# Robotics I

**January 9, 2013**

## Exercise 1

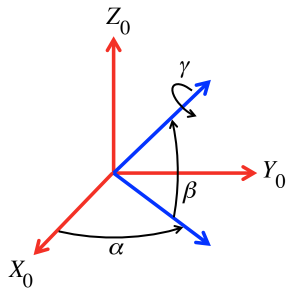


Figure 1: Definition of the three angles $\alpha$, $\beta$, and $\gamma$

Consider the orientation obtained through the sequence of three rotations specified by the angles $\alpha$, $\beta$, and $\gamma$ in Fig. 1. Pay attention to the definition of positive rotations —the figure shows a situation in which $\alpha$ and $\beta$ have some positive values in $(0, \pi/2)$.

**a)** Determine the associated rotation matrix $\boldsymbol{R}(\alpha, \beta, \gamma)$ (*direct problem*).

**b)** When the orientation is expressed by a rotation matrix $\boldsymbol{R}$, find the closed-form expressions for the minimal representation of orientation using the above set of angles $\alpha$, $\beta$, and $\gamma$ (*inverse problem*). Characterize the cases when two solutions or an infinite number of solutions exist.

**c)** Obtain the mapping between the time derivatives of the three angles in this minimal representation and the angular velocity vector, i.e.,

$$\boldsymbol{\omega} = \boldsymbol{T}(\alpha, \beta) \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix},$$

and find the singularities of the matrix $\boldsymbol{T}$. In one of these singularities, provide two numerical examples in which a desired $\boldsymbol{\omega}$ can or, respectively, cannot be realized.

## Exercise 2

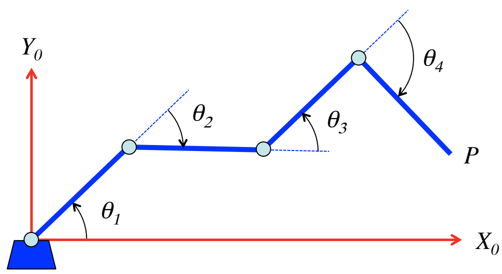


Figure 2: A planar $4R$ robot arm with unitary link lengths

A $4R$ planar robot with links of unitary length is shown in Fig. 2.

**a)** Provide the Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ relating the joint velocity $\dot{\boldsymbol{\theta}} = \begin{pmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_4 \end{pmatrix}^T \in \mathbb{R}^4$ to the linear velocity $\boldsymbol{v} = \begin{pmatrix} v_x & v_y \end{pmatrix}^T \in \mathbb{R}^2$ of the robot end-effector.

**b)** Find all singular configurations of this Jacobian.

**c)** In the configuration $\boldsymbol{\theta} = \begin{pmatrix} 0 & 0 & -\pi/4 & \pi/2 \end{pmatrix}^T$, determine the joint velocity $\dot{\boldsymbol{\theta}}$ of *minimum norm* that realizes the desired end-effector velocity $\boldsymbol{v} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$.

**Exercise 3**

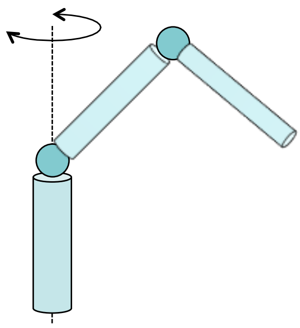

Figure 3: An anthropomorphic $3R$ robot arm

The $3R$ anthropomorphic robot in Fig. 3 is equipped with encoders at the joints for position sensing. Suppose that a small error $\Delta\theta_1$ affects the position measurement of the encoder at the first joint. What is the *maximum* norm of the error $\boldsymbol{\Delta p}$ over the *whole* workspace when estimating the end-effector position $\boldsymbol{p}$ using the encoder readings? Provide a complete explanation of your answer.

**[180 minutes; open books]**

# Solutions

January 9, 2013

**Exercise 1**

The first rotation is around axis $Z = Z_0$ (by an angle $\alpha$), and the following ones are around the moving axes $Y' = -Y_1$ (by $\beta$) and $X'' = X_2$ (by $\gamma$). The only caution is in the definition of the second angle $\beta$, which is positive counterclockwise around $-Y_1$. Thus, the associated rotation matrices are

$$
\begin{aligned}
\boldsymbol{R}_1(\alpha) &= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\boldsymbol{R}_2(\beta) &= \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \qquad \text{(note the \textit{opposite} signs of the sin terms)} \\
\boldsymbol{R}_3(\gamma) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}.
\end{aligned}
$$

The rotation matrix for the direct problem is computed as

$$
\begin{aligned}
\boldsymbol{R}(\alpha,\beta,\gamma) &= \boldsymbol{R}_1(\alpha)\boldsymbol{R}_2(\beta)\boldsymbol{R}_3(\gamma) \\
&= \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ \sin\alpha\cos\beta & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\alpha\sin\gamma - \sin\alpha\sin\beta\cos\gamma \\ \sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}. \qquad (1)
\end{aligned}
$$

For the inverse problem, let $\boldsymbol{R} = \{R_{ij}\}$. From the expressions of the elements in the last row of $\boldsymbol{R}(\alpha,\beta,\gamma)$, one has

$$
\beta = \mathrm{ATAN2}\left\{R_{31}, \pm\sqrt{R_{32}^2 + R_{33}^2}\right\}. \qquad (2)
$$

When $R_{32}^2 + R_{33}^2 \neq 0$ (or, $\cos^2\beta \neq 0$), for each of the two values of $\beta$ obtained from eq. (2) we have an associated solution

$$
\alpha = \mathrm{ATAN2}\left\{\frac{R_{21}}{\cos\beta}, \frac{R_{11}}{\cos\beta}\right\}, \qquad \gamma = \mathrm{ATAN2}\left\{\frac{R_{32}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\}.
$$

Instead, when $R_{32} = R_{33} = R_{11} = R_{21} = 0$ (or, $\cos\beta = 0$), it is $\sin\beta = \pm 1$ and thus

$$
\boldsymbol{R}(\alpha,\beta,\gamma)|_{\beta=\pm\frac{\pi}{2}} = \begin{pmatrix} 0 & -\sin(\alpha\pm\gamma) & \mp\cos(\alpha\pm\gamma) \\ 0 & \cos(\alpha\pm\gamma) & \mp\sin(\alpha\pm\gamma) \\ \pm 1 & 0 & 0 \end{pmatrix}.
$$

Therefore, we can only determine the *sum* or, respectively, the *difference* of the two angles $\alpha$ and $\gamma$, leading to an infinite number of inverse solutions. If $R_{33} = 1$, we have

$$
\beta = \frac{\pi}{2}, \qquad \alpha + \gamma = \mathrm{ATAN2}\{-R_{12}, R_{22}\}.
$$

If $R_{33} = -1$, we have

$$
\beta = -\frac{\pi}{2}, \qquad \alpha - \gamma = \mathrm{ATAN2}\{-R_{12}, R_{22}\}.
$$

Finally, the contributions of the time derivatives $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ to $\boldsymbol{\omega}$ are computed by evaluating the directions of the rotation axes $Z_0$, $-Y_1$ and $X_2$ in the original reference frame[1]:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\alpha}} + \boldsymbol{\omega}_{\dot{\beta}} + \boldsymbol{\omega}_{\dot{\gamma}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\alpha} + \begin{pmatrix} \sin\alpha \\ -\cos\alpha \\ 0 \end{pmatrix} \dot{\beta} + \begin{pmatrix} \cos\alpha\cos\beta \\ \sin\alpha\cos\beta \\ \sin\beta \end{pmatrix} \dot{\gamma} = \boldsymbol{T}(\alpha,\beta) \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}.$$

We have a singularity when $\det \boldsymbol{T}(\alpha,\beta) = \cos\beta = 0$, or $\beta = \pm\pi/2$. For instance, when $\beta = \pi/2$, the angular velocity

$$\omega = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathcal{R}\left\{\boldsymbol{T}(\alpha, \pi/2)\right\}$$

can be realized in infinite ways by choosing

$$\dot{\beta} = 0, \quad \dot{\alpha} + \dot{\gamma} = 1.$$

On the other hand, the angular velocity $\boldsymbol{\omega} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \end{pmatrix}^T$ will certainly not belong to the range of $\boldsymbol{T}$ at the current $\alpha$, and therefore cannot be realized by any choice of $\begin{pmatrix} \dot{\alpha} & \dot{\beta} & \dot{\gamma} \end{pmatrix}^T$ —such situations are always present when using any minimal representation for the orientation.

**Exercise 2**

The position $\boldsymbol{p}$ of the robot end-effector is

$$\boldsymbol{p} = \begin{pmatrix} \cos\theta_1 + \cos(\theta_1+\theta_2) + \cos(\theta_1+\theta_2+\theta_3) + \cos(\theta_1+\theta_2+\theta_3+\theta_4) \\ \sin\theta_1 + \sin(\theta_1+\theta_2) + \sin(\theta_1+\theta_2+\theta_3) + \sin(\theta_1+\theta_2+\theta_3+\theta_4) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{\theta}).$$

Therefore, its velocity $\boldsymbol{v}$ is obtained as

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} = \boldsymbol{J}(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}},$$

where the Jacobian matrix is

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} -(s_1+s_{12}+s_{123}+s_{1234}) & -(s_{12}+s_{123}+s_{1234}) & -(s_{123}+s_{1234}) & -s_{1234} \\ c_1+c_{12}+c_{123}+c_{1234} & c_{12}+c_{123}+c_{1234} & c_{123}+c_{1234} & c_{1234} \end{pmatrix}$$

and we used the compact notation $c_{ijk} = \cos(\theta_i+\theta_j+\theta_k)$ and similar.

Note that this matrix can be conveniently rewritten as

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} -s_1 & -s_{12} & -s_{123} & -s_{1234} \\ c_1 & c_{12} & c_{123} & c_{1234} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \boldsymbol{J}'(\boldsymbol{\theta})\,\boldsymbol{T}.$$

Being $\boldsymbol{T}$ nonsingular, the analysis of the rank deficiencies of $\boldsymbol{J}$ can be performed on the simplified matrix $\boldsymbol{J}'$. We have a singularity at every configuration where the six $(2\times 2)$ minors of $\boldsymbol{J}'$ vanish simultaneously. It is easy to see that this occurs if and only if

$$\sin\theta_2 = \sin\theta_3 = \sin\theta_4 = 0,$$

[1] The axis $-Y_1$ is obtained as $\boldsymbol{R}_1(\alpha)\cdot\begin{pmatrix} 0 & -1 & 0 \end{pmatrix}^T$; the axis $X_2$ is obtained as $\boldsymbol{R}_1(\alpha)\boldsymbol{R}_2(\beta)\cdot\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$. An alternative (but longer) procedure would be to extract $\boldsymbol{\omega}$ from the relation $\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}\boldsymbol{R}^T$, with $\boldsymbol{R} = \boldsymbol{R}(\alpha,\beta,\gamma)$ given by eq. (1).

namely when all the links are folded or stretched along a common radial line originating at the robot base.

The configuration $\boldsymbol{\theta} = \begin{pmatrix} 0 & 0 & -\pi/4 & \pi/2 \end{pmatrix}^T$ is a regular one, and thus any Cartesian velocity $\boldsymbol{v}$ of the end-effector can be realized (by $\infty^2$ different joint velocity vectors $\dot{\boldsymbol{\theta}}$). The minimum norm solution is found when using the pseudoinverse of $\boldsymbol{J}$, namely

$$\begin{aligned}\dot{\boldsymbol{\theta}}^* &= \boldsymbol{J}^{\#}(\boldsymbol{\theta})\,\boldsymbol{v} = \boldsymbol{J}^T(\boldsymbol{\theta})\left(\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{J}^T(\boldsymbol{\theta})\right)^{-1}\boldsymbol{v}\\ &= \begin{pmatrix} 0 & 0 & 0 & -0.7071 \\ 3.4142 & 2.4142 & 1.4142 & 0.7071 \end{pmatrix}^{\#}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.1752 \\ 0.1239 \\ 0.0726 \\ -1.4142 \end{pmatrix} \text{ [rad/s]},\end{aligned}$$

with $\left\|\dot{\boldsymbol{\theta}}^*\right\| = 0.2265$.

**Exercise 3**

The presence of a small measurement error at joint 1 affects the computation of the nominal position of the robot end-effector through the direct kinematics. This error can be seen as a displacement of the end-effector position resulting from a small angular variation of the joint. As such, the Cartesian error for sufficiently small variations can be estimated by using differential arguments, i.e., through the robot Jacobian. In the considered case, we only need to evaluate the first column of the geometric Jacobian related to the linear velocity, i.e.,

$$\boldsymbol{\Delta p} = \begin{pmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \end{pmatrix} = \boldsymbol{J}_{L1}(\boldsymbol{q})\,\Delta\theta_1 = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p}_{0,e} \end{pmatrix}\Delta\theta_1.$$

Computations are simplified when expressing all vectors in the Denavit-Hartenberg frame $RF_1$ that has the $\boldsymbol{x}_1$ and $\boldsymbol{y}_1$ axes in the plane of motion of links 2 and 3. Moreover, what really matters is the distance of the end-effector from the axis $\boldsymbol{z}_0$ of joint 1 (i.e., the component of $\boldsymbol{p}_{0,e}$ along the $\boldsymbol{x}_1$ direction). Therefore

$$\|\boldsymbol{\Delta p}\| = \left\|{}^1\boldsymbol{\Delta p}\right\| = \left\|{}^1\boldsymbol{z}_0 \times {}^1\boldsymbol{p}_{0,e}\right\| \cdot |\Delta\theta_1| = \left\|\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3) \\ * \\ 0 \end{pmatrix}\right\| \cdot |\Delta\theta_1|\,,$$

where $a_2$ and $a_3$ are DH parameters (the length of links 2 and 3) and $*$ denotes an irrelevant quantity. As a result, the maximum norm of the position error over the whole workspace is

$$\max\|\boldsymbol{\Delta p}\| = \max_{(\theta_2,\theta_3)} |a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)| \cdot |\Delta\theta_1| = (a_2+a_3)\cdot|\Delta\theta_1|\,.$$

As intuition suggests, the maximum error is obtained when the robot is stretched horizontally, with its end-effector at the limit of the robot workspace.

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