Robotics I

Test — November 10, 2009

Exercise 1

Consider a minimal representation of orientation specified by the following sequence of angles, defined around fixed axes: α around Y; β around X; γ around Z.

- Compute the associated rotation matrix $\boldsymbol{R}_{YXZ}(\alpha,\beta,\gamma)$.
- Determine all sets of angles (α,β,γ) realizing the orientation specified by the matrix

$$\boldsymbol{R} = \begin{pmatrix} 0.7392 & -0.6124 & -0.2803 \\ 0.5732 & 0.3536 & 0.7392 \\ -0.3536 & -0.7071 & 0.6124 \end{pmatrix}.$$

- Characterize all rotation matrices $\boldsymbol{R}$ for which the inverse problem yields undefined angles in the sequence.

Exercise 2

Consider the kinematic structure in Figure 1, representing a camera mounted on the head of a humanoid trunk with three revolute joints.

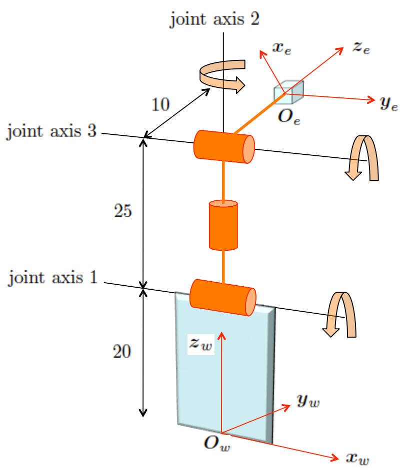

Figure 1: Kinematics of a camera head (units are in cm)

- Assign the frames according to the Denavit-Hartenberg convention in such a way that the positive (counterclockwise) joint rotations are those shown. Compute the associated table of parameters.
- Compute the expression of the rotation matrix ${}^w\boldsymbol{R}_e(\theta_1,\theta_2,\theta_3)$ relating the orientation of the given end-effector (camera) frame RF_e with respect to the world frame RF_w, placed as shown in Figure 1.
- Provide a rotation matrix ${}^w\boldsymbol{R}_e$ that can be realized by infinite pairs of values (θ_1,θ_3) and a single value of θ_2.

[120 minutes; open books]

Solutions

November 10, 2009

Exercise 1

By using the elementary rotation matrices around the coordinate axes

$$\begin{aligned}
\boldsymbol{R}_Y(\alpha) &= \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \\
\boldsymbol{R}_X(\beta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}, \\
\boldsymbol{R}_Z(\gamma) &= \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

and being the sequence of rotations defined around fixes axes, we obtain

$$\boldsymbol{R}_{YXZ}(\alpha,\beta,\gamma) = \boldsymbol{R}_Z(\gamma)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Y(\alpha),$$

or

$$\boldsymbol{R}_{YXZ}(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\beta\cos\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ -\sin\alpha\cos\beta & \sin\beta & \cos\alpha\cos\beta \end{pmatrix}.$$

The inverse mapping from a given rotation matrix

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

to the sequence of angles (α,β,γ) is given by

$$\beta = \text{ATAN2}\left\{ r_{32}, \pm\sqrt{r_{31}^2 + r_{33}^2} \right\}$$

and, provided that $r_{31}^2 + r_{33}^2 \neq 0$ (i.e., $\cos\beta \neq 0$),

$$\alpha = \text{ATAN2}\left\{ \frac{-r_{31}}{\cos\beta}, \frac{r_{33}}{\cos\beta} \right\}, \qquad \gamma = \text{ATAN2}\left\{ \frac{-r_{12}}{\cos\beta}, \frac{r_{22}}{\cos\beta} \right\}.$$

For the given data, we obtain the pair of solutions:

$$(\alpha,\beta,\gamma) = (0.5236, -0.7854, 1.0472) \text{ [rad]} = (30, -45, 60) \text{ [deg]}$$

and

$$(\alpha,\beta,\gamma) = (-2.6180, -2.3562, -2.0944) \text{ [rad]} = (-150, -135, -120) \text{ [deg]}.$$

When $r_{31} = r_{33} = 0$, β is uniquely defined whereas the other data provide only information either on the sum $\alpha + \gamma$ or on the difference $\alpha - \gamma$. In fact, for an orientation matrix of the form

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 1 & 0 \end{pmatrix},$$

i.e., with $r_{32} = 1$, we have $\beta = \pi/2$ ($\cos\beta = 0$, $\sin\beta = 1$) and thus

$$\begin{aligned} \boldsymbol{R}_{YXZ}(\alpha, \pi/2, \gamma) &= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\gamma & 0 & \sin\alpha\cos\gamma + \cos\alpha\sin\gamma \\ \cos\alpha\sin\gamma + \sin\alpha\cos\gamma & 0 & \sin\alpha\sin\gamma - \cos\alpha\cos\gamma \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha+\gamma) & 0 & \sin(\alpha+\gamma) \\ \sin(\alpha+\gamma) & 0 & -\cos(\alpha+\gamma) \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\alpha + \gamma = \text{ATAN2}\left\{r_{21}, r_{11}\right\} = \text{ATAN2}\left\{r_{13}, -r_{23}\right\}.$$

On the other hand, for an orientation matrix of the form

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & -1 & 0 \end{pmatrix},$$

i.e., with $r_{32} = -1$, we have $\beta = -\pi/2$ ($\cos\beta = 0$, $\sin\beta = -1$) and thus

$$\begin{aligned} \boldsymbol{R}_{YXZ}(\alpha, -\pi/2, \gamma) &= \begin{pmatrix} \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & 0 & \sin\alpha\cos\gamma - \cos\alpha\sin\gamma \\ \cos\alpha\sin\gamma - \sin\alpha\cos\gamma & 0 & \sin\alpha\sin\gamma + \cos\alpha\cos\gamma \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha-\gamma) & 0 & \sin(\alpha-\gamma) \\ -\sin(\alpha-\gamma) & 0 & \cos(\alpha-\gamma) \\ 0 & -1 & 0 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\alpha - \gamma = \text{ATAN2}\left\{-r_{21}, r_{11}\right\} = \text{ATAN2}\left\{r_{13}, r_{23}\right\}.$$

In both cases, the angles α and γ are not fully defined.

Exercise 2

Consider the assignment of Denavit-Hartenberg frames as in Figure 2, where the positive direction of the axes $\boldsymbol{z}_i$ ($i = 0, 1, 2$) has been chosen consistently with the requirement in the text. The shown configuration has $\theta_1 = 0$, $\theta_2 = 0$, and θ_3 equal to some positive angle between $\pi/2$ and $3\pi/4$.

The Denavit-Hartenberg parameters are given in Table 1, with $d_2 = 25$ cm. The associated

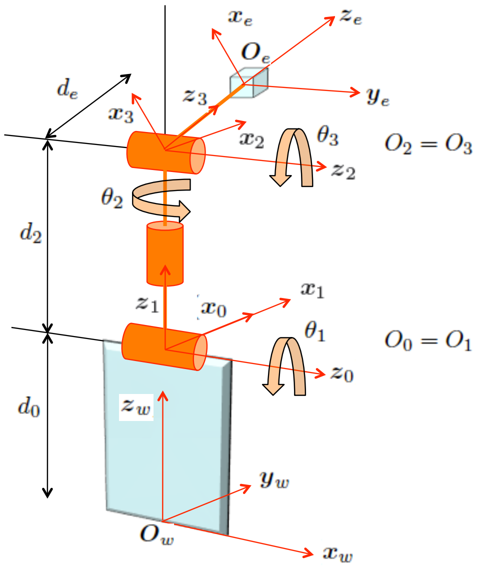

Figure 2: Denavit-Hartenberg frames

i	α_i	a_i	d_i	θ_i
1	$-\frac{\pi}{2}$	0	0	θ_1
2	$\frac{\pi}{2}$	0	d_2	θ_2
3	$\frac{\pi}{2}$	0	0	θ_3

Table 1: Denavit-Hartenberg parameters

homogeneous transformation matrices are

$$
\begin{aligned}
{}^0\boldsymbol{A}_1(\theta_1) &= \begin{pmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(\theta_1) & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^1\boldsymbol{A}_2(\theta_2) &= \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ \sin\theta_2 & 0 & -\cos\theta_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{R}_2(\theta_2) & {}^1\boldsymbol{p}_{12} \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^2\boldsymbol{A}_3(\theta_3) &= \begin{pmatrix} \cos\theta_3 & 0 & \sin\theta_3 & 0 \\ \sin\theta_3 & 0 & -\cos\theta_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^2\boldsymbol{R}_3(\theta_3) & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}.
\end{aligned}
$$

In addition, we can define the following (constant) homogenous transformation matrices

$$
{}^w\boldsymbol{T}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^w\boldsymbol{R}_0 & {}^w\boldsymbol{p}_{w0} \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

$$
{}^3\boldsymbol{T}_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_e \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^3\boldsymbol{R}_e & {}^3\boldsymbol{p}_{3e} \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

with $d_0 = 20$ cm and $d_e = 10$ cm. Note that ${}^3\boldsymbol{R}_e = \boldsymbol{I}$.

The orientation of frame RF_e w.r.t. the world frame RF_w is thus

$$
\begin{aligned}
{}^w\boldsymbol{R}_e(\boldsymbol{\theta}) &= {}^w\boldsymbol{R}_0\, {}^0\boldsymbol{R}_1(\theta_1)\, {}^1\boldsymbol{R}_2(\theta_2)\, {}^2\boldsymbol{R}_3(\theta_3)\, {}^3\boldsymbol{R}_e \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \\
&\quad \begin{pmatrix} \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 \\ \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & \sin\theta_1\sin\theta_2 & \sin\theta_1\cos\theta_2\sin\theta_3 - \cos\theta_1\cos\theta_3 \\ -\sin\theta_2\cos\theta_3 & \cos\theta_2 & -\sin\theta_2\sin\theta_3 \end{pmatrix}.
\end{aligned}
$$

One can now proceed by solving the inverse kinematics of this three-dof robotic structure for a given orientation matrix ${}^w\boldsymbol{R}_e$. In particular, we can solve for $\boldsymbol{\theta}$ the following kinematic equation

$$
{}^0\boldsymbol{R}_1(\theta_1)\, {}^1\boldsymbol{R}_2(\theta_2)\, {}^2\boldsymbol{R}_3(\theta_3) = {}^w\boldsymbol{R}_0^T\, {}^w\boldsymbol{R}_e = {}^0\boldsymbol{R}_e = \begin{pmatrix} {}^0r_{11} & {}^0r_{12} & {}^0r_{13} \\ {}^0r_{21} & {}^0r_{22} & {}^0r_{23} \\ {}^0r_{31} & r_{32} & {}^0r_{33} \end{pmatrix},
$$

where the right-hand side matrix is a constant. By similar reasoning as in Exercise 1, one can see that the inverse problem has an infinity set of values for θ_1 and θ_3 (with a prescribed sum or difference) if and only if

$$
{}^0r_{31} = {}^0r_{33} = 0 \qquad ({}^0r_{32} = \pm 1).
$$

All possible rotation matrices ${}^w\boldsymbol{R}_e$ leading to this situation are then of the form

$$
{}^w\boldsymbol{R}_e = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} {}^0r_{11} & 0 & {}^0r_{13} \\ {}^0r_{21} & 0 & {}^0r_{23} \\ 0 & \pm 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \pm 1 & 0 \\ {}^0r_{11} & 0 & {}^0r_{13} \\ {}^0r_{21} & 0 & {}^0r_{23} \end{pmatrix}.
$$

For example, one candidate is

$$
{}^w\boldsymbol{R}_e = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

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