Robotics I

October 27, 2014

Exercise 1

Consider a minimal representation of the orientation of a rigid body as given by Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ around the sequence of mobile axes XYZ.

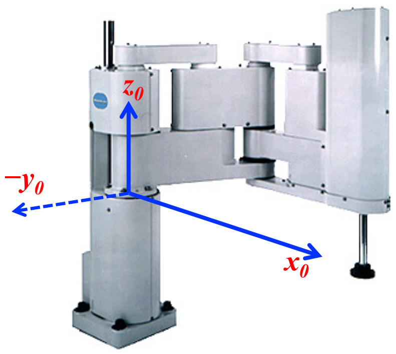
a) Determine the relation
$$\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$$
between the time derivatives of the Euler angles and the angular velocity $\boldsymbol{\omega}$ of the rigid body.

b) Find the singularities of $\boldsymbol{T}(\boldsymbol{\phi})$, and provide an example of an angular velocity vector $\boldsymbol{\omega}$ that cannot be represented in a singularity.

c) Given a rotation matrix $\boldsymbol{R} = \{r_{ij}\} \in SO(3)$, provide the analytic solution $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ to the inverse representation problem out of singularities.

Exercise 2

Consider a 4-dof SCARA robot, with base frame and Denavit-Hartenberg table as shown in Fig. 1.



i	α_i	a_i	d_i	θ_i
1	0	a_1	0	q_1
2	π	a_2	0	q_2
3	0	0	q_3	0
4	0	0	0	q_4

Figure 1: A SCARA robot and its DH table

a) Provide the 6×4 geometric Jacobian of this robot in analytic form.

b) In the following, neglect the presence of joint limits. When the robot arm is fully stretched along the $\boldsymbol{x}_0$ axis, determine, if possible, a joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ that realizes the following desired end-effector generalized velocity:
$$\begin{pmatrix} \boldsymbol{v}_d^T & \boldsymbol{\omega}_d^T \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix}^T \qquad \text{[m/s] or [rad/s]}.$$

c) In this configuration, does a solution exist for any desired generalized velocity $\begin{pmatrix} \boldsymbol{v}_d^T & \boldsymbol{\omega}_d^T \end{pmatrix}^T$, provided only that $\omega_{d,x} = \omega_{d,y} = 0$? And when a solution does exist, is it unique?

[150 minutes; open books]

Solution

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Exercise 1

a) The orientation of a rigid body using the Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ around the sequence of mobile axes XYZ is represented by the product of elementary rotation matrices

$$\boldsymbol{R}_{XYZ}(\alpha, \beta, \gamma) = \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta)\boldsymbol{R}_Z(\gamma). \tag{1}$$

The angular velocity $\boldsymbol{\omega}$ due to $\dot{\boldsymbol{\phi}}$ can be obtained as the sum of the three angular velocities contributed by, respectively, $\dot{\alpha}$ (along the unit vector $\boldsymbol{X}$), $\dot{\beta}$ (along the rotated $\boldsymbol{Y}'$), and $\dot{\gamma}$ (along the doubly rotated $\boldsymbol{Z}''$), or

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\alpha}} + \boldsymbol{\omega}_{\dot{\beta}} + \boldsymbol{\omega}_{\dot{\gamma}} = \boldsymbol{X}\dot{\alpha} + \boldsymbol{Y}'\dot{\beta} + \boldsymbol{Z}''\dot{\gamma}$$

where the unit vectors $\boldsymbol{X}$, $\boldsymbol{Y}'$ e $\boldsymbol{Z}''$ are expressed with respect to the initial reference frame. It is

$$\boldsymbol{X} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{Y}' = \boldsymbol{R}_X(\alpha) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{Z}'' = \boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, it is sufficient to compute

$$\boldsymbol{R}_X(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad \boldsymbol{R}_Y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix},$$

and

$$\boldsymbol{R}_X(\alpha)\boldsymbol{R}_Y(\beta) = \begin{pmatrix} * & * & \sin\beta \\ * & * & -\sin\alpha\cos\beta \\ * & * & \cos\alpha\cos\beta \end{pmatrix}$$

in order to obtain

$$\boldsymbol{\omega} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{\alpha} + \begin{pmatrix} 0 \\ \cos\alpha \\ \sin\alpha \end{pmatrix} \dot{\beta} + \begin{pmatrix} \sin\beta \\ -\sin\alpha\cos\beta \\ \cos\alpha\cos\beta \end{pmatrix} \dot{\gamma} = \begin{pmatrix} 1 & 0 & \sin\beta \\ 0 & \cos\alpha & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha\cos\beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.$$

Note also that, as a general property, matrix $\boldsymbol{T}$ depends only on the first two Euler angles.

b) Matrix $\boldsymbol{T}$ is singular when

$$\det \boldsymbol{T} = \cos\beta = 0 \quad \Longleftrightarrow \quad \beta = \pm\frac{\pi}{2}.$$

In this condition, an angular velocity vector (with norm k) of the form

$$\boldsymbol{\omega} = k \begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix} \notin \mathcal{R}\left\{ \boldsymbol{T}(\alpha, \pm\frac{\pi}{2}) \right\}$$

cannot be represented by any choice of $\dot{\boldsymbol{\phi}}$.

c) We need first to determine the complete expression of the rotation matrix in (1). For this, we need also the third elementary rotation matrix

$$
\boldsymbol{R}_Z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Performing the matrix multiplication in (1), we obtain 9 scalar nonlinear equations in matrix form:

$$
\begin{aligned}
\boldsymbol{R}_{XYZ}(\alpha,\beta,\gamma) &= \begin{pmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{pmatrix} \\
&= \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}.
\end{aligned}
$$

Note that only the elements of the first row and last column will be used for the solution. We have

$$
\beta = \mathrm{ATAN2}\left\{ r_{13},\ \pm\sqrt{r_{11}^2 + r_{12}^2} \right\},
$$

and, assuming $\cos\beta \neq 0$,

$$
\alpha = \mathrm{ATAN2}\left\{ -\frac{r_{23}}{\cos\beta}, \frac{r_{33}}{\cos\beta} \right\}, \qquad \gamma = \mathrm{ATAN2}\left\{ -\frac{r_{12}}{\cos\beta}, \frac{r_{11}}{\cos\beta} \right\}.
$$

For completeness, we report also the expression of the Euler XYZ rotation matrix in the singular case (this refers to T, since a rotation matrix can never be singular!), which occurs if and only if $r_{11}^2 + r_{12}^2 = 0$, i.e., $\cos\beta = 0$. Then, either we have $\beta = \pi/2$ and

$$
\boldsymbol{R}_{XYZ}(\alpha,\pi/2,\gamma) = \begin{pmatrix} 0 & 0 & 1 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \end{pmatrix},
$$

or $\beta = -\pi/2$ and

$$
\boldsymbol{R}_{XYZ}(\alpha,-\pi/2,\gamma) = \begin{pmatrix} 0 & 0 & -1 \\ \sin(\gamma-\alpha) & \cos(\gamma-\alpha) & 0 \\ \cos(\gamma-\alpha) & -\sin(\gamma-\alpha) & 0 \end{pmatrix}.
$$

In these cases, we cannot find two distinct solutions to the inverse representation problem. Rather can solve only for the sum $(\alpha+\gamma)$ or, respectively, for the difference $(\gamma-\alpha)$, obtaining thus an infinite number of solutions.

Exercise 2

a) The desired 6×4 Jacobian matrix is most efficiently computed as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{p}_{04}}{\partial q_1} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_2} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_3} & \frac{\partial \boldsymbol{p}_{04}}{\partial q_4} \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} & \boldsymbol{z}_3 \end{pmatrix},$$

namely:

- for the first three rows (linear components), by analytic derivation of the position vector $\boldsymbol{p}_{04}$ of the origin of the end-effector frame (# 4);
- for the last three rows (angular components), by the standard geometric expressions for revolute or prismatic joints.

Using the DH table in Fig. 1, we easily obtain from the products with the homogeneous transformation matrices

$$\begin{aligned}
\boldsymbol{p}_{04,\text{hom}}(\boldsymbol{q}) &= \boldsymbol{A}_1(q_1)\left(\boldsymbol{A}_2(q_2)\left(\boldsymbol{A}_3(q_3)\left(\boldsymbol{A}_4(q_4)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right)\right) \\
&= \begin{pmatrix} a_1 \cos q_1 + a_2 \cos(q_1+q_2) \\ a_1 \sin q_1 + a_2 \sin(q_1+q_2) \\ -q_3 \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_{04}(\boldsymbol{q}) \\ 1 \end{pmatrix}.
\end{aligned}$$

In the first row above, brackets have been used to indicate the most convenient order of products, especially for symbolic computations.

The sub-matrix $\boldsymbol{J}_L(\boldsymbol{q})$ is given then by

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_{04}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

where the usual compact notations have been used. Since all joint axes are parallel, for $\boldsymbol{J}_A(\boldsymbol{q})$ we have just to take into account the actual directions of the $\boldsymbol{z}_i$ axes (pointing upwards or downwards). It is

$$\boldsymbol{z}_0 = \boldsymbol{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{z}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Summarizing, the geometric Jacobian matrix is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}. \tag{2}$$

The fourth and fifth rows are structurally zero, since the end-effector of a SCARA robot cannot rotate out of the horizontal plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$. Thus, in order to be feasible, any desired $\boldsymbol{\omega}_d$ should have $\omega_{d,x} = \omega_{d,y} = 0$.

b) When the arm is fully stretched along the $\boldsymbol{x}_0$ axis, it is $q_1 = q_2 = 0$ (the values of q_3 and q_4 are irrelevant). Therefore, we need to solve the following set of linear equations (the link lengths a_1 and a_2 are left parametric):

$$\boldsymbol{J}\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{v}_d \\ \boldsymbol{\omega}_d \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_1 + a_2 & a_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Eliminating the first, fourth, and fifth equations (which are identities of the form $0 = 0$), we are left with three linear independent equations in four unknowns. Therefore, the solutions are infinite in number (more precisely, ∞^1). One possible solution is the joint velocity $\dot{\boldsymbol{q}}$ with components

$$\begin{aligned} \dot{q}_1 &= \frac{a_1 + a_2}{(a_1 + a_2)^2 + a_2^2}, \\ \dot{q}_2 &= \frac{a_2}{(a_1 + a_2)^2 + a_2^2}, \\ \dot{q}_3 &= -1, \\ \dot{q}_4 &= \dot{q}_1 + \dot{q}_2 - 1 = \frac{a_1 + 2a_2}{(a_1 + a_2)^2 + a_2^2} - 1, \end{aligned}$$

which minimizes in norm the velocity of the first two joints. Another solution $\dot{\boldsymbol{q}}$ is given by

$$\begin{aligned} \dot{q}_1 &= \frac{a_2 + 2a_1 - a_1 a_2}{2(a_1^2 + a_1 a_2 + a_2^2)}, \\ \dot{q}_2 &= \frac{a_2 - a_1 + a_1^2 + a_1 a_2}{2(a_1^2 + a_1 a_2 + a_2^2)}, \\ \dot{q}_3 &= -1, \\ \dot{q}_4 &= \frac{a_1^2 + a_1 + 2a_2}{2(a_1^2 + a_1 a_2 + a_2^2)} - 1, \end{aligned}$$

which is the joint velocity of minimum norm (considering all four joints) among all possible solutions.

c) From the previous arguments, it can be concluded that a generic desired generalized velocity, even when it has $\omega_{d,x} = \omega_{d,y} = 0$ may not solve the problem unless it has also $v_{d,x} = 0$. The first two joints of the robot are those responsible for the linear velocity of the end-effector in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$. This robot substructure is associated to the upper left 2×2 block in the Jacobian $\boldsymbol{J}$ in (2), which is singular in the considered configuration. However, since the desired planar velocity $(v_{d,x}\ v_{d,y})^T = (0\ 1)^T$ still lies in the range space of this submatrix, then the problem is solvable (as a matter of fact, it has an infinity of solutions). Otherwise, the problem would have no solution.

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