Robotics I

February 6, 2015

Exercise 1

Consider the 3R robot in Fig. 1 *(this is the same robotic structure of an exercise assigned in September 2007)*. The base frame and an *additional* end-effector frame are already specified.

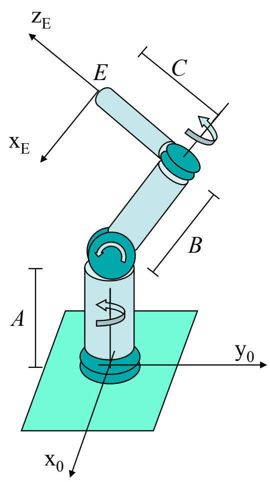

Figure 1: A robot with three revolute joints.

- Given a desired *orientation* $\boldsymbol{R}_d$ of the end-effector frame, solve the inverse kinematics problem in symbolic form. Consider also possible singular cases.
- Apply your result and determine *all* numerical solutions $\boldsymbol{q}$ for the following two sets of data:

$$\boldsymbol{R}_{d,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}; \qquad \boldsymbol{R}_{d,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}.$$

- Provide for this robot the relation between $\dot{\boldsymbol{q}}$ and the angular velocity $\boldsymbol{\omega}_E$ of the end-effector frame (expressed in the base frame).
- Determine a joint velocity $\dot{\boldsymbol{q}}$ in the configuration $\boldsymbol{q} = \boldsymbol{0}$ that produces the desired angular velocity $\boldsymbol{\omega}_{E,d} = \begin{pmatrix} 0 & 0 & 3 \end{pmatrix}^T$ [rad/s]. Has this problem a solution? If so, is it unique?
- *"This robot is of little use for positioning the end-effector in 3D space."* Do you agree with this statement? Why?

Extra • Based on the analysis you have performed, can this robot realize any *pointing* task with its end-effector axis $\boldsymbol{z}_E$? If so, is there a unique solution in the generic case? Are there singular situations? *(If you reply correctly to the extra questions, you get a bonus)*

Exercise 2

Given the two points

$$\boldsymbol{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \text{ [m]} \qquad \text{and} \qquad \boldsymbol{B} = \begin{pmatrix} 0.732 \\ 1 \end{pmatrix} \text{ [m]}$$

on the plane, connect them with the arc (of minimum length) of a circle having radius $R = 2$ [m] and parametrize this path by its *arc length* s. Design a timing law $s = s(t)$ with *trapezoidal speed profile* so as to obtain a rest-to-rest circular trajectory $\boldsymbol{p}(t)$ from $\boldsymbol{A}$ to $\boldsymbol{B}$ that performs the transfer in minimum time T under the maximum velocity and acceleration constraints

$$\|\dot{\boldsymbol{p}}(t)\| \leq V_{max}, \qquad \|\ddot{\boldsymbol{p}}(t)\| \leq A_{max}, \qquad t \in [0, T],$$

and the bound on the *normal* acceleration $\ddot{\boldsymbol{p}}_n(t)$ to the path

$$\|\ddot{\boldsymbol{p}}_n(t)\| \leq A_{n,max}, \qquad t \in [0, T].$$

Solve this Cartesian trajectory planning problem with the data

$$V_{max} = 3 \text{ [m/s]}, \qquad A_{max} = 4 \text{ [m/s}^2\text{]}, \qquad A_{n,max} = 2 \text{ [m/s}^2\text{]},$$

providing also the numerical values of the associated minimum time T.

[180 minutes; open books]

Solution

February 6, 2015

Exercise 1

For the assignment of DH frames and the associated table of parameters, see Fig. 2 and Tab. 1. We need also an additional transformation matrix ${}^3\boldsymbol{T}_E$ relating the third DH frame RF_3 to RF_E:

$$ {}^3\boldsymbol{T}_E = \begin{pmatrix} {}^3\boldsymbol{R}_E & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1} $$

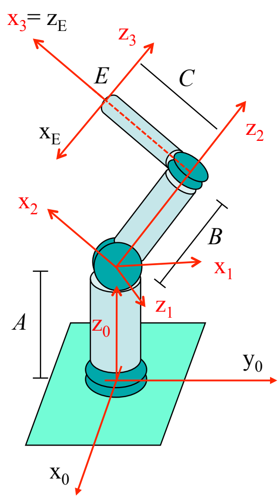

Figure 2: Denavit-Hartenberg frames for the robot of Fig. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	A	q_1
2	$\pi/2$	0	0	q_2
3	0	C	B	q_3

Table 1: Denavit-Hartenberg parameters associated to the frames chosen as in Fig. 2.

Using Tab. 1 and (1), the direct kinematics for the orientation is computed as

$$ {}^0\boldsymbol{R}_E = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_3(q_3)\,{}^3\boldsymbol{R}_E \tag{2} $$

$$ = \begin{pmatrix} -\cos q_1 \sin q_2 & \sin q_1 \cos q_3 - \cos q_1 \cos q_2 \sin q_3 & \sin q_1 \sin q_3 + \cos q_1 \cos q_2 \cos q_3 \\ -\sin q_1 \sin q_2 & -\cos q_1 \cos q_3 - \sin q_1 \cos q_2 \sin q_3 & \sin q_1 \cos q_2 \cos q_3 - \cos q_1 \sin q_3 \\ \cos q_2 & -\sin q_2 \sin q_3 & \sin q_2 \cos q_3 \end{pmatrix}, $$

which is independent of A, B, and C.

The Jacobian matrix in ${}^0\boldsymbol{\omega}_E \left(= {}^0\boldsymbol{\omega}_3\right) = \boldsymbol{J}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \left({}^0\boldsymbol{z}_0 \; {}^0\boldsymbol{z}_1 \; {}^0\boldsymbol{z}_2 \right) \dot{\boldsymbol{q}}$ is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad {}^0\boldsymbol{R}_1(q_1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & \sin q_1 & \cos q_1 \sin q_2 \\ 0 & -\cos q_1 & \sin q_1 \sin q_2 \\ 1 & 0 & -\cos q_2 \end{pmatrix},$$

with $\det \boldsymbol{J}(\boldsymbol{q}) = \sin q_2$.

For a desired orientation of the end-effector frame, represented by a given rotation matrix $\boldsymbol{R}_d = \{R_{ij}\}$, one can determine the inverse kinematics solution using the elements in (2). First, compute

$$q_2^I = \text{ATAN2}\left\{ \sqrt{R_{32}^2 + R_{33}^2}, R_{31} \right\}.$$

If $R_{32}^2 + R_{33}^2 \neq 0$, which means $\sin q_2 \neq 0$, we are in the *regular* case. A second distinct solution for q_2 is computed as

$$q_2^{II} = \text{ATAN2}\left\{ -\sqrt{R_{32}^2 + R_{33}^2}, R_{31} \right\}.$$

Moreover,

$$q_1^I = \text{ATAN2}\left\{ \frac{-R_{21}}{\sin q_2^I}, \frac{-R_{11}}{\sin q_2^I} \right\}, \qquad q_1^{II} = \text{ATAN2}\left\{ \frac{-R_{21}}{\sin q_2^{II}}, \frac{-R_{11}}{\sin q_2^{II}} \right\},$$

and

$$q_3^I = \text{ATAN2}\left\{ \frac{-R_{32}}{\sin q_2^I}, \frac{R_{33}}{\sin q_2^I} \right\}, \qquad q_3^{II} = \text{ATAN2}\left\{ \frac{-R_{32}}{\sin q_2^{II}}, \frac{R_{33}}{\sin q_2^{II}} \right\}.$$

When $R_{32} = R_{33} = 0$, we are in a *singular* situation. This occurs if and only if $\sin q_2 = 0$, thus when either $q_2 = 0$ or $q_2 = \pi$. If $q_2 = 0$, we can solve only for the difference $q_1 - q_3$:

$$\left.{}^0\boldsymbol{R}_E\right|_{q_2=0} = \begin{pmatrix} 0 & \sin(q_1 - q_3) & \cos(q_1 - q_3) \\ 0 & -\cos(q_1 - q_3) & \sin(q_1 - q_3) \\ 1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad q_{1-3} := q_1 - q_3 = \text{ATAN2}\left\{R_{23}, R_{13}\right\},$$

leading to an infinity of solutions of the form

$$\boldsymbol{q} = \begin{pmatrix} \alpha & 0 & \alpha - q_{1-3} \end{pmatrix}^T, \qquad \forall \alpha \in \mathbb{R}.$$

Similarly, when $q_2 = \pi$ we can solve only for the sum $q_1 + q_3$:

$$\left.{}^0\boldsymbol{R}_E\right|_{q_2=\pi} = \begin{pmatrix} 0 & \sin(q_1 + q_3) & -\cos(q_1 + q_3) \\ 0 & -\cos(q_1 + q_3) & -\sin(q_1 + q_3) \\ -1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad q_{1+3} := q_1 + q_3 = \text{ATAN2}\left\{R_{12}, -R_{13}\right\},$$

leading to an infinity of solutions of the form

$$\boldsymbol{q} = \begin{pmatrix} \beta & \pi & q_{1+3} - \beta \end{pmatrix}^T, \qquad \forall \beta \in \mathbb{R}.$$

Applying these results to the given data, we have that

$$\boldsymbol{R}_{d,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{a singular case with } q_2 = \pi,$$

leading to the solutions $\boldsymbol{q} = \begin{pmatrix} \beta & \pi & \pi/2 - \beta \end{pmatrix}^T$, for any β. On the other hand, $\boldsymbol{R}_{d,2} = \boldsymbol{I}$ is a regular case leading to the pair of solutions

$$\boldsymbol{q}^I = \begin{pmatrix} \pi & \pi/2 & 0 \end{pmatrix}^T, \qquad \boldsymbol{q}^{II} = \begin{pmatrix} 0 & -\pi/2 & \pi \end{pmatrix}^T.$$

The Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ is singular in the zero configuration $\boldsymbol{q} = \boldsymbol{0}$. However,

$$\boldsymbol{J}(\boldsymbol{0}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{\omega}_{E,d} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \in \mathcal{R}\left(\boldsymbol{J}(\boldsymbol{0})\right) = \text{span}\left\{ \begin{pmatrix} 0 \\ \gamma \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} \right\}.$$

Therefore, there exists an infinite number of joint velocity solutions $\dot{\boldsymbol{q}}$ providing $\boldsymbol{\omega}_{E,d}$, all having $\dot{q}_2 = 0$ and with $\dot{q}_1 - \dot{q}_3 = 3$ [rad/s]. In particular, $\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{0})\,\boldsymbol{\omega}_{E,d} = \begin{pmatrix} 1.5 & 0 & -1.5 \end{pmatrix}^T$ [rad/s] provides the minimum norm solution.

Due to its kinematics this robot has a limited use for positioning tasks, since the primary workspace is very restricted. In fact, it is a thin spherical mantle/surface, placed on top of the surface of the sphere described by the tip position of the second link (a 2R polar sub-structure).

The solution to the last (extra) question is left as an exercise.

Exercise 2

The specified path from $\boldsymbol{A}$ to $\boldsymbol{B}$ can be constructed easily in a geometric way, by defining a circumference of given radius R passing through two points, as illustrated in Fig. 3. The shortest path from $\boldsymbol{A}$ to $\boldsymbol{B}$ on the circle of radius R centered in $\boldsymbol{C}_1$ is shown as a bolded arc (the arrow indicates its clockwise rotation).

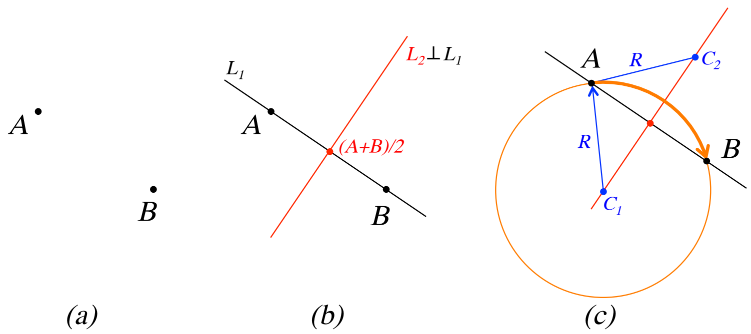

Figure 3: Geometric steps for constructing a circle of radius R through two points $\boldsymbol{A}$ and $\boldsymbol{B}$ (there are two solutions in the regular case).

In general, an *infinitesimal arc length* on a circle of radius R can be written as $ds = R\,d\theta$, where $d\theta$ is the angle spanning the arc from the circle center $\boldsymbol{C}$. Using simple trigonometry, the path parametrization by the arc length is given by

$$\boldsymbol{p}(s) = \boldsymbol{C} + R \begin{pmatrix} \cos\left(\pm\dfrac{s}{R} + \phi\right) \\ \sin\left(\pm\dfrac{s}{R} + \phi\right) \end{pmatrix}, \qquad s \in [0, L], \tag{3}$$

where the sign in $\pm$ is chosen positive if the path is traced counterclockwise, negative otherwise. L is the total arc length (from point $\boldsymbol{A}$ to point $\boldsymbol{B}$), while the phase ϕ is chosen so as to be in $\boldsymbol{A}$ for $s = 0$.

A nice feature of the problem is that one does *not* have to determine the center $\boldsymbol{C}$ of the circle, nor the circle itself, in order to satisfy all the design specifications on the trajectory! Even the path length L can be directly computed from the known formula (see, e.g., wikipedia) relating the distance d of two points $\boldsymbol{A}$ and $\boldsymbol{B}$ with the length L of the (shortest) arc of a circle of radius R passing through the two points:

$$L = R\,\theta_{AB}, \quad d = \|\boldsymbol{B} - \boldsymbol{A}\| = 2R \sin\left(\frac{\theta_{AB}}{2}\right) \quad \Rightarrow \quad L = 2R \arcsin\left(\frac{d}{2R}\right).$$

With the length L and the generic expression (3), we can solve completely the assigned problem.

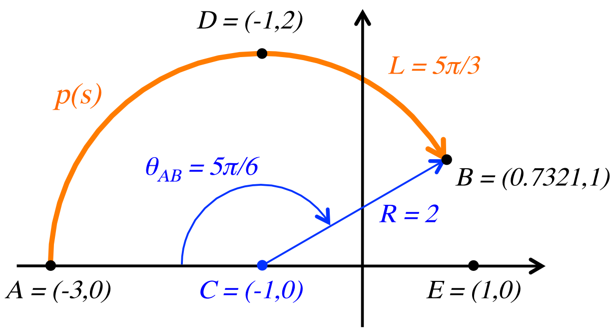

Figure 4: The actual geometric path $\boldsymbol{p}(s)$ constructed with the problem data, using a circle of radius $R = 2$ m with center at $\boldsymbol{C} = (-1, 0)$.

Nonetheless, due to the specific data values that were given, a center $\boldsymbol{C}$ can be found rather immediately by visual inspection —see Fig. 4. By imposing $\boldsymbol{p}(0) = \boldsymbol{A}$, the parametrization of the clockwise circular path becomes

$$\boldsymbol{p}(s) = \boldsymbol{C} - R \begin{pmatrix} \cos\left(-\frac{s}{R}\right) \\ \sin\left(-\frac{s}{R}\right) \end{pmatrix}, \qquad s \in [0, L], \quad L = R\,\frac{5\pi}{6} = \frac{5\pi}{3} = 5.236 \text{ [m]}, \tag{4}$$

where the phase $\phi = \pi$ chosen in the argument of the trigonometric functions in (3) leads to the minus sign in front of the first R. The length L is obtained from the angle θ_{AB} spanning the whole path (equal to 150°, if expressed in degrees) multiplied by the radius $R = 2$.

For a generic $s = s(t)$, the first and second time derivatives of $\boldsymbol{p}(s)$ in (4) are given by

$$\dot{\boldsymbol{p}} = \frac{d\boldsymbol{p}}{ds}\frac{ds}{dt} = \begin{pmatrix} \sin\left(\frac{s}{R}\right) \\ \cos\left(\frac{s}{R}\right) \end{pmatrix} \dot{s} \tag{5}$$

and

$$
\begin{aligned}
\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_t + \ddot{\boldsymbol{p}}_n &= \frac{d\boldsymbol{p}}{ds}\,\ddot{s} + \frac{d^2\boldsymbol{p}}{ds^2}\,\dot{s}^2 = \begin{pmatrix} \sin\left(\frac{s}{R}\right) \\ \cos\left(\frac{s}{R}\right) \end{pmatrix} \ddot{s} + \frac{1}{R} \begin{pmatrix} \cos\left(\frac{s}{R}\right) \\ -\sin\left(\frac{s}{R}\right) \end{pmatrix} \dot{s}^2 \\
&= \begin{pmatrix} \cos\left(\frac{s}{R}\right) & \sin\left(\frac{s}{R}\right) \\ -\sin\left(\frac{s}{R}\right) & \cos\left(\frac{s}{R}\right) \end{pmatrix} \begin{pmatrix} \dot{s}^2/R \\ \ddot{s} \end{pmatrix} = \mathrm{Rot}^T\!\left(\frac{s}{R}\right) \begin{pmatrix} \dot{s}^2/R \\ \ddot{s} \end{pmatrix},
\end{aligned}
\tag{6}
$$

with a decomposition in tangential and normal acceleration to the path, respectively $\ddot{\boldsymbol{p}}_t$ and $\ddot{\boldsymbol{p}}_n$. The 2×2 matrix $\mathrm{Rot}(\theta)$ is a planar rotation by an angle θ, acting on 2-dimensional vectors. Thanks to the used parametrization by the arc length, we have the following properties for the norms

$$
\left\| \frac{d\boldsymbol{p}}{ds} \right\| = 1 \;\Rightarrow\; \|\dot{\boldsymbol{p}}\| = |\dot{s}|,\; \|\ddot{\boldsymbol{p}}_t\| = |\ddot{s}|, \qquad \left\| \frac{d^2\boldsymbol{p}}{ds^2} \right\| = \frac{1}{R} \;\Rightarrow\; \|\ddot{\boldsymbol{p}}_n\| = \frac{\dot{s}^2}{R}, \qquad \|\ddot{\boldsymbol{p}}\| = \sqrt{\left(\frac{\dot{s}^2}{R}\right)^2 + \ddot{s}^2}.
$$

We consider now a generic a trapezoidal profile for $\dot{s}(t)$ of duration T, with symmetric initial and final acceleration/deceleration phases of absolute value $\bar{A}$ and equal duration T_s, and a central constant cruising speed $\bar{V} > 0$ to be kept for $T - 2T_s$ seconds. The four quantities $\bar{V}$, $\bar{A}$, T_s, and T have to be determined so as to cover the total path length L, while minimizing T and satisfying the constraints specified by V_{max}, A_{max}, and $A_{n,max}$.

The important thing to note is that the curvature $1/R$ of the path and the bound on the normal acceleration $\ddot{\boldsymbol{p}}_n$

$$
\|\ddot{\boldsymbol{p}}_n\| = \frac{\dot{s}^2}{R} \leq A_{n,max}
$$

may impose a more severe limit on $\dot{s}$ than the bound V_{max} on the norm of $\dot{\boldsymbol{p}}$. In fact, we have that

$$
|\dot{s}| \leq \min\left\{V_{max}, \sqrt{R\,A_{n,max}}\right\} = \min\{3, \sqrt{4}\} = 2 =: \bar{V}'. \tag{7}
$$

To evaluate the constraint on the total acceleration $\ddot{\boldsymbol{p}}$, we distinguish two situations for the tangential acceleration: constant $\ddot{s} = \pm\bar{A} \neq 0$ (in the initial and final phases) and $\ddot{s} = 0$ (in the cruise phase at constant speed). During the cruise phase, it is

$$
\|\ddot{\boldsymbol{p}}\| = \frac{\dot{s}^2}{R} \leq A_{max} \quad \Rightarrow \quad |\dot{s}| \leq \sqrt{R\,A_{max}} = \sqrt{8} =: \bar{V}''. \tag{8}
$$

As a result, combining (7) and (8), we have for the maximum constant speed $\dot{s}$ during cruising

$$
\dot{s}(t) = \bar{V} = \min\left\{\bar{V}', \bar{V}''\right\} = 2, \qquad t \in [T_s, T - T_s].
$$

In the constant acceleration phase (a specular argument applies to the constant deceleration phase), the speed increases linearly from 0 at $t = 0$ (start at rest) to $\bar{V}$ at $t = T_s$. The largest value for the norm of the total acceleration is approached when $t = T_s$. Thus, we impose satisfaction of the constraint in the worst case:

$$
\|\ddot{\boldsymbol{p}}(T_s)\| = \sqrt{\left(\frac{\bar{V}^2}{R}\right)^2 + \bar{A}^2} \leq A_{max} \quad \Rightarrow \quad \bar{A} \leq \sqrt{A_{max}^2 - \left(\frac{\bar{V}^2}{R}\right)^2} = \sqrt{12}.
$$

Since a minimum transfer time is requested, we choose the maximum feasible value of the acceleration norm (i.e., $\|\ddot{\boldsymbol{p}}(T_s)\| = A_{max}$), leading to $\bar{A} = \sqrt{12}$.

With the above values for $\bar{V}$ and $\bar{A}$, having already computed the length L of the path, we determine the remaining unknowns with the usual formulas:

$$T_s = \frac{\bar{V}}{\bar{A}} = \frac{2}{\sqrt{12}} = 0.577 \text{ [s]}, \qquad (T - T_s)\bar{V} = L \quad \Rightarrow \quad T = T_s + \frac{L}{\bar{V}} = 0.577 + \frac{5\pi}{6} = 3.195 \text{ [s]}.$$

We obtained $T > 2T_s$, confirming that the actual speed profile is trapezoidal.

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