

# Robotics I

Classroom Test — November 21, 2014

## Exercise 1 [6 points]

In the Unimation Puma 560 robot, the DC motor that drives joint 2 is mounted in the body of link 2 (upper arm) and is connected to the joint axis through a train of transmission elements (see Fig. 1). The output shaft of the motor (code 506-1612) is connected to an idler assembly (code 500-2401) through a *bevel gear*, which changes the axis of rotation by  $90^\circ$  and has a reduction ratio  $n_{bg} = 10.88$ . In turn, the idler assembly is connected via a *spur gear* (code 500-0941) to the axis of joint 2, moving thus the second link. The two engaged wheels in the spur gear have radius  $r_{in} = 1.1$  [cm] and  $r_{out} = 10.86$  [cm], respectively.

- What is the reduction ratio  $n_r$  of the complete transmission from motor 2 to link 2?
- The inertia (of the rotor) of this motor is  $J_m = 0.0002$  [kg·m<sup>2</sup>]. If the reduction ratio used by Unimation were *optimal*, what should be the target inertia of the load (link 2)?
- The maximum rated torque produced at the motor shaft is  $\tau_m = 0.3$  [Nm]. With the values found in a) and b), neglecting dissipative effects, gravity, and all other dynamic couplings, what would be the maximum angular acceleration  $\ddot{\theta}_2$  of link 2 realized by this motor/transmission unit?

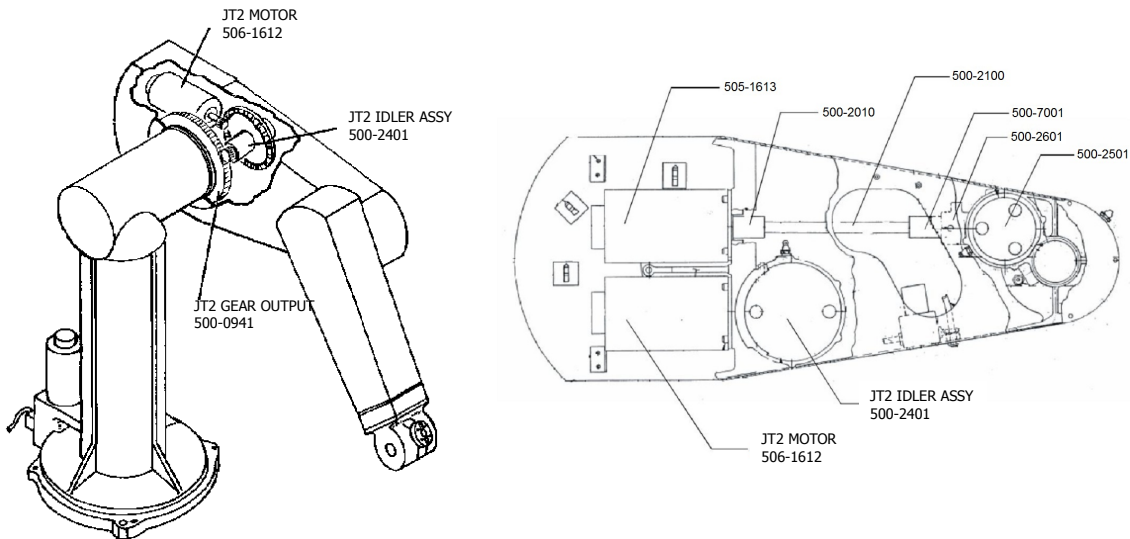


Figure 1: Two inside views of the upper arm of the Unimation Puma 560 robot, showing in particular the motor and gear train driving joint 2

(continues)

**Exercise 2 [6 points]**

Figure 2 shows again the Unimation Puma 560, a 6R articulated robot having a spherical wrist, with a set of link frames and relevant distances labeled from *A* to *E*. For the sake of drawing clarity, frames may be shown displaced from their actual placement; in particular, the origin of frame 5 is at the wrist center, while the origin of frame 6 is midway between the gripper jaws. Verify that the frame assignments are correct according to the Denavit-Hartenberg (DH) convention. If so, determine the associated table of DH parameters, using the distances *A* to *E* (with signs) and providing the values of the joint variables in the shown configuration.

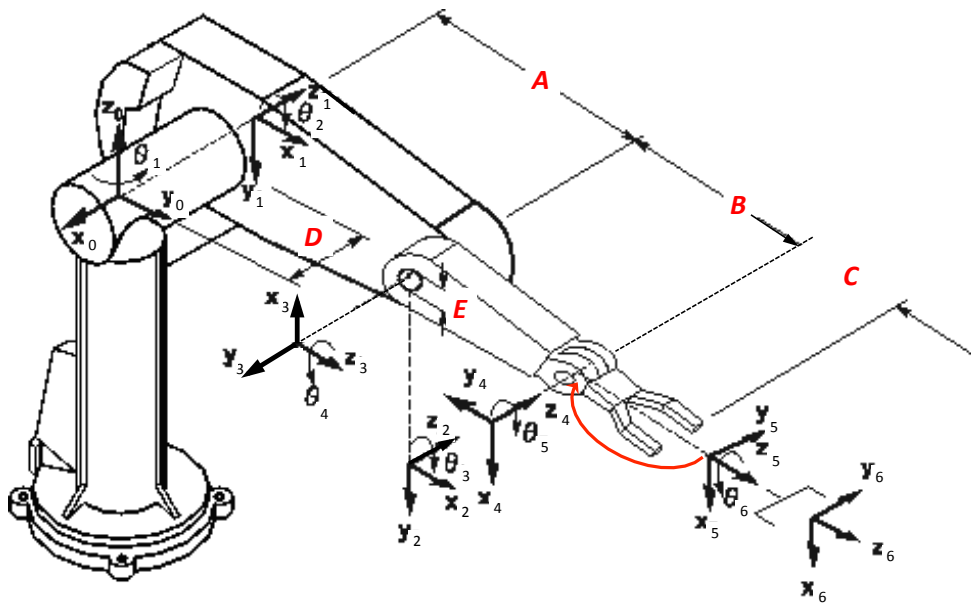


Figure 2: The Unimation Puma 560 robot with assigned link frames

**Exercise 3 [6 points]**

Consider a 3R planar manipulator with links of *generic* lengths  $l_1$ ,  $l_2$ , and  $l_3$ . Assuming that each joint has unlimited rotation, determine the primary workspace for the end-effector of this robot. Verify your analysis by drawing the workspace in the following two numerical cases (both have the same value for the sum of the link lengths):

- a)  $l_1 = 1, l_2 = 0.4, l_3 = 0.3$  [m];
- b)  $l_1 = 0.5, l_2 = 0.7, l_3 = 0.5$  [m].

(continues)

**Exercise 4 [12 points]**

A large Cartesian robot has 3 prismatic joints, followed by a spherical wrist with center  $W$ . Table 1 provides the DH parameters of this 6-dof robot. A variety of tools can be mounted on the robot end-effector, each having the Tool Center Point (TCP) placed along the approaching axis. The distance of the TCP from the wrist center  $W$  is specified by the parameter  $d_{TCP} > 0$ .

$i$	$\alpha_i$	$a_i$	$d_i$	$\theta_i$
1	$-\pi/2$	0	$q_1$	0
2	$-\pi/2$	0	$q_2$	$-\pi/2$
3	0	0	$q_3$	0
4	$-\pi/2$	0	0	$q_4$
5	$\pi/2$	0	0	$q_5$
6	0	0	$d_{TCP}$	$q_6$

Table 1: Denavit-Hartenberg parameters of a 3P-3R robot with spherical wrist

- a) Given a desired position of the TCP (with its  $d_{TCP}$ ) and a desired orientation of the robot end-effector frame (i.e., of frame 6), provide an analytic solution in closed form to the inverse kinematics problem. Is the solution unique?
- b) Apply your results to the data

$$d_{TCP} = 0.15 \text{ [m]}, \quad \mathbf{p}_d = \begin{pmatrix} 5.0 \\ 2.0 \\ 1.5 \end{pmatrix} \text{ [m]}, \quad \mathbf{R}_d = \begin{pmatrix} -0.5 & 0.5 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0.5 & -0.5 & -\sqrt{2}/2 \end{pmatrix},$$

and provide the numerical value of at least one joint configuration  $\mathbf{q} = (q_1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6)^T$  solving the inverse kinematics.

**[210 minutes; open books]**

## Solutions

November 21, 2014

### Exercise 1 [6 points]

a) The reduction ratio of the spur gear is  $n_{sg} = r_{out}/r_{in} = 10.86/1.1 = 9.87$ . The complete transmission has then reduction ratio

$$n_r = n_{bg} \cdot n_{sg} = 10.88 \cdot 9.87 = 107.38.$$

b) Assuming that this is the optimal value of the reduction ratio (i.e., the one that minimizes the motor torque needed to accelerate the link at a desired value  $\dot{\theta}_2 = a$ ), then the inertia  $J_2$  of link 2 around its rotation axis should satisfy

$$n_r^* = 107.38 = \sqrt{\frac{J_2}{J_m}} = \sqrt{\frac{J_2}{0.0002}} \implies J_2 = 0.0002 \cdot (107.38)^2 = 2.3061 \text{ [kg}\cdot\text{m}^2].$$

Note that in this analysis we have considered the inertia of the intermediate gears as negligible w.r.t. motor and link inertias. In the Puma 560, this is reasonable as the inertia of the gear train for joint 2, when reflected to the axis of motor 2, is less than 2% of the rotor inertia of the motor.

c) When using the optimal reduction ratio, there is a balanced partition of the torque produced by the motor:

$$\tau_m = J_m \ddot{\theta}_m + \frac{1}{n_r^*} J_2 \ddot{\theta}_2 = \left( J_m n_r^* + \frac{1}{n_r^*} J_2 \right) \ddot{\theta}_2 = 2 \sqrt{J_m J_2} \ddot{\theta}_2 = 2 J_m n_r^* \ddot{\theta}_2.$$

Thus, the maximum acceleration of link 2 (to be intended in absolute value) is

$$\ddot{\theta}_2 = \frac{\tau_m}{2} \frac{1}{J_m n_r^*} = 0.15 \frac{1}{0.0002 \cdot 107.38} = 6.9845 \text{ [rad/s}^2].$$

### Exercise 2 [6 points]

The assignment of link frames is feasible according to the (classical) Denavit-Hartenberg convention. The associated DH parameters are given in Table 2.

$i$	$\alpha_i$	$a_i$	$d_i$	$\theta_i$
1	$-\pi/2$	0	0	$q_1 = \pi/2$
2	0	$A > 0$	$D > 0$	$q_2 = 0$
3	$-\pi/2$	$E > 0$	0	$q_3 = -\pi/2$
4	$-\pi/2$	0	$B > 0$	$q_4 = \pi$
5	$\pi/2$	0	0	$q_5 = 0$
6	0	0	$C > 0$	$q_6 = 0$

Table 2: Denavit-Hartenberg parameters of the Unimation Puma 560 robot associated to the link frames and to the specific configuration shown in Fig. 2

Since the first two joint axes intersect, the origin  $O_1$  *must* be set at the intersection point, which is where also  $O_0$  is, and so  $a_1 = d_1 = 0$ . Also, the drawing may not be 100% clear on the sign of  $a_3$ : here, we took  $a_3 = E > 0$ , i.e., along the positive direction of  $\mathbf{x}_3$  (as it is in reality).

### Exercise 3 [6 points]

Denote the lengths of the longest and shortest links, respectively with

$$l_{max} = \max \{l_i, i = 1, 2, 3\}, \quad l_{min} = \min \{l_i, i = 1, 2, 3\},$$

and with  $l_{med}$  the length of the intermediate link of intermediate length. If two or more links have equal lengths, their relative ordering is irrelevant. No matter how the links of different length are placed within the kinematic chain, the workspace of the planar 3R will have as outer boundary a circumference of radius

$$R_{out} = l_{min} + l_{med} + l_{max} = l_1 + l_2 + l_3,$$

and as inner boundary a circumference of radius

$$R_{in} = \max \{0, l_{max} - (l_{med} + l_{min})\}.$$

This last formula means that when  $l_{max} \leq l_{med} + l_{min}$ , there will be no forbidden area internal to the workspace, which is then a circle of radius  $R_{out}$  centered at the robot base. Otherwise, there will be a circular ‘hole’ of radius  $l_{max} - (l_{med} + l_{min}) > 0$  in the center of the workspace. Consider next the two case studies.

**a)** Here,  $l_{max} = l_1 = 1$ ,  $l_{med} = l_2 = 0.4$ , and  $l_{min} = l_3 = 0.3$ . Since  $l_{max} = 1 > 0.7 = l_{med} + l_{min}$ , the workspace will be an annulus with inner radius  $R_{in} = 0.3$  and outer radius  $R_{out} = 1.7$ . Figure 3 shows the geometric construction of the workspace in this case.

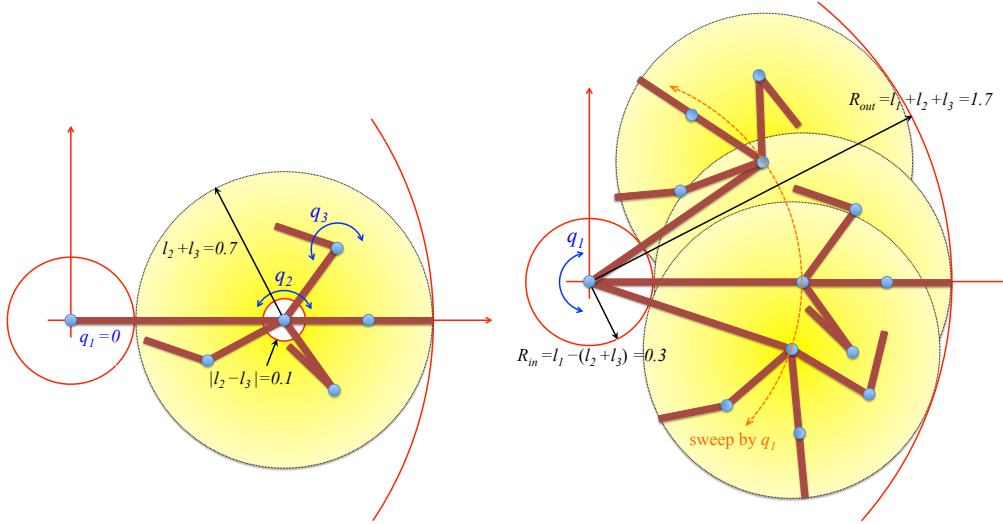


Figure 3: Construction of the primary workspace of a 3R robot with link lengths  $l_1 = 1$ ,  $l_2 = 0.4$ , and  $l_3 = 0.3$ : [Left] the workspace of the second and third link (keeping  $q_1$  fixed) contains an unreachable circular area centered at the second joint; [Right] sweeping by  $q_1$  will eliminate this area, although the workspace of the robot still contains a central ‘hole’ of radius  $R_{in} = 0.3$

**b)** In this case,  $l_{max} = l_2 = 0.7$ ,  $l_{med} = l_{min} = l_1$  (or  $l_3$ ) = 0.4. Since  $l_{max} = 0.7 < 0.8 = l_{med} + l_{min}$ , the workspace will be a full circle of radius  $R_{out} = 1.7$ .

**Exercise 4 [12 points]**

From the DH table, we first compute the direct kinematics. The first three prismatic joints provide a homogeneous transformation matrix

$${}^0\mathbf{T}_3(q_1, q_2, q_3) = \begin{pmatrix} {}^0\mathbf{R}_3 & {}^0\mathbf{p}_{03}(q_1, q_2, q_3) \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & q_3 \\ 0 & -1 & 0 & q_2 \\ 1 & 0 & 0 & q_1 \\ \mathbf{0}^T & & & 1 \end{pmatrix}.$$

It is clear that prismatic joints do not change orientation, up to a (signed) permutation of the Cartesian axes. The position of the wrist center is  $\mathbf{p}_W = {}^0\mathbf{p}_{03}(q_1, q_2, q_3)$ . Performing the complete computation (in case, by using a straightforward adaptation of the symbolic code `dirkin_SCARA.m` available on the course web page) yields

$${}^0\mathbf{T}_6(\mathbf{q}) = \begin{pmatrix} {}^0\mathbf{R}_6(q_4, q_5, q_6) & {}^0\mathbf{p}_{06}(\mathbf{q}) \\ \mathbf{0}^T & 1 \end{pmatrix} \quad (1)$$

where

$$\begin{aligned} {}^0\mathbf{R}_6(q_4, q_5, q_6) &= \begin{pmatrix} \mathbf{n}(q_4, q_5, q_6) & \mathbf{s}(q_4, q_5, q_6) & \mathbf{a}(q_4, q_5) \end{pmatrix} \\ &= \begin{pmatrix} -\sin q_5 \cos q_6 & \sin q_5 \sin q_6 & \cos q_5 \\ -\sin q_4 \cos q_5 \cos q_6 - \cos q_4 \sin q_6 & \sin q_4 \cos q_5 \sin q_6 - \cos q_4 \cos q_6 & -\sin q_4 \sin q_5 \\ \cos q_4 \cos q_5 \cos q_6 - \sin q_4 \sin q_6 & -\cos q_4 \cos q_5 \sin q_6 - \sin q_4 \cos q_6 & \cos q_4 \sin q_5 \end{pmatrix} \end{aligned} \quad (2)$$

and

$${}^0\mathbf{p}_{06}(\mathbf{q}) = \mathbf{p}(\mathbf{q}) = \begin{pmatrix} q_3 + d_{TCP} \cos q_5 \\ q_2 - d_{TCP} \sin q_4 \sin q_5 \\ q_1 + d_{TCP} \cos q_4 \sin q_5 \end{pmatrix}. \quad (3)$$

**a)** Since the robot has a spherical wrist, we can find the solution to the inverse kinematics problem in a partitioned way, first determining the values of the joint variables  $q_1$ ,  $q_2$ , and  $q_3$  for the main axes, and then (for each solution found in the first step) finding the values of the joint variables  $q_4$ ,  $q_5$ , and  $q_6$  for the spherical wrist. Indeed, things are particularly simple in the first step because this sub-structure is a PPP (Cartesian) robot. From eqs. (1–3), we obtain directly the first part of the solution by means of the expression

$$\mathbf{p}(\mathbf{q}) - d_{TCP} \mathbf{a}(q_4, q_5) = (\mathbf{p}_W) = \begin{pmatrix} q_3 \\ q_2 \\ q_1 \end{pmatrix} \quad (4)$$

There is indeed a *unique* solution to step 1 (note also the reordering of the first three variables). In the second step, since the rotation matrix  ${}^0\mathbf{R}_3$  independent from the joint variables  $q_1$ ,  $q_2$ , and  $q_3$ , it is not necessary to isolate the rotation matrix

$${}^3\mathbf{R}_6(q_4, q_5, q_6) = {}^0\mathbf{R}_3^T {}^0\mathbf{R}_6(q_4, q_5, q_6) = \dots \text{(this matrix is associated to a ZYZ Euler sequence!).}$$

In fact, we can just equate the expression (2) with that of a generic orientation matrix  $\mathbf{R} = \{r_{ij}\}$  of the end-effector frame. Using a compact notation, we have

$$\begin{pmatrix} -s_5 c_6 & s_5 s_6 & c_5 \\ -s_4 c_5 c_6 - c_4 s_6 & s_4 c_5 s_6 - c_4 c_6 & -s_4 s_5 \\ c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}. \quad (5)$$

Proceeding as in the solution of an inverse problem of minimal representation of orientation by a generic Euler sequence, we can isolate  $q_5$  from the elements in the last column or in the first row of (5) as

$$q_5 = \text{ATAN2} \left\{ \pm \sqrt{r_{23}^2 + r_{33}^2}, r_{13} \right\} \quad \text{or} \quad q_5 = \text{ATAN2} \left\{ \pm \sqrt{r_{11}^2 + r_{12}^2}, r_{13} \right\}. \quad (6)$$

Provided that  $s_5 \neq 0$ , which should be checked in advance as  $r_{23}^2 + r_{33}^2 \neq 0$  (or, equivalently,  $r_{11}^2 + r_{12}^2 \neq 0$ ) on the given data, we can solve for  $q_4$  and  $q_6$  from the last column and first row in (5):

$$q_4 = \text{ATAN2} \{-r_{23}/s_5, r_{33}/s_5\}, \quad q_6 = \text{ATAN2} \{r_{12}/s_5, -r_{11}/s_5\}. \quad (7)$$

In the regular case, *two solutions* are obtained from eqs. (4), (6), and (7). Instead, when  $s_5 = 0$  the robot is in a wrist singularity (occurring for  $q_5 = 0$  or  $\pi$ ), and two cases arise. If  $q_5 = 0$ , we can only solve for the sum of the two angles  $q_4 + q_6$ . In fact, setting  $s_5 = 0$  and  $c_5 = 1$  in (5) and considering a compatible orientation matrix  $\mathbf{R}$ , eq. (5) becomes

$$\begin{pmatrix} 0 & 0 & 1 \\ -s_{46} & -c_{46} & 0 \\ c_{46} & -s_{46} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix},$$

from which

$$q_4 + q_6 = \text{ATAN2} \{-r_{21}, r_{31}\}.$$

Similarly, if  $q_5 = \pi$  we can only solve for the difference  $q_4 - q_6$  of the two remaining joint angles of the robot wrist. In both cases, we will have a (simple) infinity of inverse solutions. Note that often the situation  $q_5 = \pi$  is ruled out by the presence of a limited range for joint 5 around its zero value.

**b)** From the given numerical data for the desired pose, with  $\mathbf{a}_d$  being the third column of  $\mathbf{R}_d$ , we perform the same operation as in eq. (4) and obtain the values (fully specified by the data, without the need to compute  $q_4$  and  $q_5$  first)

$$\begin{pmatrix} q_3 \\ q_2 \\ q_1 \end{pmatrix} = \mathbf{p}_d - d_{TCP} \mathbf{a}_d = \begin{pmatrix} 5.0 \\ 2.0 \\ 1.5 \end{pmatrix} - 0.15 \begin{pmatrix} -\sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 5.1061 \\ 2.0 \\ 1.6061 \end{pmatrix} \text{ [m]}. \quad (8)$$

Moreover, since  $r_{23,d}^2 + r_{33,d}^2 = 0.5 \neq 0$ , the robot is not in a wrist singularity. Applying then eqs. (6–7), we obtain

$$q_5^{(+,-)} = \text{ATAN2} \left\{ \pm 1/\sqrt{2}, -1/\sqrt{2} \right\} = \pm \frac{3\pi}{4} \text{ [rad]}$$

and

$$q_4^{(+,-)} = \text{ATAN2} \{0, \mp 1\} = \{\pi, 0\} \text{ [rad]}, \quad q_6^{(+,-)} = \text{ATAN2} \left\{ \pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2} \right\} = \{\pi/4, -3\pi/4\} \text{ [rad]}.$$

As a result, there are two regular solutions for the robot wrist angles. The complete inverse kinematics solutions are then

$$(q_1 \quad q_2 \quad q_3 \quad q_4^+ \quad q_5^+ \quad q_6^+) = \left( 1.6061 \quad 2.0 \quad 5.1061 \quad \pi \quad \frac{3\pi}{4} \quad \frac{\pi}{4} \right)$$

and

$$(q_1 \quad q_2 \quad q_3 \quad q_4^- \quad q_5^- \quad q_6^-) = \left( 1.6061 \quad 2.0 \quad 5.1061 \quad 0 \quad -\frac{3\pi}{4} \quad -\frac{3\pi}{4} \right).$$

*Hint:* It is always good practice to feed the obtained solutions into the direct kinematics and check if the results coincide with what should be expected.

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