Robotics 1

Position and orientation of rigid bodies

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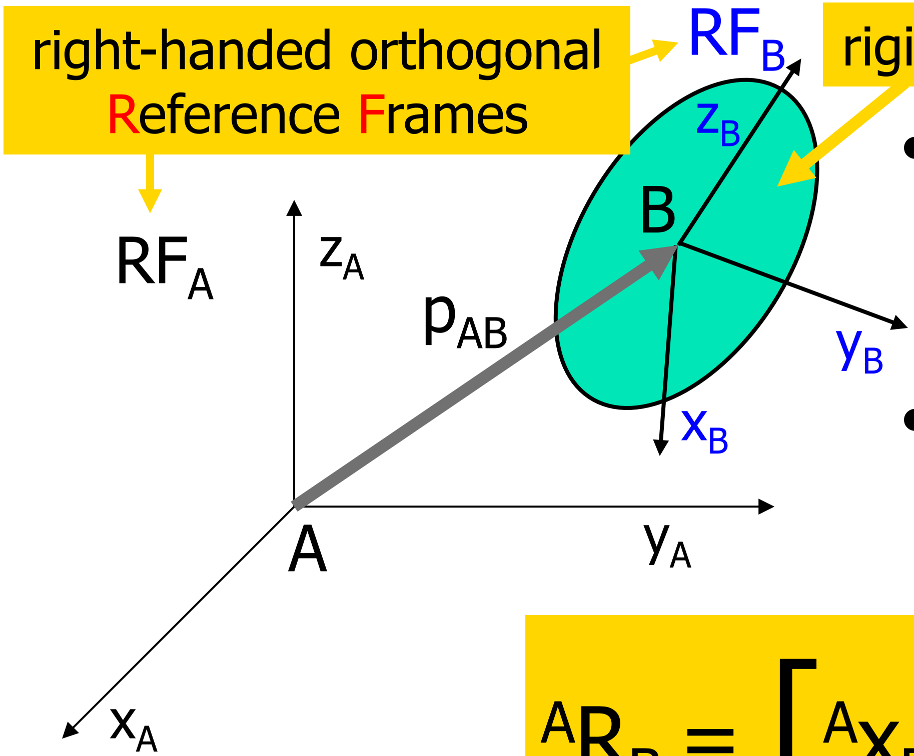
Position and orientation



- **position:** $^A p_{AB}$ (vector $\in \mathbb{R}^3$), expressed in RF_A (use of coordinates other than Cartesian is possible, e.g. cylindrical or spherical)
- **orientation:** orthonormal 3x3 matrix ($R^T = R^{-1} \Rightarrow\ {}^A R_B\, {}^B R_A = I$), with det = +1

$$^A R_B = \left[\, ^A x_B \;\; ^A y_B \;\; ^A z_B \,\right]$$

- $x_A\ y_A\ z_A$ ($x_B\ y_B\ z_B$) are unit vectors (with unitary norm) of frame RF_A (RF_B)
- components in $^A R_B$ are the direction cosines of the axes of RF_B with respect to (w.r.t.) RF_A

Rotation matrix

$$ {}^{A}R_{B} = \begin{bmatrix} x_A^T\, x_B & x_A^T\, y_B & x_A^T\, z_B \\ y_A^T\, x_B & y_A^T\, y_B & y_A^T\, z_B \\ z_A^T\, x_B & z_A^T\, y_B & z_A^T\, z_B \end{bmatrix} $$

direction cosine of z_B w.r.t. x_A

orthonormal, with det = +1

algebraic structure of a group SO(3) (neutral element = I; inverse element = R^T)

chain rule property

$$ \boxed{{}^{k}R_{i} \cdot {}^{i}R_{j} = {}^{k}R_{j}} $$

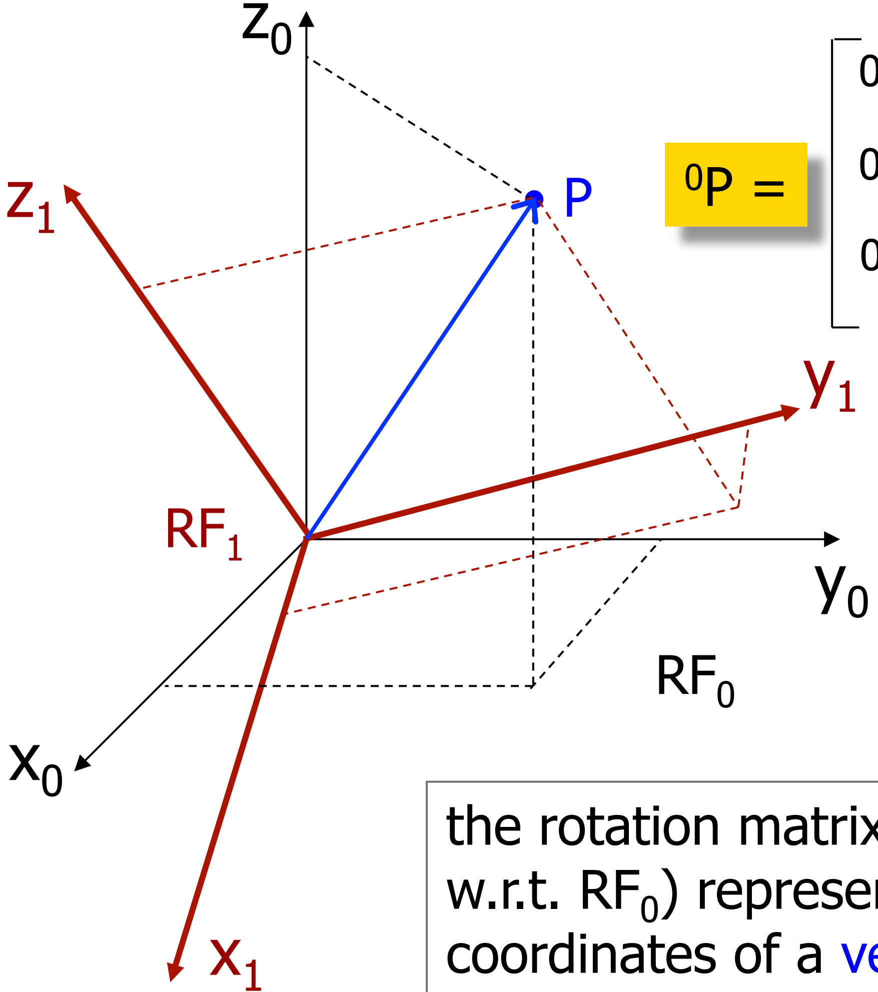
orientation of RF_i w.r.t. RF_k

orientation of RF_j w.r.t. RF_i

orientation of RF_j w.r.t. RF_k

NOTE: in general, the product of rotation matrices does not commute!

Change of coordinates



$$ {}^0P = \begin{bmatrix} {}^0p_x \\ {}^0p_y \\ {}^0p_z \end{bmatrix} = {}^1p_x\, {}^0\mathbf{x}_1 + {}^1p_y\, {}^0\mathbf{y}_1 + {}^1p_z\, {}^0\mathbf{z}_1 $$

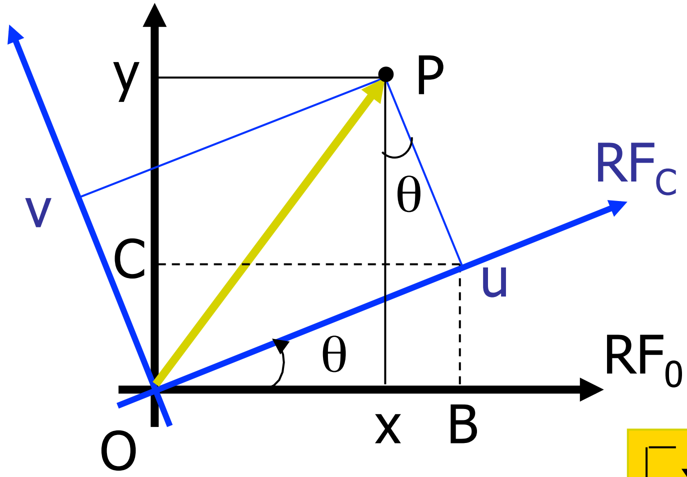
$$ = \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{z}_1 \end{bmatrix} \begin{bmatrix} {}^1p_x \\ {}^1p_y \\ {}^1p_z \end{bmatrix} $$

$$ = {}^0R_1\, {}^1P $$

the rotation matrix ${}^0R_1$ (i.e., the orientation of RF_1 w.r.t. RF_0) represents also the change of coordinates of a vector from RF_1 to RF_0

Ex: Orientation of frames in a plane

(elementary rotation around z-axis)



$$x = OB - xB = u \cos\theta - v \sin\theta$$
$$y = OC + Cy = u \sin\theta + v \cos\theta$$
$$z = w$$

or...

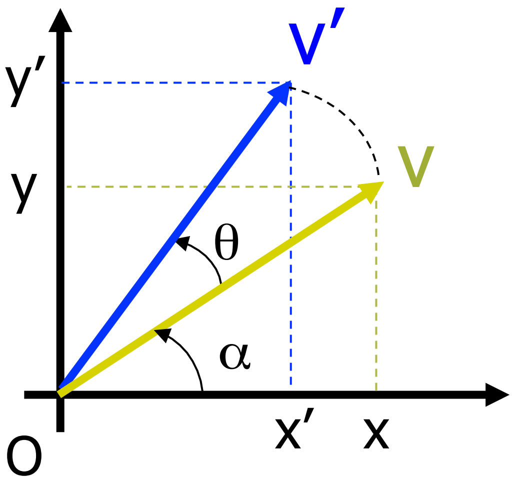
$$\begin{array}{c} \\ {}^{0}OP \rightarrow \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{array}{c} \quad {}^{0}x_C \downarrow \qquad {}^{0}y_C \downarrow \qquad {}^{0}z_C \downarrow \\ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_z(\theta) \begin{array}{c} {}^{C}OP \downarrow \\ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \end{array}$$

$$\boxed{R_z(-\theta) = R_z^T(\theta)}$$

similarly:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Ex: Rotation of a vector around z



$$x = |v| \cos \alpha$$
$$y = |v| \sin \alpha$$

$$x' = |v| \cos(\alpha + \theta) = |v| (\cos \alpha \cos \theta - \sin \alpha \sin \theta)$$
$$= x \cos \theta - y \sin \theta$$

$$y' = |v| \sin(\alpha + \theta) = |v| (\sin \alpha \cos \theta + \cos \alpha \sin \theta)$$
$$= x \sin \theta + y \cos \theta$$

$$z' = z$$

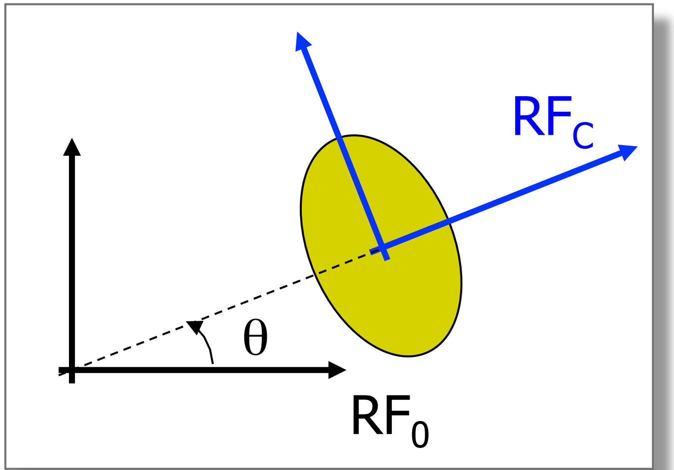
or...

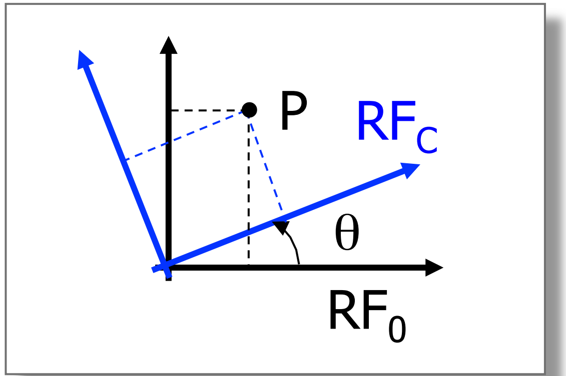
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_z(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

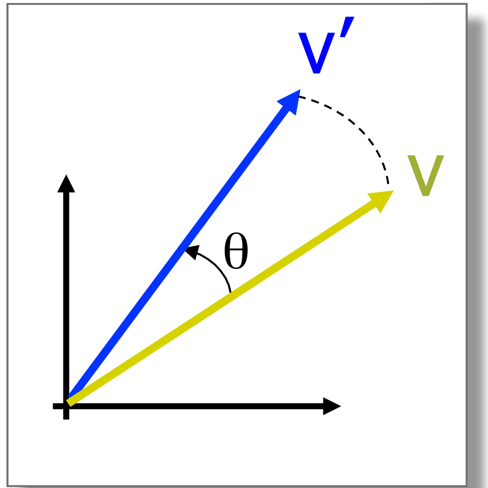
...as before!

Equivalent interpretations of a rotation matrix

the same rotation matrix, e.g., $R_z(\theta)$, may represent:







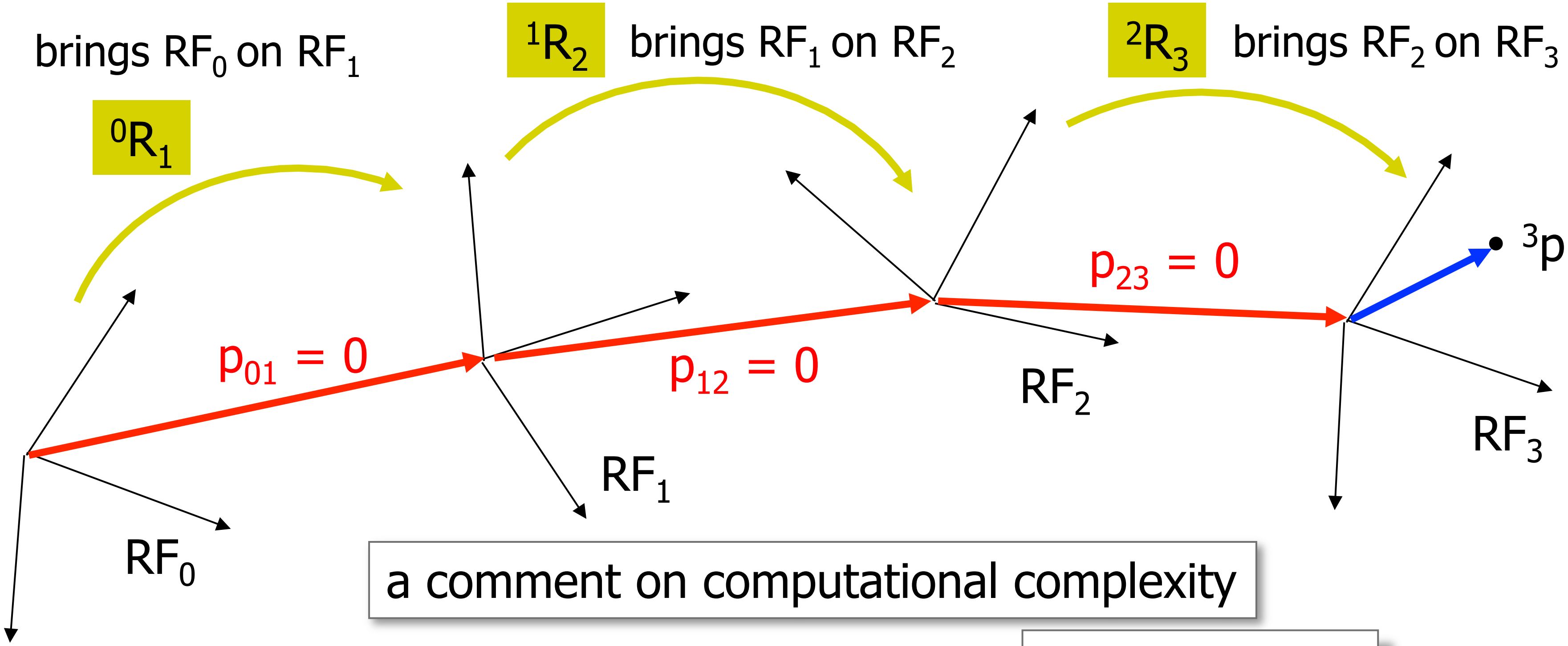
the orientation of a rigid body with respect to a reference frame RF_0
ex: $[{}^0x_c \; {}^0y_c \; {}^0z_c] = R_z(\theta)$

the change of coordinates from RF_C to RF_0
ex: ${}^0P = R_z(\theta)\,{}^CP$

the vector rotation operator
ex: $v' = R_z(\theta)\, v$

the rotation matrix ${}^0R_C$ is an operator superposing frame RF_0 to frame RF_C

Composition of rotations



a comment on computational complexity

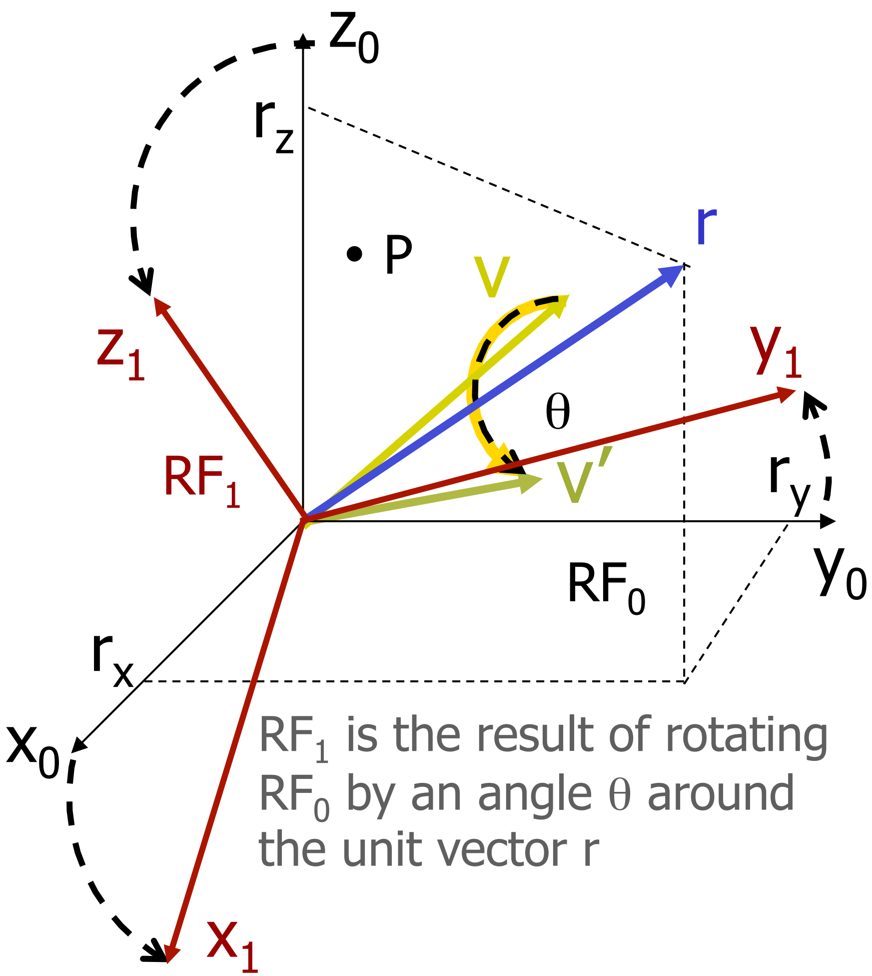
$${}^0p = ({}^0R_1\,{}^1R_2\,{}^2R_3)\,{}^3p = {}^0R_3\,{}^3p$$

63 products
42 summations

$${}^0p = {}^0R_1\,\underbrace{({}^1R_2\,\underbrace{({}^2R_3\,{}^3p)}_{{}^2p})}_{{}^1p}$$

27 products
18 summations

Axis/angle representation



RF$_1$ is the result of rotating RF$_0$ by an angle θ around the unit vector r

DATA

- unit vector r ($\|r\| = 1$)
- θ (positive if counterclockwise, as seen from an "observer" oriented like r with head placed on the arrow)

DIRECT PROBLEM

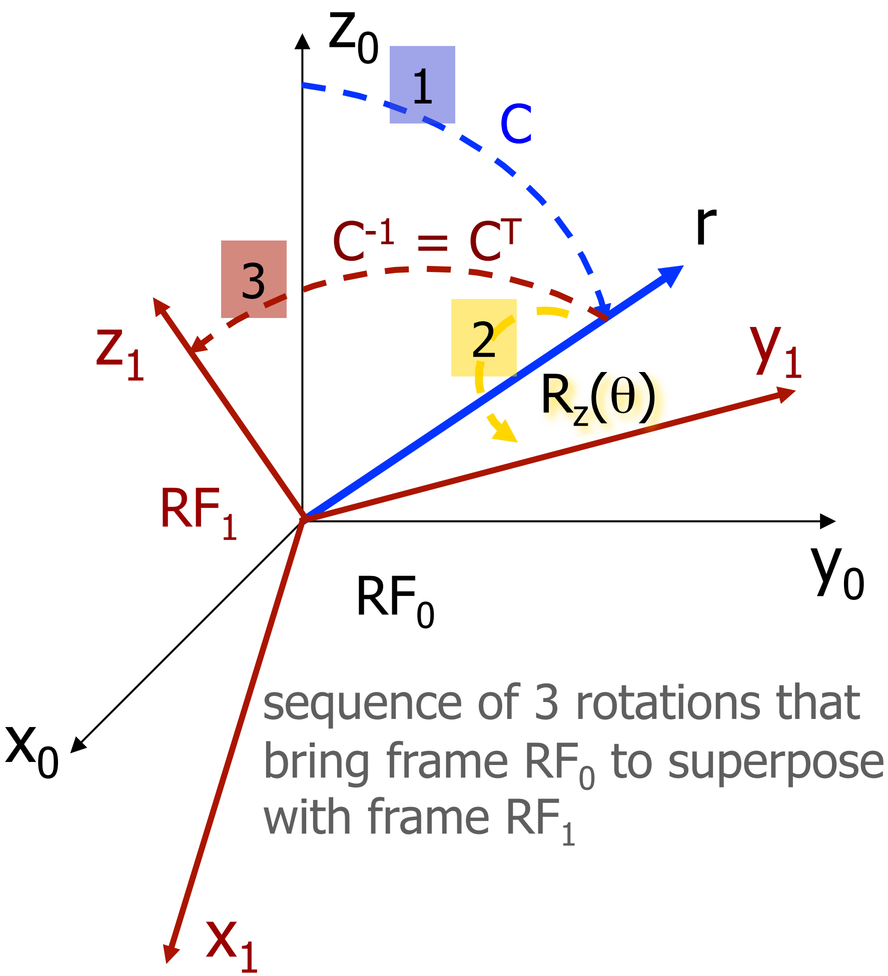
find

$$R(\theta,r) = [{}^0x_1 \ {}^0y_1 \ {}^0z_1]$$

such that

$${}^0P = R(\theta,r)\,{}^1P \qquad {}^0v' = R(\theta,r)\,{}^0v$$

Axis/angle: Direct problem



sequence of 3 rotations that bring frame RF_0 to superpose with frame RF_1

$$R(\theta,r) = C\, R_z(\theta)\, C^T$$

sequence of three rotations

$$C = \begin{bmatrix} n & s & r \end{bmatrix}$$

after the first rotation the z-axis coincides with r

n and s are orthogonal unit vectors such that

$n \times s = r$, or

$$n_y s_z - s_y n_z = r_x$$
$$n_z s_x - s_z n_x = r_y$$
$$n_x s_y - s_x n_y = r_z$$

Axis/angle: Direct problem

solution

$$R(\theta,r) = C\,R_z(\theta)\,C^T$$

$$R(\theta,r) = \begin{bmatrix} n & s & r \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n^T \\ s^T \\ r^T \end{bmatrix}$$

hint: use
- outer product of two vectors
- dyadic form of a matrix
- matrix product as product of dyads

$$= r\,r^T + (n\,n^T + s\,s^T)\,c\theta + (s\,n^T - n\,s^T)\,s\theta$$

taking into account that

$C\,C^T = n\,n^T + s\,s^T + r\,r^T = I$, and that

$$s\,n^T - n\,s^T = \begin{bmatrix} 0 & -r_z & r_y \\ & 0 & -r_x \\ \text{skew-symm} & & 0 \end{bmatrix} = S(r)$$

skew-symmetric(r):
$r \times v = S(r)v = -S(v)r$

depends only on r and θ !!

$$R(\theta,r) = r\,r^T + (I - r\,r^T)\,c\theta + S(r)\,s\theta = R^T(-\theta,r) = R(-\theta,-r)$$

Final expression of R(θ,r)

developing computations…

$$R(\theta,r) =$$

$$\begin{bmatrix} r_x^2(1-\cos\theta)+\cos\theta & r_x r_y(1-\cos\theta)-r_z\sin\theta & r_x r_z(1-\cos\theta)+r_y\sin\theta \\ r_x r_y(1-\cos\theta)+r_z\sin\theta & r_y^2(1-\cos\theta)+\cos\theta & r_y r_z(1-\cos\theta)-r_x\sin\theta \\ r_x r_z(1-\cos\theta)-r_y\sin\theta & r_y r_z(1-\cos\theta)+r_x\sin\theta & r_z^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

Axis/angle: a simple example

$$R(\theta,r) = r\,r^T + (I - r\,r^T)\,c\theta + S(r)\,s\theta$$

$$r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = z_0$$

$$R(\theta,r) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} c\theta + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s\theta$$

$$= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_z(\theta)$$

Axis/angle: proof of Rodriguez formula

$$v' = R(\theta, r)\, v$$

$$v' = v \cos\theta + (r \times v) \sin\theta + (1 - \cos\theta)(r^T v)\, r$$

proof:

$$R(\theta, r)\, v = (r\, r^T + (I - r\, r^T) \cos\theta + S(r) \sin\theta) v$$

$$= r\, r^T v\, (1 - \cos\theta) + v \cos\theta + (r \times v) \sin\theta$$

q.e.d.

Properties of $R(\theta,r)$

1. $R(\theta,r)r = r$ (r is the invariant axis in this rotation)
2. when r is one of the coordinate axes, R boils down to one of the known elementary rotation matrices
3. $(\theta,r) \rightarrow R$ is not an injective map: $R(\theta,r) = R(-\theta,-r)$
4. $\det R = +1 = \Pi\, \lambda_i$ (eigenvalues)
5. $\mathrm{tr}(R) = \mathrm{tr}(r\,r^T) + \mathrm{tr}(I - r\,r^T)c\theta = 1 + 2\,c\theta = \Sigma\, \lambda_i$

identities in brown hold for any matrix!

1. $\Rightarrow \lambda_1 = 1$

4. & 5. $\Rightarrow \lambda_2 + \lambda_3 = 2\,c\theta \;\Rightarrow\; \lambda^2 - 2\,c\theta\,\lambda + 1 = 0$

$$\Rightarrow\; \lambda_{2,3} = c\theta \pm \sqrt{c^2\theta - 1} = c\theta \pm i\, s\theta = e^{\pm i\,\theta}$$

all eigenvalues λ have unitary module ($\Leftarrow$ R orthonormal)

Axis/angle: Inverse problem

GIVEN a rotation matrix R,
FIND a unit vector r and an angle θ such that

$$R = r\,r^T + (I - r\,r^T)\cos\theta + S(r)\sin\theta = R(\theta,r)$$

Note first that $tr(R) = R_{11} + R_{22} + R_{33} = 1 + 2\cos\theta$; so, one could solve

$$\theta = \arccos \frac{R_{11} + R_{22} + R_{33} - 1}{2}$$

but:

- provides only values in $[0,\pi]$ (thus, never negative angles θ …)
- loss of numerical accuracy for $\theta \rightarrow 0$

Axis/angle: Inverse problem

solution

from

$$R - R^T = \begin{bmatrix} 0 & R_{12}-R_{21} & R_{13}-R_{31} \\ \text{skew-symm} & 0 & R_{23}-R_{32} \\ & & 0 \end{bmatrix} = 2 \sin\theta \begin{bmatrix} 0 & -r_z & r_y \\ \text{skew-symm} & 0 & -r_x \\ & & 0 \end{bmatrix}$$

it follows

$$\|r\| = 1 \;\Rightarrow\; \sin\theta = \pm\frac{1}{2}\sqrt{(R_{12}-R_{21})^2 + (R_{13}-R_{31})^2 + (R_{23}-R_{32})^2} \quad (*)$$

(**)

$$\boxed{\theta = \text{ATAN2}\left\{\pm\sqrt{(R_{12}-R_{21})^2 + (R_{13}-R_{31})^2 + (R_{23}-R_{32})^2},\; R_{11}+R_{22}+R_{33}-1\right\}}$$

see next slide

$$\boxed{r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32}-R_{23} \\ R_{13}-R_{31} \\ R_{21}-R_{12} \end{bmatrix}} \Leftarrow$$

can be used only if

$$\boxed{\sin\theta \neq 0}$$

(test made in advance on the expression (*) of sin θ in terms of the R_{ij}'s)

ATAN2 function

- arctangent with output values "in the four quadrants"
  - two input arguments
  - takes values in [-π ,+π]
  - undefined only for (0,0)
- uses the sign of both arguments to define the output quadrant
- based on arctan function with output values in [-π /2,+π /2]
- available in main languages (C++, Matlab, …)

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \pi + \arctan(\frac{y}{x}) & y \geq 0, x < 0 \\ -\pi + \arctan(\frac{y}{x}) & y < 0, x < 0 \\ \frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

Singular cases
(use when $\sin\theta = 0$)

- if $\theta = 0$ from (**), there is no given solution for r (rotation axis is undefined)
- if $\theta = \pm\pi$ from (**), then set $\sin\theta = 0$, $\cos\theta = -1$

$$\Rightarrow \quad R = 2 r r^T - I$$

$$r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \pm\sqrt{(R_{11}+1)/2} \\ \pm\sqrt{(R_{22}+1)/2} \\ \pm\sqrt{(R_{33}+1)/2} \end{bmatrix} \quad \text{with} \quad \begin{array}{|l|} \hline r_x r_y = R_{12}/2 \\ r_x r_z = R_{13}/2 \\ r_y r_z = R_{23}/2 \\ \hline \end{array} \leftarrow$$

resolving multiple signs ambiguities (always two solutions, of opposite sign)

exercise: determine the *two* solutions (r, θ) for $R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

Unit quaternion

- to eliminate undetermined and singular cases arising in the axis/angle representation, one can use the *unit quaternion* representation

$$\mathcal{Q} = \{\underbrace{\eta}_{\text{a scalar}}, \underbrace{\boldsymbol{\varepsilon}}_{\text{3-dim vector}}\} = \{\cos(\theta/2), \sin(\theta/2)\, \mathbf{r}\}$$

- $\eta^2 + \|\boldsymbol{\varepsilon}\|^2 = 1$ (thus, "unit ...")
- $(\theta, \mathbf{r})$ and $(-\theta, -\mathbf{r})$ gives the same quaternion $\mathcal{Q}$
- the absence of rotation is associated to $\mathcal{Q} = \{1, \mathbf{0}\}$
- unit quaternions can be composed with special rules (in a similar way as in a product of rotation matrices)

$$\mathcal{Q}_1 * \mathcal{Q}_2 = \{\eta_1\eta_2 - \boldsymbol{\varepsilon}_1^T\boldsymbol{\varepsilon}_2,\ \eta_1\boldsymbol{\varepsilon}_2 + \eta_2\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2\}$$