Robotics I

June 15, 2010

Exercise 1

For a planar RP robot, consider a class of one-dimensional tasks defined only in terms of the y-component of the end-effector Cartesian position

$$y = p_y(q_1, q_2).$$

a) Study the singularity conditions for the robot performing this class of tasks.

b) Given a desired task trajectory $y_d(t)$, admitting second time derivative, provide the expression of a kinematic control law that is able to zero the task error $e = y_d - y$ in an exponential way starting fron any initial robot condition $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0))$, when the available control commands are the joint accelerations $\ddot{\boldsymbol{q}}$.

Exercise 2

For a minimal representation of the orientation of a rigid body given by Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ around the sequence of mobile axes $YX'Z''$, determine the relation

$$\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$$

between the time derivatives of the Euler angles and the angular velocity $\boldsymbol{\omega}$ of the rigid body. Find the singularities of $\boldsymbol{T}(\boldsymbol{\phi})$, and provide an example of an angular velocity vector $\boldsymbol{\omega}$ that cannot be represented in a singularity.

[90 minutes; open books]

Solutions

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Exercise 1

The direct kinematics associated to the end-effector position of the RP robot is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix},$$

where a 'natural' set of coordinates has been chosen, with q_1 being the angle between the $\boldsymbol{x}_0$ axis and the second link of the robot[1].

Being the task defined only in terms of the p_y component, it is

$$\dot{p}_y = \begin{pmatrix} q_2 \cos q_1 & \sin q_1 \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

and

$$\ddot{p}_y = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \begin{pmatrix} \dot{q}_2 \cos q_1 - q_2 \sin q_1 \dot{q}_1 & \cos q_1 \dot{q}_1 \end{pmatrix} \dot{\boldsymbol{q}}.$$

The task Jacobian $\boldsymbol{J}$ is then singular when

$$\sin q_1 = 0 \quad \text{AND} \quad q_2 = 0.$$

In this case, the rank of the $\boldsymbol{J}$ matrix is zero and the one-dimensional task cannot be correctly performed. Out of singularities, all the joint accelerations $\ddot{\boldsymbol{q}}$ that realize a desired $\ddot{y}_d$ can be written in the form

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q}) \left(\ddot{y}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \right) + \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}) \right) \ddot{\boldsymbol{q}}_0,$$

being the task redundant ($M = 1$) for the RP robot ($N = 2$). Setting $\ddot{\boldsymbol{q}}_0 = \boldsymbol{0}$ one obtains the solution with minimum joint acceleration norm. Assuming full rank (equal to 1) for the task Jacobian $\boldsymbol{J}$, its pseudoinverse has the explicit expression

$$\boldsymbol{J}^{\#}(\boldsymbol{q}) = \frac{1}{q_2^2 \cos^2 q_1 + \sin^2 q_1} \begin{pmatrix} q_2 \cos q_1 \\ \sin q_1 \end{pmatrix}.$$

A kinematic control law with the requested performance is defined by

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q}) \left(\ddot{y}_d + k_d(\dot{y}_d - \dot{p}_y) + k_p(y_d - p_y) - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \right),$$

where $k_d > 0$ and $k_p > 0$ and we set for simplicity $\ddot{\boldsymbol{q}}_0 = \boldsymbol{0}$. A more convenient choice would be to include an acceleration $\ddot{\boldsymbol{q}}_0 = -\boldsymbol{K}_D\dot{\boldsymbol{q}}$, with a diagonal, positive definite matrix $\boldsymbol{K}_D$, in the null space of the task Jacobian. As a matter of fact, such additional term allows to damp possible increases of internal joint velocity without perturbing the task.

[1] When using the Denavit-Hartenberg formalism, one would define $q_2^{\mathrm{DH}} = q_2 \pm \frac{\pi}{2}$. The rest of the developments follows accordingly in a similar way.

Exercise 2

The orientation of a rigid body is represented, using the Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ around the sequence of mobile axes $YX'Z''$, by the product of elementary rotation matrices

$$\boldsymbol{R} = \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_{X'}(\beta)\boldsymbol{R}_{Z''}(\gamma).$$

The angular velocity $\boldsymbol{\omega}$ due to $\dot{\boldsymbol{\phi}}$ can be obtained as the sum of the three angular velocities contributed by, respectively, $\dot{\alpha}$ (along the unit vector $\boldsymbol{Y}$), $\dot{\beta}$ (along $\boldsymbol{X}'$), and $\dot{\gamma}$ (along $\boldsymbol{Z}''$)

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\alpha}} + \boldsymbol{\omega}_{\dot{\beta}} + \boldsymbol{\omega}_{\dot{\gamma}} = \boldsymbol{Y}\dot{\alpha} + \boldsymbol{X}'\dot{\beta} + \boldsymbol{Z}''\dot{\gamma}$$

where the unit vectors $\boldsymbol{Y}$, $\boldsymbol{X}$ e $\boldsymbol{Z}''$ are expressed with respect to the initial reference frame. It is

$$\boldsymbol{Y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{X}' = \boldsymbol{R}_Y(\alpha) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{Z}'' = \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_{X'}(\beta) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, it is sufficient to compute

$$\boldsymbol{R}_Y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \quad \boldsymbol{R}_{X'}(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix},$$

$$\boldsymbol{R}_Y(\alpha)\boldsymbol{R}_{X'}(\beta) = \begin{pmatrix} * & * & \sin\alpha\cos\beta \\ * & * & -\sin\beta \\ * & * & \cos\alpha\cos\beta \end{pmatrix}$$

in order to obtain

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\dot{\alpha} + \begin{pmatrix} \cos\alpha \\ 0 \\ -\sin\alpha \end{pmatrix}\dot{\beta} + \begin{pmatrix} \sin\alpha\cos\beta \\ -\sin\beta \\ \cos\alpha\cos\beta \end{pmatrix}\dot{\gamma} = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha\cos\beta \\ 1 & 0 & -\sin\beta \\ 0 & -\sin\alpha & \cos\alpha\cos\beta \end{pmatrix}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.$$

Note also, as a general property, that matrix $\boldsymbol{T}$ depends only on the first two Euler angles. Matrix $\boldsymbol{T}$ is singular when

$$\det \boldsymbol{T} = -\cos\beta = 0 \quad \Longleftrightarrow \quad \beta = \pm\frac{\pi}{2}.$$

In this condition, an angular velocity vector (with norm k) of the form

$$\boldsymbol{\omega} = k\begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} \notin \mathcal{R}\left\{\boldsymbol{T}(\alpha, \pm\frac{\pi}{2})\right\}$$

cannot be represented by any choice of $\dot{\boldsymbol{\phi}}$.

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