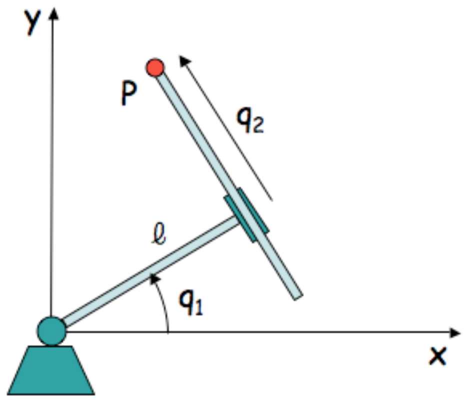
Robotics I

July 10, 2009

Exercise 1



Consider the planar RP robot shown in the figure, where ℓ is the length of the first link and the generalized coordinates to be used are indicated. Let $\boldsymbol{p} = (p_x \; p_y)^T$ be the position of the end-effector P.

- Solve the inverse kinematic problem for this robot, providing the number and type of the solutions for varying positions of P.
- Draw the robot primary workspace (with dimensions) in the case when the joint variables are bounded as: $q_1 \in [-\pi/2, +\pi/2]$, $q_2 \in [-L, +L]$. Discuss the presence of singularities on the boundaries of the workspace.

Exercise 2

Let an initial pose A and a final pose B be given in the robot Cartesian space, with the locations of the associated frame represented by the homogeneous transformation matrices:

$$
{}^0\boldsymbol{T}_A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^0\boldsymbol{T}_B = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

- Plan a *coordinated* motion trajectory from pose A to pose B along a straight path from O_A to O_B in a time $T = 2$ sec, and with zero initial and final linear and angular velocities.
- Provide the numerical value at time $t = T/2$ of the linear velocity (of the origin of moving frame) and of the angular velocity.

[120 minutes; open books]

Solutions

July 10, 2009

Exercise 1

The robot direct kinematics is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \ell \cos q_1 - q_2 \sin q_1 \\ \ell \sin q_1 + q_2 \cos q_1 \end{pmatrix}.$$

Rewriting this in the form

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \boldsymbol{R}(q_1) \begin{pmatrix} \ell \\ q_2 \end{pmatrix},$$

where $\boldsymbol{R}(\theta)$ is the 2×2 planar rotation matrix by an angle θ, it immediately follows that

$$\boldsymbol{p}^T \boldsymbol{p} = p_x^2 + p_y^2 = \begin{pmatrix} \ell & q_2 \end{pmatrix}^T \boldsymbol{R}^T(q_1) \boldsymbol{R}(q_1) \begin{pmatrix} \ell \\ q_2 \end{pmatrix} = \ell^2 + q_2^2,$$

and hence

$$q_2 = \pm \sqrt{p_x^2 + p_y^2 - \ell^2}.$$

Depending on whether $\|\boldsymbol{p}\|$ is larger, equal to, or smaller than ℓ, there will be respectively two, one (singular), or no solutions. In this analysis, no joint limits are taken into account (in particular, the one for the prismatic joint).

Once q_2 is determined, in order to find the analytic expression of the solution for the first joint variable, we can rewrite the direct kinematics as

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} -q_2 & \ell \\ \ell & q_2 \end{pmatrix} \begin{pmatrix} \sin q_1 \\ \cos q_1 \end{pmatrix},$$

where the matrix that appears is always non-singular (with determinant equal to $-(q_2^2 + \ell^2) < 0$). This yields

$$\begin{pmatrix} \sin q_1 \\ \cos q_1 \end{pmatrix} = \frac{1}{q_2^2 + \ell^2} \begin{pmatrix} \ell p_y - q_2 p_x \\ \ell p_x + q_2 p_y \end{pmatrix}.$$

Therefore,

$$q_1 = \text{ATAN2}\{\ell p_y - q_2 p_x, \ell p_x + q_2 p_y\},$$

where, in the regular case, the two solutions found for q_2 have to be replaced. In the singular case, one has only $q_2 = 0$ and thus the single associated solution $q_1 = \text{ATAN2}\{p_y, p_x\}$.

The robot workspace is shown in Figure 1. The case when only the prismatic joint variable is limited ($|q_2| \leq L$) is shown on the left, while the full requested case is given on the right. The radius of the inner and outer circumferences are $r = \ell$ and $R = \sqrt{\ell^2 + L^2}$. Note that on the external boundary of the workspace (arc of the circumference of radius R), as well as on the two straight segments belonging to the boundary, the analytic 2×2 robot Jacobian is non-singular; in fact, these limitations to the workspace are imposed by the joint limits and not by the kinematic configuration of the robot itself. Equivalently, on the parts of the workspace boundary where the Jacobian is full rank the space of admissible Cartesian velocities is still two-dimensional (though with unilaterally constrained along certain directions).

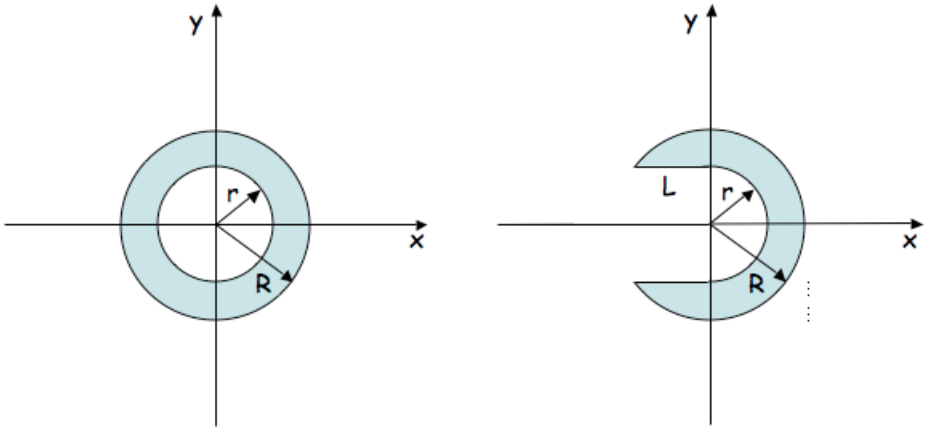


Figure 1: Robot workspace, with $r = \ell$ and $R = \sqrt{\ell^2 + L^2}$; on the left, the case when only the second joint range is limited ($|q_2| \leq L$); on the right, the full case including also $|q_1| \leq \pi/2$

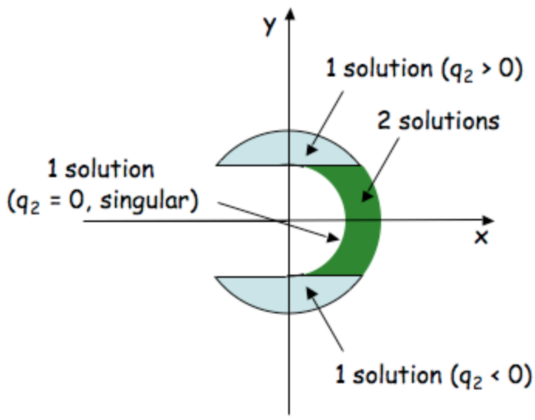


Figure 2: Number of inverse solutions in the various regions of the workspace

Finally, Figure 2 shows the partition of the workspace in terms of number of inverse kinematics solutions when joint limits are present. In particular, two solutions exist in the deep green region, including its two straight boundaries and the arc of the external circumference (on the internal one, there is only one, singular solution).

Exercise 2

The distance between the origins O_A and O_B of the frames associated to the initial and final poses is

$$L = \|{}^0\boldsymbol{p}_{0B} - {}^0\boldsymbol{p}_{0A}\| = \left\| \begin{pmatrix} 1 \\ 0 \\ -0.5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ -1 \\ -1.5 \end{pmatrix} \right\| = \sqrt{3.25}.$$

The linear path for the origin of the motion frame can be parametrized as

$$\boldsymbol{p}(s) = {}^0\boldsymbol{p}_{0A} + \frac{s}{L}\left({}^0\boldsymbol{p}_{0B} - {}^0\boldsymbol{p}_{0A}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{s}{\sqrt{3.25}} \begin{pmatrix} 0 \\ -1 \\ -1.5 \end{pmatrix}, \qquad s \in [0, L].$$

The relative rotation between pose A and pose B is given by

$${}^A\boldsymbol{R}_B = {}^0\boldsymbol{R}_A^T\, {}^0\boldsymbol{R}_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

From this, one can plan a reorientation motion using the axis/angle method. We need to compute the unit vector ${}^A\boldsymbol{r}$ (defined with respect to the initial frame) and thee angle θ_{AB} satisfying $\boldsymbol{R}({}^A\boldsymbol{r}, \theta_{AB}) = {}^A\boldsymbol{R}_B$. Denoting with r_{ij} the elements of the rotation matrix ${}^A\boldsymbol{R}_B$, from the inverse formulas of the axis/angle method we obtain

$$\begin{aligned} \theta_{AB} &= \text{ATAN2}\{+\sqrt{(r_{21}-r_{12})^2 + (r_{13}-r_{31})^2 + (r_{23}-r_{32})^2}, r_{11}+r_{22}+r_{33}-1\} \\ &= \text{ATAN2}\{\sqrt{3}, -1\} = \frac{2}{3}\pi = 2.0944 \text{ rad } (= 120^\circ) \end{aligned}$$

and

$${}^A\boldsymbol{r} = \frac{1}{2\sin\theta_{AB}} \begin{pmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 0.5774 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad (\text{being } \|{}^A\boldsymbol{r}\| = 1).$$

Note that only one of the two possible solutions is used. The orientation path can be (still linearly) parametrized as

$$\theta(s) = \frac{s}{L}\,\theta_{AB}, \qquad s \in [0, L].$$

The absolute orientation for a given value of parameter s will be

$$\boldsymbol{R}(s) = {}^0\boldsymbol{R}_A\, \boldsymbol{R}({}^A\boldsymbol{r}, \theta(s)).$$

Having use the same parameter s for the position and orientation paths, planning a single timing law $s = s(t)$, with $t \in [0, T]$, will automatically yield a coordinated motion: the translation and the rotation between the initial and final poses will be completed simultaneously.

For the timing law, the simplest choice is a cubic (bi-normalized) polynomial with zero time derivative in $t = 0$ and $t = T$. It is

$$s(t) = L\left[-2\left(\frac{t}{T}\right)^3 + 3\left(\frac{t}{T}\right)^2\right], \qquad t \in [0, T].$$

Its time derivative is

$$\dot{s}(t) = \frac{6L}{T}\left[\left(\frac{t}{T}\right) - \left(\frac{t}{T}\right)^2\right],$$

and thus

$$\dot{s}(T/2) = \frac{3L}{2T}.$$

The linear and angular velocity during the transfer motion are

$${}^0\dot{\boldsymbol{p}}(t) = \frac{d\boldsymbol{p}}{ds}\dot{s}(t) = \frac{\dot{s}(t)}{L}\begin{pmatrix} 0 \\ -1 \\ -1.5 \end{pmatrix}$$

and

$${}^A\boldsymbol{\omega}(t) = {}^A\boldsymbol{r}\,\frac{d\theta(s)}{ds}\,\dot{s}(t) = \frac{\dot{s}(t)}{L}\,\theta_{AB}\,{}^A\boldsymbol{r}.$$

At $t = T/2 = 1$ sec, we have

$${}^0\dot{\boldsymbol{p}}(1) = \frac{3}{2}\begin{pmatrix} 0 \\ -1 \\ -1.5 \end{pmatrix}, \qquad {}^A\boldsymbol{\omega}(1) = \frac{3}{2}\cdot 2.0944 \cdot 0.5774 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 1.8138 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ rad/sec},$$

and finally

$${}^0\boldsymbol{\omega}(1) = {}^0\boldsymbol{R}_A\,{}^A\boldsymbol{\omega}(1) = 1.8138\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \text{ rad/sec}.$$

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