Robotics I

February 25, 2011

Consider a 3R anthropomorphic robot whose direct kinematics is given by

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \cos q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2 + q_3)\right) \\ \sin q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2 + q_3)\right) \\ d_1 + a_2 \sin q_2 + a_3 \sin(q_2 + q_3) \end{pmatrix}, \tag{1}$$

where

$$d_1 = 0.5, \qquad a_2 = 1.5, \quad a_3 = 1. \tag{2}$$

Let the following two Cartesian positions be assigned for the robot end-effector:

$$\boldsymbol{p}_1 = \begin{pmatrix} 0 \\ 1.0607 \\ 2.5607 \end{pmatrix}, \qquad \boldsymbol{p}_2 = \begin{pmatrix} 1.5309 \\ 1.5309 \\ 0.25 \end{pmatrix}. \tag{3}$$

a) Design a infinite cyclic trajectory of period $T = 2$ s in the *joint space* such that:

- the robot end-effector is in $\boldsymbol{p}_1$ at $t = 0, T, 2T, \ldots$ with zero velocity, and in $\boldsymbol{p}_2$ at $t = T/2, 3T/2, 5T/2, \ldots$ with zero velocity;
- the trajectory is guaranteed to be *smooth* (i.e., continuously differentiable up to any order) at all times.

  Hint: It is sufficient to define the trajectory only in the first period $[0, T]$.

b) How do we check that the designed trajectory remains always in the robot workspace? Are there problems associated with the possible crossing of kinematic singularities?

c) Design a cyclic trajectory with duration $T = 2$ s that traces the *same joint path* of the trajectory determined in a) and is such that:

- the robot end-effector starts in $\boldsymbol{p}_1$ at $t = 0$ with initial zero velocity and acceleration, passes in $\boldsymbol{p}_2$ at $t = T/2$ with zero velocity, and returns in $\boldsymbol{p}_1$ at $t = T$ with final zero velocity and acceleration;
- the trajectory is guaranteed to be at least continuous up to the acceleration for $t \in [0, T]$.

[150 minutes; open books]

Solution

February 25, 2011

As a preliminary step, since a joint trajectory solution is sought, we need to associate joint configurations to the end-effector positions $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in (3). For each position, there are four different solutions to the inverse kinematics (see Homework 2, 2010/11, on the Robotics 1 course web site: the 3R robot considered here is a special case, with offset $a_1 = 0$). Using the robot kinematic parameters given in (2), a configuration corresponding to $\boldsymbol{p}_1$ is

$$\boldsymbol{q}_1 = \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \end{pmatrix}^T = \begin{pmatrix} \frac{\pi}{2} & \frac{\pi}{4} & \frac{\pi}{4} \end{pmatrix}^T \approx \begin{pmatrix} 1.5708 & 0.7854 & 0.7854 \end{pmatrix}^T \quad [\text{rad}], \tag{4}$$

and one corresponding to $\boldsymbol{p}_2$ is

$$\boldsymbol{q}_2 = \begin{pmatrix} q_{2,1} & q_{2,2} & q_{2,3} \end{pmatrix}^T = \begin{pmatrix} \frac{\pi}{4} & -\frac{\pi}{6} & \frac{\pi}{3} \end{pmatrix}^T \approx \begin{pmatrix} 0.7854 & -0.5236 & 1.0472 \end{pmatrix}^T \quad [\text{rad}]. \tag{5}$$

Note that these two inverse kinematic solutions belong to the same class, namely having the 'elbow down' ($q_3 > 0$). Any other pair of feasible inverse kinematic solutions could have been chosen. The following developments remain the same.

Problem a) requires the use of a smooth and periodic joint trajectory, with period T. Thus, it is convenient to adopt a class of trigonometric paths parametrized by a scalar λ. Assigning a timing law $\lambda = \lambda(t)$ will define the trajectory completely. As suggested in the text, it is sufficient to design a suitable trajectory for the first period $[0, T]$ (and then to repeat it indefinitely). The following derivation is useful also for problem c).

Let $\lambda(t)$ be a monotonically increasing function of time, satisfying the boundary conditions

$$\lambda(0) = 0, \qquad \lambda(T) = 1. \tag{6}$$

In view of the zero velocity conditions at $t = 0$ and $t = T$, we choose for each joint i ($i = 1, 2, 3$)

$$q_i(t) = q_{1,i} + A_i\big(1 - \cos\left(2\pi\lambda(t)\right)\big), \qquad \text{with } A_i \neq 0 \text{ yet to be defined.} \tag{7}$$

The trajectory (7) guarantees automatically the passage for $\boldsymbol{p}_1$, i.e.,

$$q_i(0) = q_i(T) = q_{1,i}, \qquad \text{for } i = 1, 2, 3.$$

Its velocity is

$$\dot{q}_i(t) = 2\pi A_i \sin\left(2\pi\lambda(t)\right) \dot{\lambda}(t), \tag{8}$$

implying, for any form of $\lambda(t)$,

$$\dot{q}_i(0) = \dot{q}_i(T) = 0.$$

For later use, the acceleration is

$$\ddot{q}_i(t) = 2\pi A_i \sin\left(2\pi\lambda(t)\right) \ddot{\lambda}(t) - 4\pi^2 A_i \cos\left(2\pi\lambda(t)\right) \dot{\lambda}^2(t). \tag{9}$$

For problem a), it is sufficient to use $\lambda(t) = t/T$ (a normalization in time). From (7) and (8)

$$q_i(t) = q_{1,i} + A_i\left(1 - \cos\frac{2\pi t}{T}\right), \qquad \dot{q}_i(t) = \frac{2\pi}{T} A_i \sin\frac{2\pi t}{T},$$

which imply at $t = T/2$

$$q_i\left(\frac{T}{2}\right) = q_{1,i} + 2A_i, \qquad \dot{q}_i\left(\frac{T}{2}\right) = 0.$$

Choosing then

$$A_i = \frac{q_{2,i} - q_{1,i}}{2}, \qquad \text{for } i = 1, 2, 3,$$

guarantees automatically the passage for $\boldsymbol{p}_2$ at $t = T/2$ with zero velocity.

However, note that the associated acceleration is from (9)

$$\ddot{q}_i(t) = \left(\frac{2\pi}{T}\right)^2 A_i \cos\frac{2\pi t}{T},$$

and thus

$$\ddot{q}_i(0) = \ddot{q}_i(T) = \left(\frac{2\pi}{T}\right)^2 A_i \neq 0, \qquad \text{for } i = 1, 2, 3.$$

The acceleration (and, similarly, all time derivatives of higher but even order) will not be zero at the boundaries of each period. This makes the present solution unsuitable for problem c). On the other hand, it is easy to see that continuity of all time derivatives up to any order (i.e., smoothness) will hold at any time, even in the passage from one period to the other (i.e., at $t = kT$, for $k = 1, 2, \ldots$). We note that this would not have been the case if using two cubic polynomials (in time) to address the same problem, one from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$ in time $T/2$ and another for the symmetric reverse motion. Such an approach would be sufficient for satisfying the boundary conditions on the specified motion, but would fail to guarantee continuity of jerk when reversing the motion in $\boldsymbol{q}_2$. Similarly, using a single higher-order polynomial (from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$ and back to $\boldsymbol{q}_1$) would lead to a discontinuity in some derivative at $t = kT$.

For $T = 2$ s, Figures 1–3 show the position, velocity, acceleration, jerk (third derivative), and snap (fourth derivative) of the designed joint trajectory. Figure 4 shows the resulting robot end-effector path $\boldsymbol{p}(\lambda)$, obtained from eq. (1) evaluated along the designed trajectory $\boldsymbol{q}(t)$ for $t \in [0, T]$. Two different 3D-views are shown. Note that the path is traced twice (forth and back) from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ (in $T/2 = 1$ s) and vice versa (in $T/2 = 1$ s).

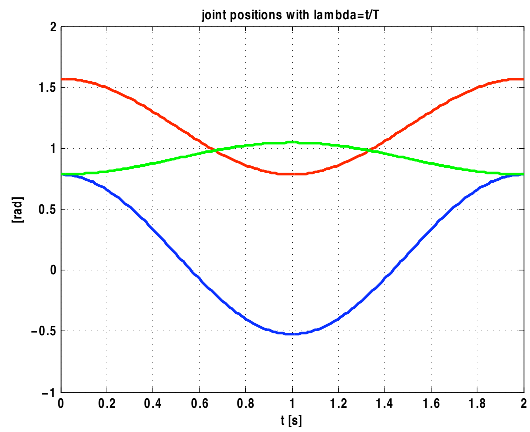

Figure 1: Solution trajectory for problem a) — joint 1 (red), joint 2 (blue), joint 3 (green)

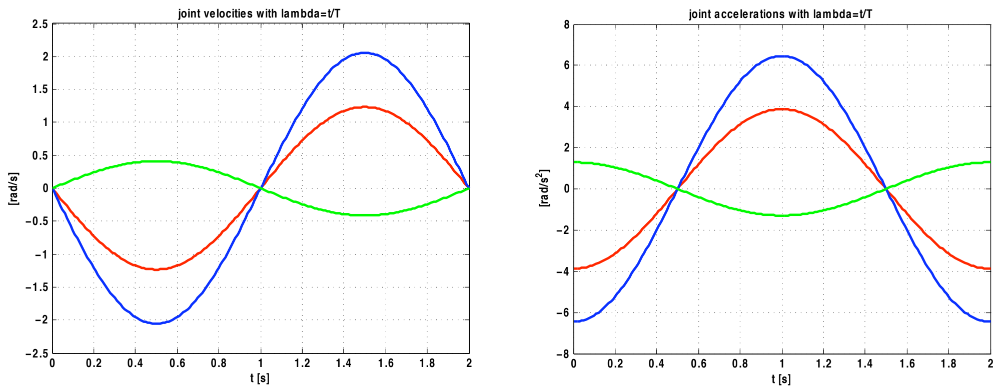

Figure 2: Velocity and acceleration of the trajectory for problem a)

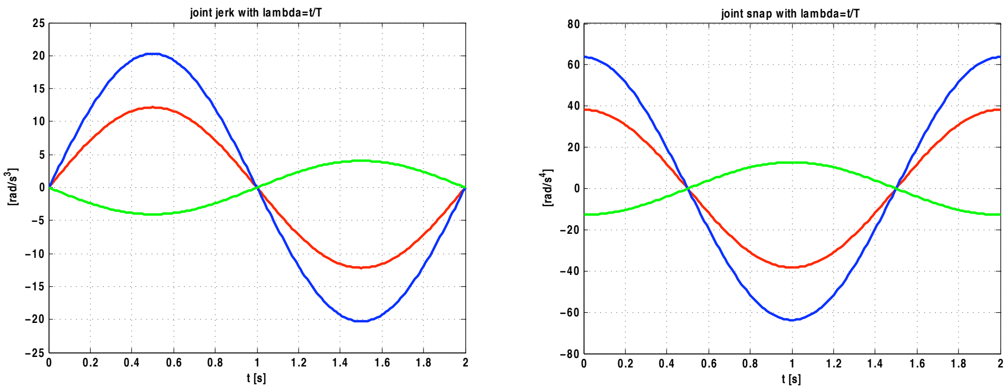

Figure 3: Jerk and snap of the trajectory for problem a)

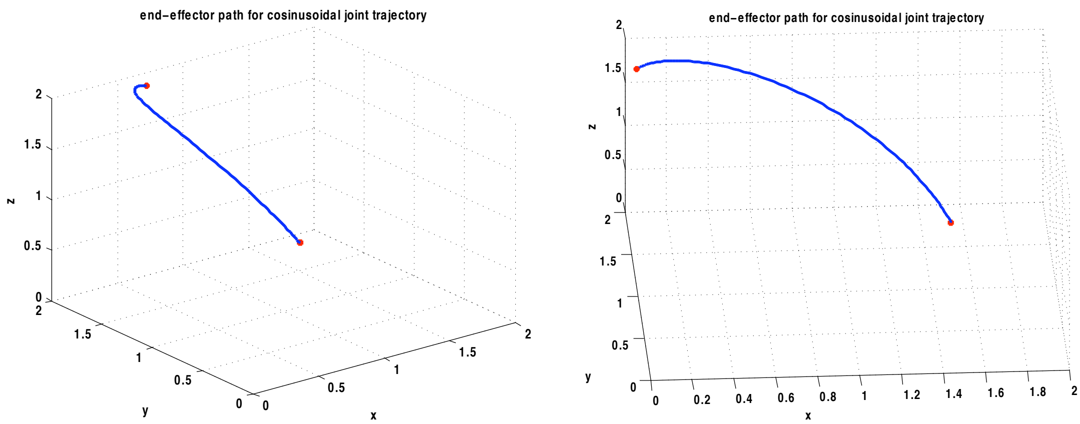

Figure 4: Two 3D-views of the end-effector path with the joint trajectory of problem a) — the red point on the left/top is $\boldsymbol{p}_1$, the other is $\boldsymbol{p}_2$

The two questions of problem b) have both a trivial answer. Since trajectories are being defined at the joint level, they will always be feasible and the end-effector cannot leave the workspace at any time. Moreover, singularity crossing is not an issue: apart from statically transforming the two Cartesian positions $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in the joint space, inversion of a Cartesian trajectory, and thus of the robot Jacobian, is not needed neither for planning nor for control purposes. As a matter of fact, the designed joint trajectory does not cross any singularity of the considered manipulator, thanks to the choice (4–5) for the inverse kinematics. A different choice, e.g., between the elbow down configuration in $\boldsymbol{p}_1$ as in (4) and the elbow up configuration

$$\boldsymbol{q}_2^{\mathrm{up}} = \begin{pmatrix} \frac{\pi}{4} & 0.2937 & -\frac{\pi}{3} \end{pmatrix}^T \quad [\mathrm{rad}]. \tag{10}$$

locating the end-effector in $\boldsymbol{p}_2$, would have led to crossing twice the singularity $q_3 = 0$, with the third link fully stretched. Still, this would have been irrelevant.

To address problem c), a slightly more general form for the time parameterization $\lambda(t)$ is needed. Taking into account the expression (9) of the joint acceleration, we note that is sufficient to impose for the time derivative of $\lambda(t)$

$$\dot{\lambda}(0) = 0, \qquad \dot{\lambda}(T) = 0 \tag{11}$$

for zeroing also the acceleration in $t = 0$ and $t = T$. In fact, being the sine function equal to zero in these two instants, the value of $\ddot{\lambda}$ will not affect the acceleration. Therefore, the boundary conditions (6) and (11) can be satisfied by the following cubic polynomial (normalized in time)

$$\lambda(t) = -2\left(\frac{t}{T}\right)^3 + 3\left(\frac{t}{T}\right)^2, \qquad \text{for } t \in [0, T], \tag{12}$$

having first two derivatives

$$\dot{\lambda}(t) = \frac{6}{T}\left(\left(\frac{t}{T}\right) - \left(\frac{t}{T}\right)^2\right), \qquad \ddot{\lambda}(t) = \frac{6}{T^2}\left(1 - 2\left(\frac{t}{T}\right)\right). \tag{13}$$

The joint trajectory is then defined by (7) with (12). Accordingly, its velocity and acceleration will be defined by (8) and (9), with (13).

The new joint trajectory is shown in Figs. 5–7, where the time derivatives are reported up to the fourth order (snap). As expected, the starting and ending phases are somewhat slower at the expense of larger velocity peaks with respect to the previous case. For instance, comparing the left sides of Figs. 2 and 6, we see that the velocity of joint 2 (in blue) goes from a peak (in absolute value) slightly larger than 2 rad/s to more than 2.7 rad/s. It is easy to realize that, when this trajectory is repeated for any multiple of the same period T, one obtains again a periodic solution that is smooth everywhere. Therefore, this trajectory is another solution to problem a), with the additional property of having also zero acceleration when the end-effector is in $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. Finally, as a check, Figure 8 confirms that the resulting robot end-effector path $\boldsymbol{p}(\lambda)$ has not changed.

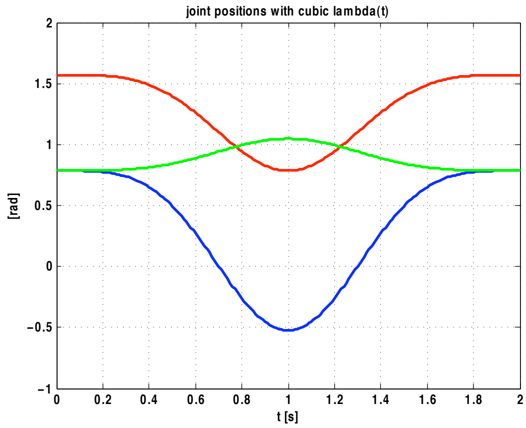

Figure 5: Solution trajectory for problem c) — joint 1 (red), joint 2 (blue), joint 3 (green)

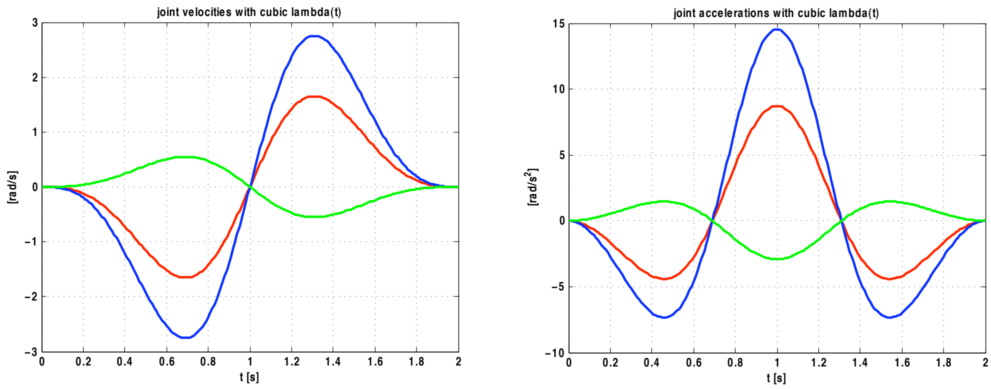

Figure 6: Velocity and acceleration of the trajectory for problem c)

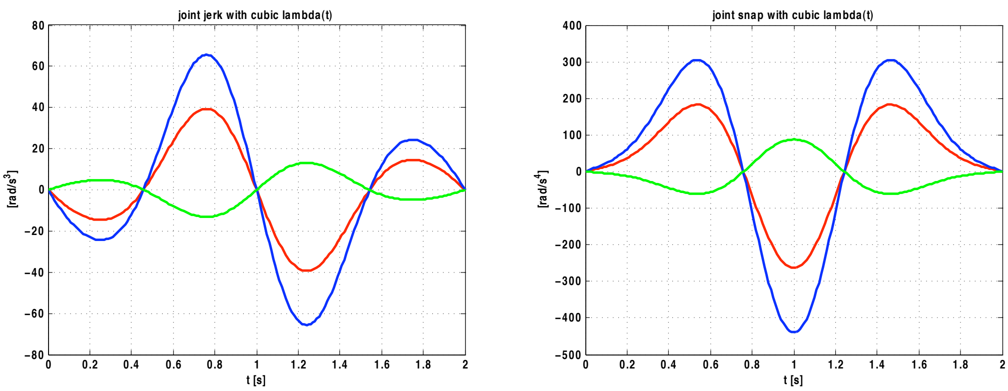

Figure 7: Jerk and snap of the trajectory for problem c)

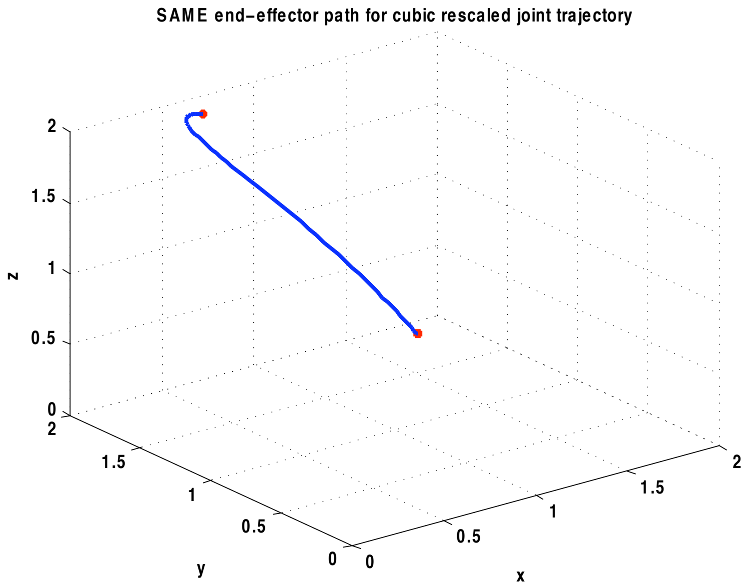

Figure 8: The same end-effector path is obtained with the trajectory of problem c) — the red point on the left/top is $\boldsymbol{p}_1$, the other is $\boldsymbol{p}_2$

* * * * *