# Mobility analysis

- $\rho(J) = \rho(J(q))$, $\Re(J) = \Re(J(q))$, $\aleph(J^T) = \aleph(J^T(q))$ are locally defined, i.e., they depend on the current configuration q
- $\Re(J(q))$ = subspace of all "generalized" velocities (with linear and/or angular components) that can be instantaneously realized by the robot end-effector when varying the joint velocities at the configuration q
- if J(q) has max rank (typically = m) in the configuration q, the robot end-effector can be moved in any direction of the task space $R^m$
- if $\rho(J(q)) < m$, there exist directions in $R^m$ along which the robot end-effector cannot move (instantaneously!)
  - these directions lie in $\aleph(J^T(q))$, namely the complement of $\Re(J(q))$ to the task space $R^m$, which is of dimension $m - \rho(J(q))$
- when $\aleph(J(q)) \neq \{0\}$, there exist non-zero joint velocities that produce zero end-effector velocity ("self motions")
  - this always happens for $m<n$, i.e., when the robot is redundant for the task

# Kinematic singularities

- **configurations where the Jacobian loses rank**
  $\Leftrightarrow$ loss of instantaneous mobility of the robot end-effector
- for $m = n$, they correspond to Cartesian poses at which the number of solutions of the inverse kinematics problem differs from the "generic" case
- "in" a singular configuration, we cannot find a joint velocity that realizes a desired end-effector velocity in an arbitrary direction of the task space
- "close" to a singularity, large joint velocities may be needed to realize some (even small) velocity of the end-effector
- finding and analyzing in advance all singularities of a robot helps in avoiding them during trajectory planning and motion control
  - when $m = n$: find the configurations $q$ such that $\det J(q) = 0$
  - when $m < n$: find the configurations $q$ such that all $m \times m$ minors of $J$ are singular (or, equivalently, such that $\det [J(q)\, J^T(q)] = 0$)
- finding all singular configurations of a robot with a large number of joints, or the actual "distance" from a singularity, is a hard computational task

# Singularities of planar 2R arm



direct kinematics

$$p_x = l_1\, c_1 + l_2\, c_{12}$$

$$p_y = l_1\, s_1 + l_2\, s_{12}$$

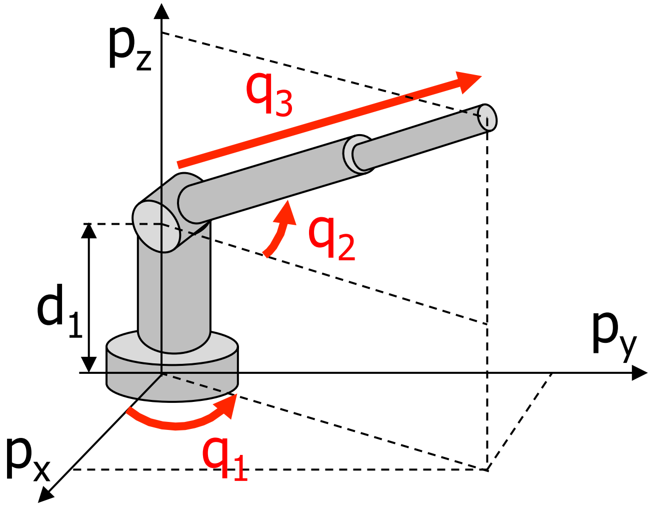
analytical Jacobian

$$\dot{p} = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix} \dot{q} = J(q)\, \dot{q}$$

$$\boxed{\det J(q) = l_1 l_2 s_2}$$

- singularities: arm is stretched ($q_2 = 0$) or folded ($q_2 = \pi$)
- singular configurations correspond here to Cartesian points on the boundary of the workspace
- in general, these singularities separate regions in the joint space with distinct inverse kinematic solutions (e.g., “elbow up” or “down”)

# Singularities of polar (RRP) arm



$$\boxed{\det J(q) = q_3^2\, c_2}$$

direct kinematics

$$p_x = q_3\, c_2\, c_1$$

$$p_y = q_3\, c_2\, s_1$$

$$p_z = d_1 + q_3\, s_2$$

analytical Jacobian

$$\dot{p} = \begin{pmatrix} -q_3 s_1 c_2 & -q_3 c_1 s_2 & c_1 c_2 \\ q_3 c_1 c_2 & -q_3 s_1 s_2 & s_1 c_2 \\ 0 & q_3 c_2 & s_2 \end{pmatrix} \dot{q} = J(q)\, \dot{q}$$

- singularities
  - E-E is along the z axis ($q_2 = \pm\pi/2$): simple singularity ⇒ rank J = 2
  - third link is fully retracted ($q_3 = 0$): double singularity ⇒ rank J drops to 1
- all singular configurations correspond here to Cartesian points internal to the workspace (supposing no limits for the prismatic joint)

# Singularities of robots with spherical wrist

- $n = 6$, last three joints are revolute and their axes intersect at a point
- without loss of generality, we set $O_6 = W =$ center of spherical wrist (i.e., choose $d_6 = 0$ in the DH table)

$$J(q) = \begin{pmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{pmatrix}$$

- since $\det J(q_1,\ldots,q_5) = \det J_{11} \cdot \det J_{22}$, there is a decoupling property
  - $\det J_{11}(q_1,\ldots,q_3) = 0$ provides the arm singularities
  - $\det J_{22}(q_4, q_5) = 0$ provides the wrist singularities
- being $J_{22} = [z_3\ z_4\ z_5]$ (in the geometric Jacobian), wrist singularities correspond to when $z_3$, $z_4$ and $z_5$ become linearly dependent vectors

  $\Rightarrow$ when either $q_5 = 0$ or $q_5 = \pm\pi/2$
- inversion of J is simpler (block triangular structure)
- the determinant of J will never depend on $q_1$: why?