Robotics I

Test — November 29, 2013

Exercise 1 [6 points]

A DC motor is used to actuate a single robot link that rotates in the horizontal plane around a joint axis passing through its base. The motor is connected to the link by means of two transmission/reduction elements placed in series: a spur gear (SG) made of two toothed wheels, and a harmonic drive (HD). The output shaft of the motor drives the smaller toothed wheel (of radius r_1). The output axis of the larger wheel (of radius $r_2 > r_1$) is connected to the wave generator of the HD. Finally, the output axis of the flexspline is the joint axis of the robot link. The motor delivers a maximum torque $T_{m,max} = 2.2$ [Nm], while the inertia of its rotor is $J_m = 0.0012$ [kg·m^2]. The smaller wheel of the gear has radius $r_1 = 2$ [cm]. The flexspline of the HD has 70 outer teeth. Finally, the link has an inertia $J_\ell = 5.88$ [kg·m^2] around its rotation axis (at the link base).

a) Neglecting dissipative effects and other inertial loads except rotor and link inertias, determine the radius r_2 of the larger toothed wheel of the spur gear so that the reduction ratio $n > 1$ of the complete transmission is *optimal* in terms of motor torque needed to accelerate the link.

b) With the resulting optimal value n^*, determine the maximum angular acceleration $\ddot{\theta}_{\ell,max}$ of the link that can be realized using this motor/transmission unit.

Exercise 2 [12 points]

The K-1207 robot developed by Robotics Research Co. (USA) is a modular 7-dof manipulator having all revolute joints and *no* spherical wrist or shoulder. Figure 1 shows a picture of the robot (mounted on an inclined base) and a few snaphots of the robot in motion (mounted on a vertical base), in order to illustrate its dexterity.

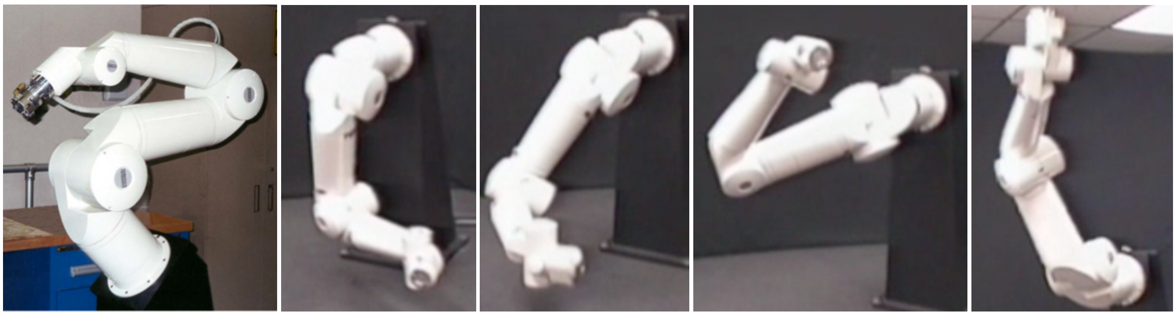

Figure 1: The Robotics Research K-1207 robot

A drawing of the K-1207 robot for a particular configuration is shown in Fig. 2, with indication of the seven revolute joint axes and the physical sizes (in inches). Origin O_0 and axis $\boldsymbol{x}_0$ of the base frame as well as origin O_7 and axis $\boldsymbol{z}_7$ of the last frame are assigned as in the figure.

a) Assign the link frames and the table of parameters according to the Denavit-Hartenberg convention (use the extra sheet for your sketch of the frames).

b) Provide the numerical values of the constant parameters and of the joint variables associated to the configuration shown in Fig. 2.

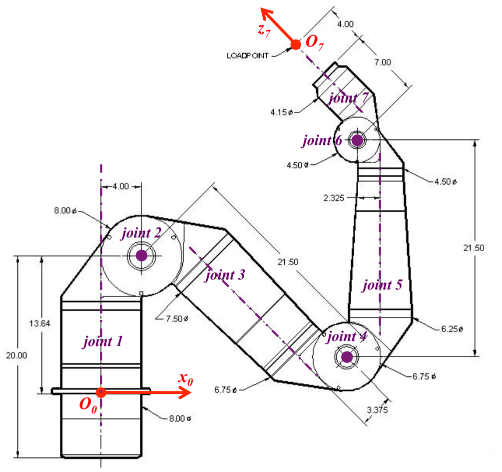

Figure 2: Drawing of the K-1207 robot: In this configuration, joint axes 1, 3, 5, and 7 are on a common plane, while joint axes 2, 4, and 6 are normal to this plane

Exercise 3 [12 points]

Consider the planar RPR robot shown in Fig. 3, and the definition of joint variables $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$ given therein. The three-dimensional task vector is $\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p}^T & \alpha \end{pmatrix}^T = \boldsymbol{f}(\boldsymbol{q})$, where $\boldsymbol{p} = \begin{pmatrix} p_x & p_y \end{pmatrix}^T$ is the position of the end-effector and α is the orientation angle of the last link w.r.t. the $\boldsymbol{x}_0$ axis. Assume that $q_2 \geq 0$ holds for the prismatic joint variable.

a) Solve the inverse kinematics problem for a given $\boldsymbol{r}_d$, providing the expression of all feasible solutions in closed form.

b) Compute the solutions $\boldsymbol{q}$ associated to $\boldsymbol{r}_d = \begin{pmatrix} -2 & -2 & \pi/2 \end{pmatrix}^T$ [m,m,rad] (i.e., such that $\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{r}_d$) using the data $L_1 = 1$ [m] and $L_3 = 0.7$ [m], and sketch the robot configurations.

c) Draw the primary workspace in the plane of robot motion for generic values of L_1 and L_3, assuming that the prismatic joint range is bounded as $|q_2| \leq D$ (with $D > 0$).

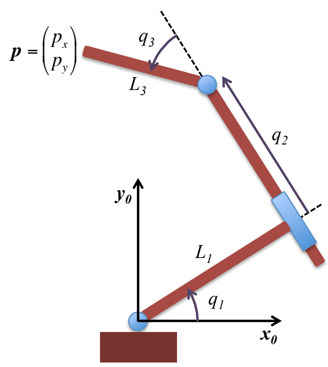

Figure 3: A planar RPR robot with the definition of joint variables

[210 minutes; open books]

Solutions

November 29, 2013

Exercise 1 [6 points]

a) The reduction ratio of the gear is obtained by equating the absolute value of the linear velocities of the two wheels at the point of contact between the meshing teeth (the wheels rotate in opposite directions). Denoting by ω_i the angular velocity of toothed wheel i ($1 =$ input, $2 =$ output),

$$|\omega_1|\, r_1 = |\omega_2|\, r_2 \qquad \Rightarrow \qquad n_{SG} = \left|\frac{\omega_1}{\omega_2}\right| = \frac{r_2}{r_1} = \frac{r_2}{2}.$$

The reduction ratio of the harmonic drive is

$$n_{HD} = \frac{\#\,\text{teeth of flexspline}}{\#\,\text{teeth of circular spline} - \#\,\text{teeth of flexspline}} = \frac{70}{2} = 35,$$

since the number of (inner) teeth of the circular spline always exceeds that (in the outer side) of the flexspline by 2. The complete transmission has then reduction ratio

$$n = n_{SG} \cdot n_{HD} = \frac{r_2}{2} \cdot 35 = 17.5\, r_2.$$

The optimal value of the reduction ratio that minimizes the motor torque needed to accelerate the link at a desired value $\ddot{\theta}_\ell$ is

$$n^* = \sqrt{\frac{J_\ell}{J_m}} = \sqrt{\frac{5.88}{0.0012}} = \sqrt{4900} = 70.$$

Thus, such reduction ratio is obtained by choosing $r_2 = 70/17.5 = 4$ [cm] ($n_{SG} = 2$).

b) The torque balance for the complete motor/transmission/load system is then

$$T_m = J_m \ddot{\theta}_m + \frac{1}{n^*} J_\ell \ddot{\theta}_\ell = J_m \left(n^* \ddot{\theta}_\ell\right) + \frac{1}{n^*} J_\ell \ddot{\theta}_\ell = \left(J_m \sqrt{\frac{J_\ell}{J_m}} + J_\ell \sqrt{\frac{J_m}{J_\ell}}\right) \ddot{\theta}_\ell = 2\sqrt{J_\ell J_m}\, \ddot{\theta}_\ell$$

$$\left(\text{or also } \ldots = \frac{1}{n^*}\left({n^*}^2 J_m + J_\ell\right) \ddot{\theta}_\ell = \frac{2\, J_\ell}{n^*}\, \ddot{\theta}_\ell\right).$$

As a result, the maximum angular acceleration of the link is

$$\ddot{\theta}_{\ell,max} = \frac{T_{m,max}}{2\sqrt{J_\ell J_m}} \left(= \frac{n^* T_{m,max}}{2\, J_\ell}\right) = \frac{2.2}{0.168} = 13.095 \ \ [\text{rad/s}^2].$$

Exercise 2 [12 points]

a) A feasible assignment of the Denavit-Hartenberg frames is shown in Figure 4. The associated parameters are given in Table 1. We note that the mechanical (modular) structure of joints 2, 4, and 6 leads to the kinematic identities $|a_1| = |a_2|$, $|a_3| = |a_4|$, and $|a_5| = |a_6|$ (the absolute value is needed because of different possible choices for the positive directions of the $\boldsymbol{x}_i$ axes).

b) The numerical values of the constant parameters a_i and d_i are specified (in inches) in the same Table, together with the values of the joint variables $q_i = \theta_i \in (-\pi, +\pi]$ when the robot is in the shown configuration. In this configuration, the $\boldsymbol{z}_i$ axes not lying in the plane (i.e., for $i = 2, 4, 6$) are pointing outwards.

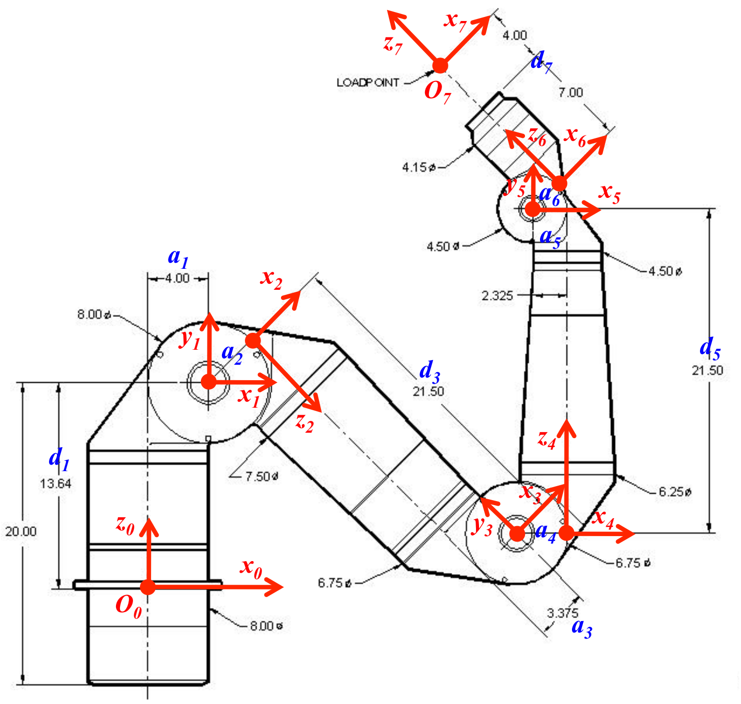

Figure 4: Assignment of Denavit-Hartenberg frames for the K-1207 robot

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	$a_1 = 4.00$"	$d_1 = 13.64$"	$q_1 = 0$
2	$\pi/2$	$a_2 = 4.00$"	0	$q_2 = \pi/4$
3	$-\pi/2$	$a_3 = 3.375$"	$d_3 = 21.50$"	$q_3 = 0$
4	$-\pi/2$	$a_4 = 3.375$"	0	$q_4 = -\pi/4$
5	$\pi/2$	$a_5 = -2.25$"	$d_5 = 21.50$"	$q_5 = 0$
6	$-\pi/2$	$a_6 = 2.25$"	0	$q_6 = \pi/4$
7	0	0	$d_7 = 11.00$"	$q_7 = 0$

Table 1: Denavit-Hartenberg parameters for the K-1207 robot (associated to the frames in Fig. 4)

Exercise 3 [12 points]

a) The direct kinematics for the given task is

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p} \\ \alpha \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} L_1 \cos q_1 + q_2 \cos\left(q_1 + \frac{\pi}{2}\right) + L_3 \cos\left(q_1 + q_3 + \frac{\pi}{2}\right) \\ L_1 \sin q_1 + q_2 \sin\left(q_1 + \frac{\pi}{2}\right) + L_3 \sin\left(q_1 + q_3 + \frac{\pi}{2}\right) \\ q_1 + q_3 + \frac{\pi}{2} \end{pmatrix}$$

$$= \begin{pmatrix} L_1 \cos q_1 - q_2 \sin q_1 - L_3 \sin(q_1 + q_3) \\ L_1 \sin q_1 + q_2 \cos q_1 + L_3 \cos(q_1 + q_3) \\ q_1 + q_3 + \frac{\pi}{2} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).$$

From a given $\boldsymbol{r}$ $(= \boldsymbol{r}_d)$, we can easily write the position $\boldsymbol{w}$ of the end-point of the second link as

$$\boldsymbol{w} = \begin{pmatrix} w_x \\ w_y \end{pmatrix} = \boldsymbol{p} - L_3 \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} L_1 \cos q_1 - q_2 \sin q_1 \\ L_1 \sin q_1 + q_2 \cos q_1 \end{pmatrix}.$$

Squaring and summing the components of $\boldsymbol{w}$ yields

$$q_2 = +\sqrt{w_x^2 + w_y^2 - L_1^2} = +\sqrt{(p_x - L_3 \cos\alpha)^2 + (p_y - L_3 \sin\alpha)^2 - L_1^2}. \tag{1}$$

Only the positive sign has been kept in (1), since we have assumed that only $q_2 \geq 0$ is feasible. Indeed, the value of q_2 is real if and only if $\|\boldsymbol{w}\| \geq L_1$ (namely, when the position $\boldsymbol{w}$ of the tip of the second link is in the workspace of the sub-structure RP made by the first two joints and links). With q_2 from (1), we can always solve the following linear system for $\cos q_1$ and $\sin q_1$

$$\begin{pmatrix} L_1 & -q_2 \\ q_2 & L_1 \end{pmatrix} \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \begin{pmatrix} w_x \\ w_y \end{pmatrix},$$

yielding

$$\begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \frac{1}{L_1^2 + q_2^2} \begin{pmatrix} L_1 w_x + q_2 w_y \\ L_1 w_y - q_2 w_x \end{pmatrix}.$$

Being $L_1^2 + q_2^2 > 0$, we can skip division by this quantity when evaluating q_1 with the ATAN2 function. Thus,

$$\begin{aligned} q_1 &= \text{ATAN2}\left\{L_1 w_y - q_2 w_x, L_1 w_x + q_2 w_y\right\} \\ &= \text{ATAN2}\left\{L_1 (p_y - L_3 \sin\alpha) - q_2 (p_x - L_3 \cos\alpha), L_1 (p_x - L_3 \cos\alpha) + q_2 (p_y - L_3 \sin\alpha)\right\} \end{aligned} \tag{2}$$

Finally, we have

$$q_3 = \alpha - q_1 - \frac{\pi}{2}. \tag{3}$$

There is only one feasible solution to the inverse kinematics problem, as given by eqs. (1)–(3).

b) With the data $L_1 = 1$ [m], $L_3 = 0.7$ [m], and $\boldsymbol{r}_d = \begin{pmatrix} p_x & p_y & \alpha \end{pmatrix}^T = \begin{pmatrix} -2 & -2 & \pi/2 \end{pmatrix}^T$ [m,m,rad], the above formulas yield (the resulting robot configuration is sketched in Fig. 5)

$$\begin{aligned} \boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T &= \begin{pmatrix} 2.8062 & 3.2078 & -2.8062 \end{pmatrix}^T \quad \text{[rad, m, rad]} \\ &= \begin{pmatrix} 160.7855 & 3.2078 & -160.7855 \end{pmatrix}^T \quad \text{[deg, m, deg]}. \end{aligned} \tag{4}$$

c) In order to determine the robot primary workspace for $|q_2| \leq D$, we have to distinguish two cases. When $L_1 \geq L_3$, the primary workspace is an annulus with inner and outer radius given by

$$R_{in,L_1 \geq L_3} = L_1 - L_3 \geq 0, \qquad R_{out} = \sqrt{L_1^2 + D^2} + L_3 > 0.$$

Figure 6 shows the actual construction of the workspace in the case of a length L_1 strictly larger than L_3. In particular, for equal link lengths $L_1 = L_3$, the workspace is a circle of radius R_{out}. When $L_1 < L_3$, the additional mobility provided by the prismatic joint allows to reduce, at least in part, the 'hole' (of radius $L_3 - L_1$) at the center that would characterize the workspace of a 2R planar arm. It is easy to see that, starting with $q_2 = 0$ and with the third link folded on the first one, the robot end-effector can access this inner part by progressively extending the second joint and *pivoting* with the third link (or with its prolongation) about the origin. When the second joint reaches its limit, the end-effector will be at a distance $|R_{in,L_1<L_3}|$ from the origin, with

$$R_{in,L_1<L_3} = L_3 - \sqrt{L_1^2 + D^2}, \qquad \Rightarrow \qquad R_{in,L_1<L_3} = 0 \iff D^2 = L_3^2 - L_1^2.$$

Therefore, the workspace will be an annulus with inner radius $R_{in,L_1<L_3} > 0$ and outer radius R_{out} when $D^2 < L_3^2 - L_1^2$, or a circle of radius R_{out} when $D^2 \geq L_3^2 - L_1^2$.

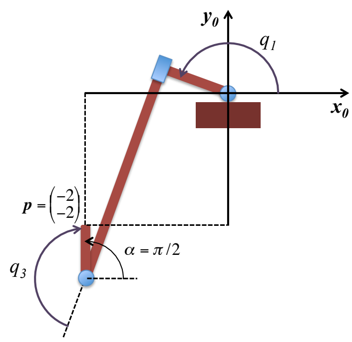

Figure 5: The inverse kinematic solution (4) associated to $\boldsymbol{r}_d = (-2 \;\; -2 \;\; \pi/2)^T$ [m, m, rad]

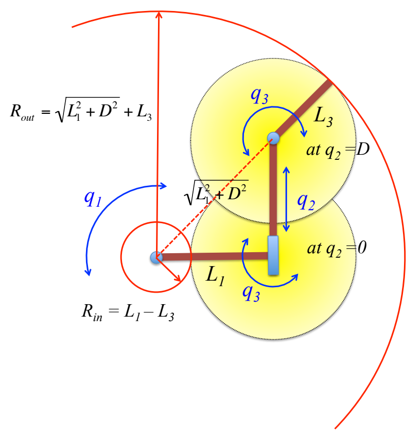

Figure 6: Primary workspace of RPR robot with prismatic joint in range $[-D, +D]$, for $L_1 > L_3$

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