Robotics I

September 15, 2010

For a revolute robot joint, consider the rest-to-rest motion $q = q(t)$ defined by the jerk profile $\dddot{q}(t)$ shown in Fig. 1, with given $j_{max} > 0$. The motion starts from $q(0) = q_0$ at time $t = 0$, with zero initial velocity ($\dot{q}(0) = 0$) and zero initial acceleration ($\ddot{q}(0) = 0$).

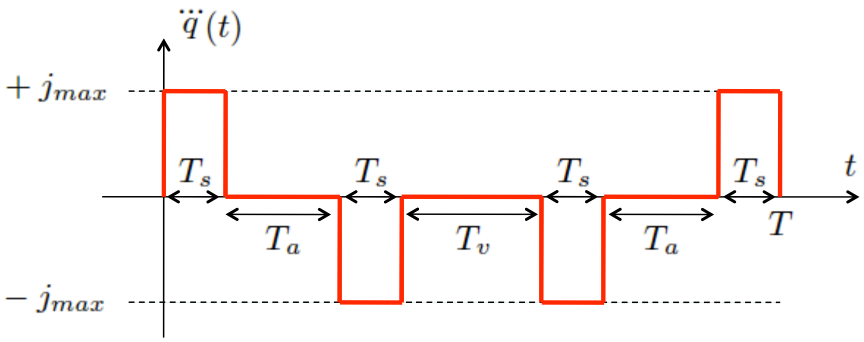

Figure 1: Jerk profile

i) Let the bounds $|\dot{q}(t)| \leq v_{max}$, $|\ddot{q}(t)| \leq a_{max}$ (with $v_{max} > 0$ and $a_{max} > 0$) be assigned, as well as the time interval $T_v \geq 0$. Under the assumption

$$\frac{v_{max}}{a_{max}} - \frac{a_{max}}{j_{max}} \geq 0,$$

determine the analytic expression of the *maximum feasible displacement* $\Delta q = q(T) - q_0$ that can be realized. Provide the numerical solution for

$$j_{max} = 12 \text{ [rad/s}^3\text{]} \qquad a_{max} = 5 \text{ [rad/s}^2\text{]} \qquad v_{max} = 3 \text{ [rad/s}^3\text{]} \qquad T_v = 2 \text{ [s]}.$$

ii) Let the bounds $|\dot{q}(t)| \leq v_{max}$, $|\ddot{q}(t)| \leq a_{max}$ (with $v_{max} > 0$ and $a_{max} > 0$) be assigned, as well as the total displacement $\Delta q > 0$. Under the assumptions

$$\frac{v_{max}}{a_{max}} - \frac{a_{max}}{j_{max}} \geq 0 \qquad \Delta q \geq v_{max} \left(\frac{v_{max}}{a_{max}} + \frac{a_{max}}{j_{max}} \right),$$

determine the analytic expression of the *minimum feasible motion time* T that can be realized. Provide the numerical solution for

$$j_{max} = 10 \text{ [rad/s}^3\text{]} \qquad a_{max} = 4 \text{ [rad/s}^2\text{]} \qquad v_{max} = 2 \text{ [rad/s}^3\text{]} \qquad \Delta q = 3 \text{ [rad]}.$$

[90 minutes; open books]

Solution

September 15, 2010

The solution is obtained by integration of the jerk profile, using the given initial conditions at time $t = 0$ and then the suitable boundary conditions at the instants of jerk switching. In addition, due to the symmetry of the trajectory derivatives with respect to $T/2$, it is sufficient to analyze only the first half of the motion. Without loss of generality, we set $q_0 = 0$ (only the displacement w.r.t. the initial position matters). We will also see that the assumptions made on the velocity, acceleration, and jerk bounds, as well as on the total displacement assigned in problem *ii)*, guarantee that none of the motion segments will vanish.

- *First segment:* $\dddot{q}(t) = j_{max}$, for $t \in [0, T_s)$

$$\begin{aligned}
\ddot{q}(t) &= j_{max}\, t & & & \ddot{q}(T_s) &= j_{max}\, T_s = a_{max} \quad \Rightarrow \quad T_s = \frac{a_{max}}{j_{max}} \\
\dot{q}(t) &= \tfrac{1}{2}\, j_{max}\, t^2 & \Rightarrow & & \dot{q}(T_s) &= \tfrac{1}{2}\, j_{max}\, T_s^2 \\
q(t) &= \tfrac{1}{6}\, j_{max}\, t^3 & & & q(T_s) &= \tfrac{1}{6}\, j_{max}\, T_s^3
\end{aligned}$$

- *Second segment:* $\dddot{q}(t) = 0$, for $t \in [T_s, T_s + T_a)$

$$\begin{aligned}
\ddot{q}(t) &= a_{max} \\
\dot{q}(t) &= \tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\,(t - T_s) \\
q(t) &= \tfrac{1}{6}\, j_{max}\, T_s^3 + \tfrac{1}{2}\, j_{max}\, T_s^2 (t - T_s) + \tfrac{1}{2}\, a_{max}\,(t - T_s)^2
\end{aligned}$$

$$\Rightarrow \quad \begin{aligned}
\ddot{q}(T_s + T_a) &= a_{max} \\
\dot{q}(T_s + T_a) &= \tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a \\
q(T_s + T_a) &= \tfrac{1}{6}\, j_{max}\, T_s^3 + \tfrac{1}{2}\, j_{max}\, T_s^2 T_a + \tfrac{1}{2}\, a_{max}\, T_a^2
\end{aligned}$$

- *Third segment:* $\dddot{q}(t) = -j_{max}$, for $t \in [T_s + T_a, 2T_s + T_a)$

$$\begin{aligned}
\ddot{q}(t) &= a_{max} - j_{max}\,(t - (T_s + T_a)) \\
\dot{q}(t) &= \tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a + a_{max}\,(t - (T_s + T_a)) - \tfrac{1}{2}\, j_{max} (t - (T_s + T_a))^2 \\
q(t) &= \tfrac{1}{6}\, j_{max}\, T_s^3 + \tfrac{1}{2}\, j_{max}\, T_s^2 T_a + \tfrac{1}{2}\, a_{max}\, T_a^2 + (\tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a)(t - (T_s + T_a)) \\
&\quad + \tfrac{1}{2}\, a_{max}\,(t - (T_s + T_a))^2 - \tfrac{1}{6}\, j_{max} (t - (T_s + T_a))^3
\end{aligned}$$

$$\Rightarrow \quad \begin{aligned}
\ddot{q}(2T_s + T_a) &= a_{max} - j_{max}\, T_s = 0 \\
\dot{q}(2T_s + T_a) &= \tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a + a_{max}\, T_s - \tfrac{1}{2}\, j_{max}\, T_s^2 = v_{max} \quad \Rightarrow \quad T_a = \frac{v_{max}}{a_{max}} - \frac{a_{max}}{j_{max}} \\
q(2T_s + T_a) &= \tfrac{1}{6}\, j_{max}\, T_s^3 + \tfrac{1}{2}\, j_{max}\, T_s^2 T_a + \tfrac{1}{2}\, a_{max}\, T_a^2 + (\tfrac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a)\, T_s \\
&\quad + \tfrac{1}{2}\, a_{max}\, T_s^2 - \tfrac{1}{6}\, j_{max}\, T_s^3
\end{aligned}$$

- *First half of fourth segment:* $\dddot{q}(t) = 0$, for $t \in [2T_s + T_a, 2T_s + T_a + T_v/2)$.

$$\begin{aligned}
\ddot{q}(t) &= 0 \\
\dot{q}(t) &= v_{max} \\
q(t) &= q(2T_s + T_a) + v_{max}(t - (2T_s + T_a))
\end{aligned}$$

$$\Rightarrow \quad \begin{aligned}
\ddot{q}\left(2T_s + T_a + \frac{T_v}{2}\right) &= 0 \\
\dot{q}\left(2T_s + T_a + \frac{T_v}{2}\right) &= v_{max} \\
q\left(2T_s + T_a + \frac{T_v}{2}\right) &= q(2T_s + T_a) + v_{max}\left(\frac{T_v}{2}\right)
\end{aligned}$$

Since we have that $\frac{T}{2} = 2T_s + T_a + \frac{T_v}{2}$, due to the symmetry of the trajectory, we have

$$q\left(\frac{T}{2}\right) = \frac{\Delta q}{2},$$

or

$$\frac{1}{2}\, j_{max}\, T_s^2 T_a + \frac{1}{2}\, a_{max}\, T_a^2 + (\frac{1}{2}\, j_{max}\, T_s^2 + a_{max}\, T_a)\, T_s + \frac{1}{2}\, a_{max}\, T_s^2 + v_{max}\left(\frac{T_v}{2}\right) = \frac{\Delta q}{2}.$$

Substituting the expressions of T_s and T_a and simplifying, we obtain finally

$$T_v = \frac{\Delta q}{v_{max}} - \frac{v_{max}}{a_{max}} - \frac{a_{max}}{j_{max}}. \tag{1}$$

While

$$T_s = \frac{a_{max}}{j_{max}} > 0$$

always hold, we note that the assumptions made on the relative amplitudes of the bounds v_{max}, a_{max}, j_{max} and on Δq simultaneously guarantee that

$$T_a \geq 0, \qquad T_v \geq 0.$$

As a result, the total motion time is given by

$$T = T_v + 2T_a + 4T_s = \frac{\Delta q}{v_{max}} + \frac{v_{max}}{a_{max}} + \frac{a_{max}}{j_{max}}, \tag{2}$$

which is the minimum feasible time under the made assumptions. For specific choices of data, some of the motion segments may collapse, and the actual duration of each of them (and thus the total motion time) should be computed accordingly.

If T_v is assigned, the maximum feasible displacement is obtained from (1) as

$$\Delta q = v_{max}\left(T_v + \frac{v_{max}}{a_{max}} + \frac{a_{max}}{j_{max}}\right). \tag{3}$$

Plugging the data of problem *i)* in eq. (3) yields $\Delta q = 9.05$ [rad] (with a total time $T = 4.033$ [s]). The obtained profiles of position, velocity, acceleration, and jerk are shown in Fig. 2. With the data of problem *ii)*, from eq. (2) we have $T = 2.4$ [s]. The associated profiles of position, velocity, acceleration, and jerk are shown in Fig. 3. Matlab sources are available.

* * * * *

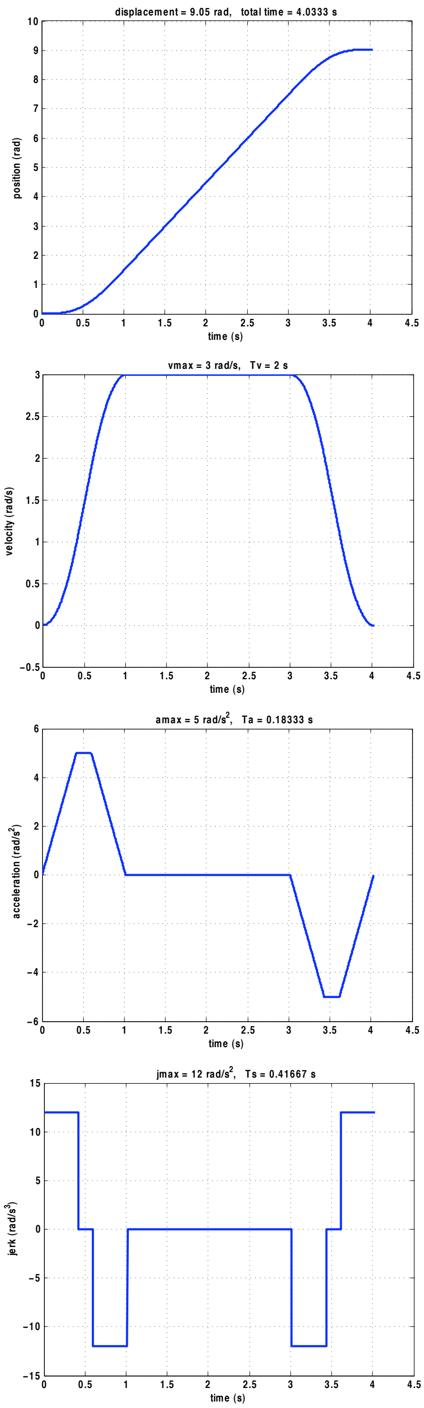

Figure 2: Position, velocity, acceleration, and jerk profiles for the solution to problem *i)*

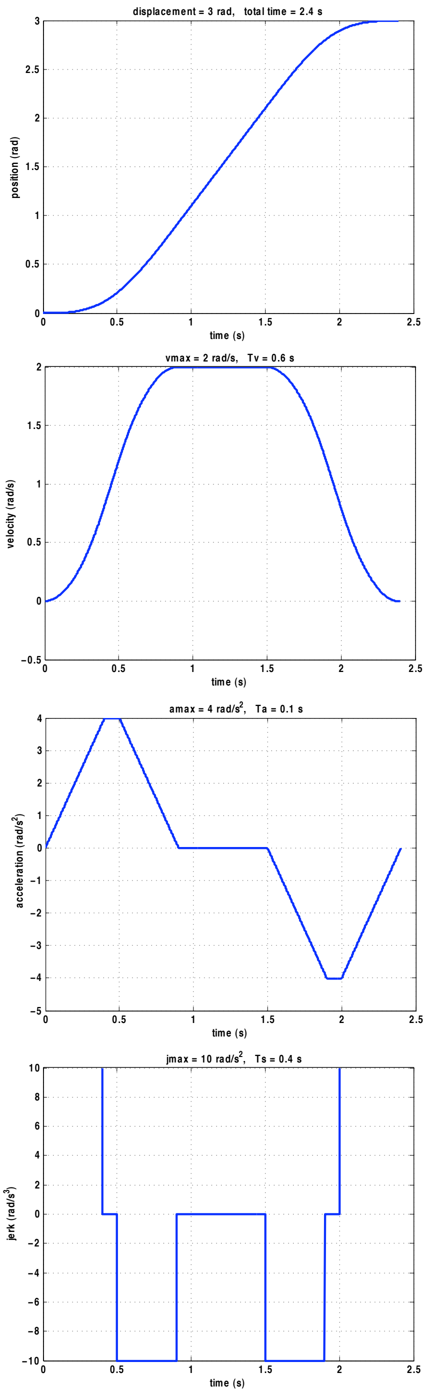

Figure 3: Position, velocity, acceleration, and jerk profiles for the solution to problem *ii)*