Robotics I

February 6, 2015

Exercise 1

Consider the 3R robot in Fig. 1 *(this is the same robotic structure of an exercise assigned in September 2007)*. The base frame and an *additional* end-effector frame are already specified.

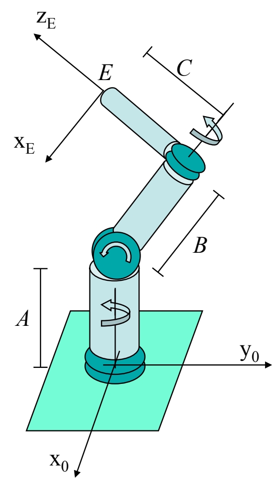


Figure 1: A robot with three revolute joints.

- Given a desired *orientation* $\boldsymbol{R}_d$ of the end-effector frame, solve the inverse kinematics problem in symbolic form. Consider also possible singular cases.
- Apply your result and determine *all* numerical solutions $\boldsymbol{q}$ for the following two sets of data:

$$\boldsymbol{R}_{d,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{R}_{d,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}.$$

- Provide for this robot the relation between $\dot{\boldsymbol{q}}$ and the angular velocity $\boldsymbol{\omega}_E$ of the end-effector frame (expressed in the base frame).
- Determine a joint velocity $\dot{\boldsymbol{q}}$ in the configuration $\boldsymbol{q} = \boldsymbol{0}$ that produces the desired angular velocity $\boldsymbol{\omega}_{E,d} = \begin{pmatrix} 0 & 0 & 3 \end{pmatrix}^T$ [rad/s]. Has this problem a solution? If so, is it unique?
- *"This robot is of little use for positioning the end-effector in 3D space."* Do you agree with this statement? Why?

Extra
- Based on the analysis you have performed, can this robot realize any *pointing* task with its end-effector axis $\boldsymbol{z}_E$? If so, is there a unique solution in the generic case? Are there singular situations? *(If you reply correctly to the extra questions, you get a bonus)*

Exercise 2

Given the two points

$$\boldsymbol{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \text{[m]} \qquad \text{and} \qquad \boldsymbol{B} = \begin{pmatrix} 0.732 \\ 1 \end{pmatrix} \text{[m]}$$

on the plane, connect them with the arc (of minimum length) of a circle having radius $R = 2$ [m] and parametrize this path by its *arc length* s. Design a timing law $s = s(t)$ with *trapezoidal speed profile* so as to obtain a rest-to-rest circular trajectory $\boldsymbol{p}(t)$ from $\boldsymbol{A}$ to $\boldsymbol{B}$ that performs the transfer in minimum time T under the maximum velocity and acceleration constraints

$$\|\dot{\boldsymbol{p}}(t)\| \leq V_{max}, \qquad \|\ddot{\boldsymbol{p}}(t)\| \leq A_{max}, \qquad t \in [0, T],$$

and the bound on the *normal* acceleration $\ddot{\boldsymbol{p}}_n(t)$ to the path

$$\|\ddot{\boldsymbol{p}}_n(t)\| \leq A_{n,max}, \qquad t \in [0, T].$$

Solve this Cartesian trajectory planning problem with the data

$$V_{max} = 3\,\text{[m/s]}, \qquad A_{max} = 4\,\text{[m/s}^2\text{]}, \qquad A_{n,max} = 2\,\text{[m/s}^2\text{]},$$

providing also the numerical values of the associated minimum time T.

[180 minutes; open books]

Solution

February 6, 2015

Exercise 1

As usual, the first step is to assign the DH frames and fill the associated table of parameters, see Fig. 2 and Tab. 1. Note that the end-effector frame RF_E cannot be the last reference frame RF_3 of the DH frame assignment. In fact, its orientation cannot be generated by a suitable choice of feasible DH parameters (in the last row of the table), since the $\boldsymbol{x}_E$ axis is **not** incident and orthogonal to the last defined joint axis, i.e., $\boldsymbol{z}_2$. Therefore, we need also an additional (constant) transformation matrix ${}^3\boldsymbol{T}_E$ relating RF_3 to RF_E.

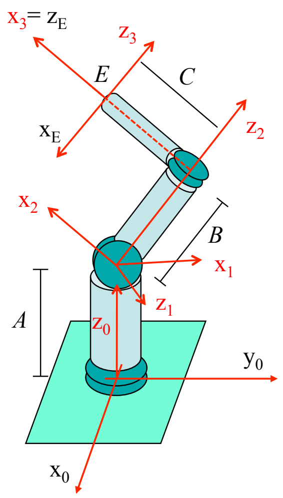


Figure 2: Denavit-Hartenberg frames for the robot of Fig. 1.

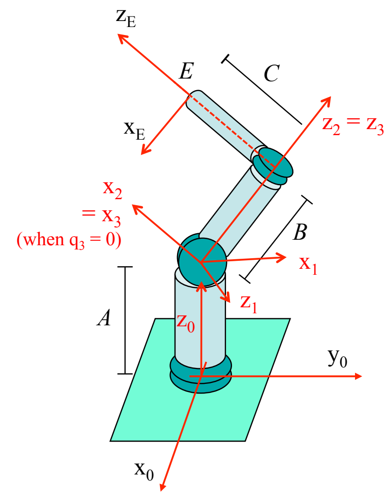
i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	A	q_1
2	$\pi/2$	0	0	q_2
3	0	C	B	q_3

Table 1: Denavit-Hartenberg parameters associated to the frames chosen as in Fig. 2.

Using Tab. 1, we compute the homogeneous matrices ${}^{i-1}\boldsymbol{A}_i(q_i)$, for $i = 1, 2, 3$. The additional constant transformation matrix from RF_3 to the specified end-effector frame RF_E is

$$
{}^3\boldsymbol{T}_E = \begin{pmatrix} {}^3\boldsymbol{R}_E & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Indeed, there are other possible assignments of DH frames. In particular, one could also place the origin of the last DH frame RF_3 coincident with that of frame RF_2 (at the robot shoulder). Such situation is shown in Fig. 3, together with the new DH table and the (different) transformation matrix ${}^3\boldsymbol{T}_E$. The last row of the DH table is made of zeros, except for $\theta_3 = q_3$.



i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	A	q_1
2	$\pi/2$	0	0	q_2
3	0	0	0	q_3

$$ {}^3T_E = \begin{bmatrix} 0 & 0 & 1 & C \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & B \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^3R_E & {}^3p_{3E} \\ 0^T & 1 \end{bmatrix} $$

Figure 3: An alternative assignment of the last Denavit-Hartenberg frame RF_3, with the associated table and additional transformation to the end-effector frame RF_E.

The complete direct kinematics[1] is given by

$$ {}^0\boldsymbol{T}_E(q_1,q_2,q_3) = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3)\,{}^3\boldsymbol{T}_E = \begin{pmatrix} {}^0\boldsymbol{R}_E(q_1,q_2,q_3) & {}^0\boldsymbol{p}_{0E}(q_1,q_2,q_3) \\ \boldsymbol{0}^T & 1 \end{pmatrix}, $$

with

$$ {}^0\boldsymbol{R}_E = \begin{pmatrix} {}^0\boldsymbol{x}_E(q_1,q_2) & {}^0\boldsymbol{y}_E(q_1,q_2,q_3) & {}^0\boldsymbol{z}_E(q_1,q_2,q_3) \end{pmatrix} \tag{1} $$

$$ = \begin{pmatrix} -\cos q_1 \sin q_2 & \sin q_1 \cos q_3 - \cos q_1 \cos q_2 \sin q_3 & \sin q_1 \sin q_3 + \cos q_1 \cos q_2 \cos q_3 \\ -\sin q_1 \sin q_2 & -\cos q_1 \cos q_3 - \sin q_1 \cos q_2 \sin q_3 & \sin q_1 \cos q_2 \cos q_3 - \cos q_1 \sin q_3 \\ \cos q_2 & -\sin q_2 \sin q_3 & \sin q_2 \cos q_3 \end{pmatrix} $$

and

$$ {}^0\boldsymbol{p}_{0E} = \begin{pmatrix} B\cos q_1 \sin q_2 + C \sin q_1 \sin q_3 + C \cos q_1 \cos q_2 \cos q_3 \\ B \sin q_1 \sin q_2 - C \cos q_1 \sin q_3 + C \sin q_1 \cos q_2 \cos q_3 \\ A - B\cos q_2 + C \sin q_2 \cos q_3 \end{pmatrix}. $$

During the computations, we saved also

$$ {}^0\boldsymbol{z}_1 = {}^0\boldsymbol{R}_1(q_1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix}, \quad {}^0\boldsymbol{z}_2 = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos q_1 \sin q_2 \\ \sin q_1 \sin q_2 \\ -\cos q_2 \end{pmatrix}. $$

[1]The outcome is exactly the same in the two situations of Fig. 2 and Fig. 3.

Thus, the Jacobian matrix relating the joint velocity $\dot{\boldsymbol{q}}$ to the angular velocity ${}^0\boldsymbol{\omega}_E$ of the end-effector frame is given by

$$ {}^0\boldsymbol{\omega}_E \left(= {}^0\boldsymbol{\omega}_3\right) = \boldsymbol{J}(\boldsymbol{q})\,\dot{\boldsymbol{q}}, \qquad \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{z}_1 & {}^0\boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & \sin q_1 & \cos q_1 \sin q_2 \\ 0 & -\cos q_1 & \sin q_1 \sin q_2 \\ 1 & 0 & -\cos q_2 \end{pmatrix}, $$

with $\det \boldsymbol{J}(\boldsymbol{q}) = \sin q_2$. This dependence shows that only the angle q_2 (the only one that is intrinsically defined by the spatial disposition of the joint axes of this robot) matters for the angular mobility of the end effector.

Note that the actual values of A, B, and C play no role at all in the orientation kinematics (as we could have expected), since these length parameters appear only in the expression of ${}^0\boldsymbol{p}_{0E}$. Therefore, we can proceed directly to the solution of the inverse kinematics problem for the orientation without this information.

For a desired orientation of the end-effector frame, represented by a given rotation matrix $\boldsymbol{R}_d = \{R_{ij}\}$, one can determine the inverse kinematics solution using the elements in the third row and first column of (1). First, compute

$$ q_2^I = \text{ATAN2}\left\{ \sqrt{R_{32}^2 + R_{33}^2}, R_{31} \right\}. $$

If $R_{32}^2 + R_{33}^2 \neq 0$, which means $\sin q_2 \neq 0$, we are in the *regular* case. A second distinct solution for q_2 is computed as

$$ q_2^{II} = \text{ATAN2}\left\{ -\sqrt{R_{32}^2 + R_{33}^2}, R_{31} \right\}. $$

Moreover,

$$ q_1^I = \text{ATAN2}\left\{ \frac{-R_{21}}{\sin q_2^I}, \frac{-R_{11}}{\sin q_2^I} \right\}, \qquad q_1^{II} = \text{ATAN2}\left\{ \frac{-R_{21}}{\sin q_2^{II}}, \frac{-R_{11}}{\sin q_2^{II}} \right\}, $$

and

$$ q_3^I = \text{ATAN2}\left\{ \frac{-R_{32}}{\sin q_2^I}, \frac{R_{33}}{\sin q_2^I} \right\}, \qquad q_3^{II} = \text{ATAN2}\left\{ \frac{-R_{32}}{\sin q_2^{II}}, \frac{R_{33}}{\sin q_2^{II}} \right\}. $$

When $R_{32} = R_{33} = 0$, we are in a *singular* situation. This occurs if and only if $\sin q_2 = 0$, thus when either $q_2 = 0$ or $q_2 = \pi$. If $q_2 = 0$, we can solve only for the difference $q_1 - q_3$:

$$ {}^0\boldsymbol{R}_E\big|_{q_2=0} = \begin{pmatrix} 0 & \sin(q_1 - q_3) & \cos(q_1 - q_3) \\ 0 & -\cos(q_1 - q_3) & \sin(q_1 - q_3) \\ 1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad q_{1-3} := q_1 - q_3 = \text{ATAN2}\left\{R_{23}, R_{13}\right\}, $$

leading to an infinity of solutions of the form

$$ \boldsymbol{q} = \begin{pmatrix} \alpha & 0 & \alpha - q_{1-3} \end{pmatrix}^T, \qquad \forall \alpha \in \mathbb{R}. $$

Similarly, when $q_2 = \pi$ we can solve only for the sum $q_1 + q_3$:

$$ {}^0\boldsymbol{R}_E\big|_{q_2=\pi} = \begin{pmatrix} 0 & \sin(q_1 + q_3) & -\cos(q_1 + q_3) \\ 0 & -\cos(q_1 + q_3) & -\sin(q_1 + q_3) \\ -1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad q_{1+3} := q_1 + q_3 = \text{ATAN2}\left\{R_{12}, -R_{13}\right\}, $$

leading to an infinity of solutions of the form

$$\boldsymbol{q} = \begin{pmatrix} \beta & \pi & q_{1+3} - \beta \end{pmatrix}^T, \qquad \forall \beta \in \mathbb{R}.$$

Applying these results to the given data, we have that

$$\boldsymbol{R}_{d,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{a singular case with } q_2 = \pi,$$

leading to the solutions $\boldsymbol{q} = \begin{pmatrix} \beta & \pi & \pi/2 - \beta \end{pmatrix}^T$, for any β. On the other hand, $\boldsymbol{R}_{d,2} = \boldsymbol{I}$ is a regular case leading to the pair of solutions

$$\boldsymbol{q}^I = \begin{pmatrix} \pi & \pi/2 & 0 \end{pmatrix}^T, \qquad \boldsymbol{q}^{II} = \begin{pmatrix} 0 & -\pi/2 & \pi \end{pmatrix}^T.$$

The Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ is singular in the zero configuration $\boldsymbol{q} = \boldsymbol{0}$. However, it takes a form that allows to realize the given angular velocity $\boldsymbol{\omega}_{E.d}$. In fact,

$$\boldsymbol{J}(\boldsymbol{0}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{\omega}_{E,d} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \in \mathcal{R}\left(\boldsymbol{J}(\boldsymbol{0})\right) = \text{span}\left\{ \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} \right\}.$$

Therefore, there exists an infinite number of joint velocity solutions $\dot{\boldsymbol{q}}$ providing $\boldsymbol{\omega}_{E,d}$, all having $\dot{q}_2 = 0$ and with $\dot{q}_1 - \dot{q}_3 = 3$ [rad/s]. In particular, $\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{0})\,\boldsymbol{\omega}_{E,d} = \begin{pmatrix} 1.5 & 0 & -1.5 \end{pmatrix}^T$ [rad/s] provides the minimum norm solution.

Due to its kinematics this robot has a limited use for positioning tasks, since the primary workspace is very restricted. In fact, it is a thin spherical mantle/surface, placed on top of the surface of the sphere described by the tip position of the second link (a 2R polar sub-structure).

The solution to the last (extra) question is left as an exercise.

Exercise 2

We construct first the specified path from $\boldsymbol{A}$ to $\boldsymbol{B}$. This can be done easily in a geometric way by defining a circumference of given radius R passing through two points, as illustrated in Fig. 4 (the construction needs only a ruled set square and a compass): *(a)* given the points $\boldsymbol{A}$ and $\boldsymbol{B}$, *(b)* draw the line L_1 through them, define the midpoint $(\boldsymbol{A}+\boldsymbol{B})/2$ of the segment $\overline{AB}$, and draw the line L_2 orthogonal[2] to L_1 and passing through the midpoint (L_2 contains all points that are equidistant from $\boldsymbol{A}$ and $\boldsymbol{B}$); *(c)* the center of the circle will be on L_2, at a distance that can be determined by Pythagoras theorem. There are in general two solutions $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, which are fully equivalent in the present context. The shortest path from $\boldsymbol{A}$ to $\boldsymbol{B}$ on the circle of radius R centered in $\boldsymbol{C}_1$ is shown as a bolded arc (the arrow indicates its clockwise rotation in this case).

Indeed, we may prefer an algebraic solution. The above procedure can be simply programmed as a Matlab function, called by passing the coordinates of points $\boldsymbol{A}$ and $\boldsymbol{B}$, and the radius $R > 0$. For simplicity, in the following piece of code we have not considered the possible non-regular cases (e.g., when $2R$ is less than the distance $d = \|\boldsymbol{B} - \boldsymbol{A}\|$).

[2] All orthogonal lines to a line $ax + by + c = 0$ in the plane xy can be written as $ay - bx + d = 0$.

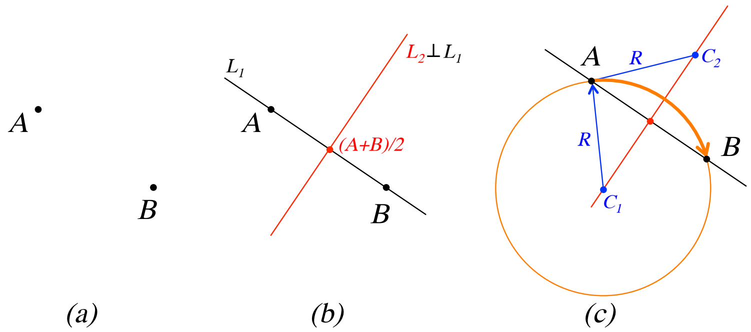


Figure 4: Geometric steps for constructing a circle of radius R through two points $\boldsymbol{A}$ and $\boldsymbol{B}$. There are two solutions in general.

```
function CircleCenter(xA,yA,xB,yB,R)
d = sqrt((xB-xA)^2+(yB-yA)^2);
xM = (xA+xB)/2;
yM = (yA+yB)/2;
% only the regular case of two solutions
disp('first center')
cx1 = xM - sqrt(R^2-(d/2)^2)*(yA-yB)/d
cy1 = yM - sqrt(R^2-(d/2)^2)*(xB-xA)/d
disp('second center')
cx2 = xM + sqrt(R^2-(d/2)^2)*(yA-yB)/d
cy2 = yM + sqrt(R^2-(d/2)^2)*(xB-xA)/d
```

Using this code on the problem data, we obtain the two centers

$$\boldsymbol{C}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \text{ [m]} \qquad \text{and} \qquad \boldsymbol{C}_2 = \begin{pmatrix} -1.268 \\ 1 \end{pmatrix} \text{ [m]}.$$

We choose (arbitrarily) $\boldsymbol{C} = \boldsymbol{C}_1$. Due to the specific data values that were given, this solution could have been found rather immediately also by visual inspection —see Fig. 5.

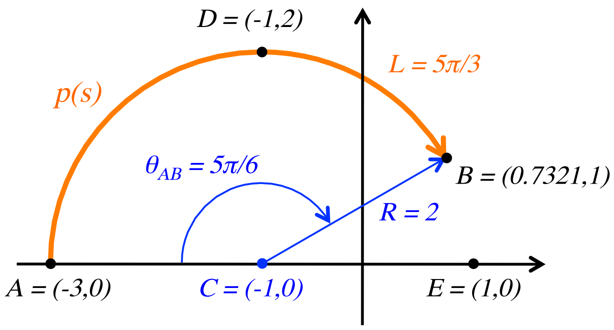


Figure 5: The actual geometric path $\boldsymbol{p}(s)$ constructed with the problem data, using the circle of radius $R = 2$ m with center at $\boldsymbol{C} = (-1, 0)$.

The *infinitesimal arc length* on a circle of radius R can be written as $ds = R\,d\theta$, where $d\theta$ is the angle spanning the arc. Moreover, the path is traced clockwise, which is the *negative* convention for the angles. Using simple trigonometry, the path parametrization by the arc length is given then in general by

$$\boldsymbol{p}(s) = \boldsymbol{C} + R \begin{pmatrix} \cos\left(-\dfrac{s}{R} + \phi\right) \\ \sin\left(-\dfrac{s}{R} + \phi\right) \end{pmatrix}, \qquad s \in [0, L]. \tag{2}$$

By imposing $\boldsymbol{p}(0) = \boldsymbol{A}$, the required path parametrization becomes

$$\boldsymbol{p}(s) = \boldsymbol{C} - R \begin{pmatrix} \cos\left(-\dfrac{s}{R}\right) \\ \sin\left(-\dfrac{s}{R}\right) \end{pmatrix}, \qquad s \in [0, L], \quad L = R\,\frac{5\pi}{6} = \frac{5\pi}{3} = 5.236 \text{ [m]}, \tag{3}$$

where the phase $\phi = \pi$ of the trigonometric functions has been tuned suitably (giving the minus sign in front of R, outside the parentheses). Note also that the total length L is obtained from angle θ_{AB} spanning the whole path (equal to 150°, if expressed in degrees) multiplied by the radius $R = 2$.

While the above computations may appear cumbersome, a nice feature of the problem is that one does *not* have to determine the center $\boldsymbol{C}$ of the circle, nor the circle itself, in order to satisfy all the design specifications on the trajectory! We need $\boldsymbol{C}$ only to define/draw the actual path $\boldsymbol{p}(s)$. Even the path length L can be directly computed from the known formula (see, e.g., wikipedia) relating the distance d of two points with the length L of the (shortest) arc of a circle of radius R passing through the two points:

$$L = R\,\theta, \quad d = \|\boldsymbol{B} - \boldsymbol{A}\| = 2R\sin\left(\frac{\theta}{2}\right) \quad \Rightarrow \quad L = 2R\arcsin\left(\frac{d}{2R}\right) \ (= 5.236 \text{ [m]}).$$

With the length L and the generic expression (2), we can solve completely the assigned problem. Nonetheless, in the following we shall continue with the simpler path expression obtained in (3).

For a generic $s = s(t)$, the first and second time derivatives of $\boldsymbol{p}(s)$ in (3) are given by

$$\dot{\boldsymbol{p}} = \frac{d\boldsymbol{p}}{ds}\frac{ds}{dt} = \begin{pmatrix} \sin\left(\dfrac{s}{R}\right) \\ \cos\left(\dfrac{s}{R}\right) \end{pmatrix} \dot{s} \tag{4}$$

and

$$\begin{aligned} \ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_t + \ddot{\boldsymbol{p}}_n &= \frac{d\boldsymbol{p}}{ds}\,\ddot{s} + \frac{d^2\boldsymbol{p}}{ds^2}\,\dot{s}^2 = \begin{pmatrix} \sin\left(\dfrac{s}{R}\right) \\ \cos\left(\dfrac{s}{R}\right) \end{pmatrix} \ddot{s} + \frac{1}{R} \begin{pmatrix} \cos\left(\dfrac{s}{R}\right) \\ -\sin\left(\dfrac{s}{R}\right) \end{pmatrix} \dot{s}^2 \\ &= \begin{pmatrix} \cos\left(\dfrac{s}{R}\right) & \sin\left(\dfrac{s}{R}\right) \\ -\sin\left(\dfrac{s}{R}\right) & \cos\left(\dfrac{s}{R}\right) \end{pmatrix} \begin{pmatrix} \dot{s}^2/R \\ \ddot{s} \end{pmatrix} = \mathrm{Rot}^T\!\left(\frac{s}{R}\right) \begin{pmatrix} \dot{s}^2/R \\ \ddot{s} \end{pmatrix}, \end{aligned} \tag{5}$$

with a decomposition in tangential and normal acceleration to the path, respectively $\ddot{\boldsymbol{p}}_t$ and $\ddot{\boldsymbol{p}}_n$. The 2×2 matrix $\mathrm{Rot}(\theta)$ is a planar rotation by an angle θ, acting on 2-dimensional vectors. Thanks to the used parametrization by the arc length, we have the following properties for the norms

$$\left\|\frac{d\boldsymbol{p}}{ds}\right\| = 1 \ \Rightarrow \ \|\dot{\boldsymbol{p}}\| = |\dot{s}|, \ \|\ddot{\boldsymbol{p}}_t\| = |\ddot{s}|, \qquad \left\|\frac{d^2\boldsymbol{p}}{ds^2}\right\| = \frac{1}{R} \ \Rightarrow \ \|\ddot{\boldsymbol{p}}_n\| = \frac{\dot{s}^2}{R}, \quad \|\ddot{\boldsymbol{p}}\| = \sqrt{\left(\frac{\dot{s}^2}{R}\right)^2 + \ddot{s}^{\,2}}.$$

As requested, we consider now a generic a trapezoidal profile for $\dot{s}(t)$ (i.e., a bang-coast-bang profile for $\ddot{s}(t)$) of duration T, with symmetric initial and final acceleration/deceleration phases of absolute value $\bar{A}$ and equal duration T_s, and a central constant cruising speed $\bar{V} > 0$ to be kept for $T - 2T_s$ seconds (assuming $T - 2T_s \geq 0$, which needs to be checked at the end). The four quantities $\bar{V}$, $\bar{A}$, T_s, and T have to be determined so as to cover the total path length L, while minimizing T and satisfying the constraints specified by V_{max}, A_{max}, and $A_{n,max}$.

The important thing to note is that the curvature $1/R$ of the path and the bound on the normal acceleration $\ddot{\boldsymbol{p}}_n$

$$\|\ddot{\boldsymbol{p}}_n\| = \frac{\dot{s}^2}{R} \leq A_{n,max}$$

may impose a more severe limit on $\dot{s}$ than the bound V_{max} on the norm of $\dot{\boldsymbol{p}}$. In fact, we have that

$$|\dot{s}| \leq \min\left\{V_{max}, \sqrt{R\,A_{n,max}}\right\} = \min\{3, \sqrt{4}\} = 2 =: \bar{V}'. \tag{6}$$

To evaluate the constraint on the total acceleration $\ddot{\boldsymbol{p}}$, we distinguish two situations for the tangential acceleration: constant $\ddot{s} = \pm\bar{A} \neq 0$ (in the initial and final phases) and $\ddot{s} = 0$ (in the cruise phase at constant speed). During the cruise phase, it is

$$\|\ddot{\boldsymbol{p}}\| = \frac{\dot{s}^2}{R} \leq A_{max} \quad \Rightarrow \quad |\dot{s}| \leq \sqrt{R\,A_{max}} = \sqrt{8} =: \bar{V}''. \tag{7}$$

As a result, combining (6) and (7), we have for the maximum speed $\dot{s}(t)$ during cruising

$$\dot{s}(t) = \bar{V} = \min\left\{\bar{V}', \bar{V}''\right\} = 2, \qquad t \in [T_s, T - T_s].$$

In the constant acceleration phase (a specular argument applies to the constant deceleration phase), the speed increases linearly from 0 at $t = 0$ (start at rest) to $\bar{V}$ at $t = T_s$. The largest value for the norm of the total acceleration is approached when $t = T_s$. Thus, we impose satisfaction of the constraint in the worst case:

$$\|\ddot{\boldsymbol{p}}(T_s)\| = \sqrt{\left(\frac{\bar{V}^2}{R}\right)^2 + \bar{A}^2} \leq A_{max} \quad \Rightarrow \quad \bar{A} \leq \sqrt{A_{max}^2 - \left(\frac{\bar{V}^2}{R}\right)^2} = \sqrt{12}.$$

Since a minimum transfer time is requested, we choose the maximum feasible value of the acceleration norm (i.e., $\|\ddot{\boldsymbol{p}}(T_s)\| = A_{max}$), leading to $\bar{A} = \sqrt{12}$.

With the above values for $\bar{V}$ and $\bar{A}$, having already computed the length L of the path, we determine the remaining unknowns with the usual formulas:

$$T_s = \frac{\bar{V}}{\bar{A}} = \frac{2}{\sqrt{12}} = 0.577 \text{ [s]}, \qquad (T - T_s)\bar{V} = L \quad \Rightarrow \quad T = T_s + \frac{L}{\bar{V}} = 0.577 + \frac{5\pi}{6} = 3.195 \text{ [s]}.$$

We obtained $T > 2T_s$, confirming that the actual speed profile is trapezoidal.

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