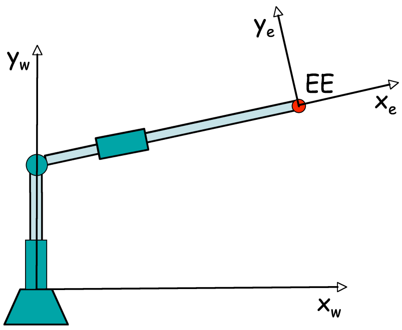
Robotics I

June 10, 2009



Consider the planar PRP robot with $n = 3$ joints in the figure above. The world reference frame $RF_w = (\boldsymbol{x}_w, \boldsymbol{y}_w, \boldsymbol{z}_w)$ and the end-effector frame $RF_e = (\boldsymbol{x}_e, \boldsymbol{y}_e, \boldsymbol{z}_e)$ are also shown.

- Assign the robot reference frames according to the Denavit-Hartenberg (DH) convention and write down the associated table of parameters. Moreover, specify the (constant) transformation matrices ${}^wT_0$, between the world frame and frame 0 of DH, and ${}^3T_e$, between frame 3 of DH and the end-effector frame.

- Based on the variables $\boldsymbol{q}$ defined in the Denavit-Hartenberg convention, compute the analytical Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for a task involving only the end-effector position in the plane of motion, and analyze its singular configurations. With the robot in a singular configuration $\boldsymbol{q}_0$, define a set of base vectors for each of the following four linear subspaces:

$$\mathcal{R}\Big(\boldsymbol{J}(\boldsymbol{q}_0)\Big) \qquad \mathcal{N}\Big(\boldsymbol{J}(\boldsymbol{q}_0)\Big) \qquad \mathcal{R}\left(\boldsymbol{J}^T(\boldsymbol{q}_0)\right) \qquad \mathcal{N}\left(\boldsymbol{J}^T(\boldsymbol{q}_0)\right).$$

- For a motion task of dimension $m = 3$ specified for the robot end-effector, consider the use of a kinematic control law in the task space,

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\left(\dot{\boldsymbol{r}}_d + \boldsymbol{K}_P(\boldsymbol{r}_d - \boldsymbol{f}(\boldsymbol{q})\right), \tag{1}$$

  where $\boldsymbol{r} \in \mathbb{R}^3$ includes the position (in the plane) as well as the orientation of the end-effector (i.e., the angle ϕ between the horizontal axis $\boldsymbol{x}_w$ and the axis $\boldsymbol{x}_e$) and $\boldsymbol{f}(\boldsymbol{q})$ is the direct kinematic function associated to these task variables. Assume that the (positive definite) gain matrix $\boldsymbol{K}_P$ is chosen as diagonal, and let the joint velocities be bounded as $|\dot{q}_i| \leq V_i$, with given values $V_i > 0$ $(i = 1, 2, 3)$.

  a) With the desired task velocity being $\dot{\boldsymbol{r}}_d = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}^T$, determine a joint configuration $\boldsymbol{q}^*$ which is nonsingular for the task (and 'matched' to it, i.e., $\boldsymbol{e} = \boldsymbol{r}_d - \boldsymbol{f}(\boldsymbol{q}^*) = \boldsymbol{0}$ for a suitable $\boldsymbol{r}_d$) and is such that the desired task *can never* be realized without violating one of the bounds on the joint velocities.

  b) Let the robot initial configuration be $\boldsymbol{q}(0) = \begin{pmatrix} 1.2 & \pi/2 & 1 \end{pmatrix}^T$, and let $\boldsymbol{r}_d(0) = \begin{pmatrix} 1.5 & 1.5 & -\pi/4 \end{pmatrix}^T$ and $\dot{\boldsymbol{r}}_d(0) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$ be the specified task at time $t = 0$. Define the numerical values of the diagonal matrix $\boldsymbol{K}_P$ in the control law (1) so that the initial task error $\boldsymbol{e}(0)$ will be reduced as fast as possible *without violating* the following bounds on the joint velocities: $V_1 = 5$ [m/s], $V_2 = \pi$ [rad/s], and $V_3 = 4$ [m/s].

[180 minutes; open books]

Solution

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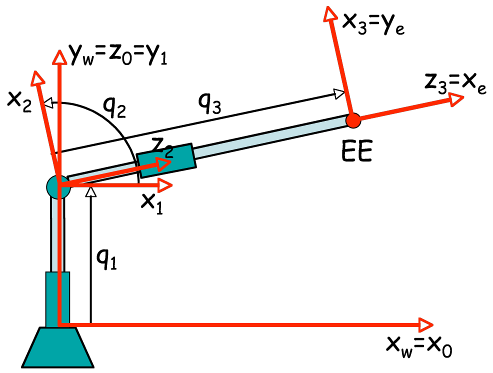
A possible assignment of the Denavit-Hartenberg frames is shown in the figure below, together with the associated table of parameters.



i	α_i	a_i	d_i	θ_i
1	$\frac{\pi}{2}$	0	q_1	0
2	$\frac{\pi}{2}$	0	0	q_2
3	0	0	q_3	0

From this, it is easy to obtain the general expression of the direct kinematics for this robot:

$$
\begin{aligned}
{}^0\boldsymbol{T}_3(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{R}_3(\boldsymbol{q}) & {}^0\boldsymbol{p}_3(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} &= {}^0A_1(q_1)\, {}^1A_2(q_2)\, {}^2A_3(q_3) \\
&= \begin{pmatrix} \cos q_2 & 0 & \sin q_2 & q_3 \sin q_2 \\ 0 & -1 & 0 & 0 \\ \sin q_2 & 0 & -\cos q_2 & q_1 - q_3 \cos q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Use of the additional transformations between the frames defined in the problem text leads to

$$
\begin{aligned}
{}^w\boldsymbol{T}_e(\boldsymbol{q}) = {}^w\boldsymbol{T}_0 {}^0\boldsymbol{T}_3(\boldsymbol{q}) {}^3\boldsymbol{T}_e &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} {}^0\boldsymbol{T}_3(\boldsymbol{q}) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \sin q_2 & \cos q_2 & 0 & q_3 \sin q_2 \\ -\cos q_2 & \sin q_2 & 0 & q_1 - q_3 \cos q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^w\boldsymbol{R}_e(\boldsymbol{q}) & {}^w\boldsymbol{p}_e(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
\end{aligned}
$$

from which one can obtain the kinematic functions of interest (which could be derived also by direct inspection, once the joint variables have been defined according to the Denavit-Hartenberg convention). To this end, note that the final rotation matrix ${}^w\boldsymbol{R}_e(\boldsymbol{q})$ takes the form of an elementary rotation matrix by an angle $\phi = q_2 - \pi/2$ around the world axis $\boldsymbol{z}_w$. In fact, it is

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\big(q_2 - \frac{\pi}{2}\big) & -\sin\big(q_2 - \frac{\pi}{2}\big) & 0 \\ \sin\big(q_2 - \frac{\pi}{2}\big) & \cos\big((q_2 - \frac{\pi}{2}\big) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sin q_2 & \cos q_2 & 0 \\ -\cos q_2 & \sin q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = {}^w\boldsymbol{R}_e(q_2).$$

For the (first) task involving only the end-effector position on the plane, it is

$$\boldsymbol{r}_1 = \boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} {}^w p_x \\ {}^w p_y \end{pmatrix} = \begin{pmatrix} q_3 \sin q_2 \\ q_1 - q_3 \cos q_2 \end{pmatrix},$$

and the analytical (2×3) Jacobian matrix is

$$\boldsymbol{J}_1(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_1(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & q_3 \cos q_2 & \sin q_2 \\ 1 & q_3 \sin q_2 & -\cos q_2 \end{pmatrix}.$$

Analyzing the three minors of $\boldsymbol{J}_1(\boldsymbol{q})$, this matrix looses rank if and only if $\sin q_2 = 0$ and $q_3 = 0$, i.e., when the third robot link is oriented along the vertical direction and the third joint is completely retracted. In such a configuration, the Jacobian becomes

$$\boldsymbol{J}_1(\boldsymbol{q}_0) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \mp 1 \end{pmatrix},$$

where the upper sign corresponds to the case $q_2 = 0$ and the lower sign to the case $q_2 = \pi$. The four linear subspaces indicated in the text are spanned by the following basis vectors:

$$\mathcal{N}\big(\boldsymbol{J}(\boldsymbol{q}_0)\big) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ \mp 1 \end{pmatrix} \right\} \qquad \mathcal{R}\left(\boldsymbol{J}^T(\boldsymbol{q}_0)\right) = \left\{ \begin{pmatrix} 1 \\ 0 \\ \mp 1 \end{pmatrix} \right\} \qquad \text{in } \mathbb{R}^3,$$

$$\mathcal{R}\big(\boldsymbol{J}(\boldsymbol{q}_0)\big) = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{N}\left(\boldsymbol{J}^T(\boldsymbol{q}_0)\right) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \qquad \text{in } \mathbb{R}^2.$$

For the end-effector planar positioning and orientation task (of dimension $m = 3$), it is

$$\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} {}^w p_x \\ {}^w p_y \\ \phi \end{pmatrix} = \begin{pmatrix} q_3 \sin q_2 \\ q_1 - q_3 \cos q_2 \\ q_2 - \pi/2 \end{pmatrix}. \tag{2}$$

The analytical (3×3) Jacobian matrix associated to this task,

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & q_3 \cos q_2 & \sin q_2 \\ 1 & q_3 \sin q_2 & -\cos q_2 \\ 0 & 1 & 0 \end{pmatrix},$$

is singular if and only if $\sin q_2 = 0$.

Under the condition of question a), namely with $\sin q_2^* \neq 0$, it is

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q}^*)\dot{\boldsymbol{r}}_d = \frac{1}{\sin q_2^*}\begin{pmatrix} \cos q_2^* & \sin q_2^* & -q_3^* \\ 0 & 0 & \sin q_2^* \\ 1 & 0 & -q_3^* \cos q_2^* \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sin q_2^*}\begin{pmatrix} q_3^* \\ -\sin q_2^* \\ q_3^* \cos q_2^* \end{pmatrix}.$$

It is then easy to see that, by sufficiently extending the prismatic joint 3, the robot will violate the velocity bound at the first joint for any assigned value $V_1 > 0$. More specifically, it is

$$q_3^* > V_1 \cdot |\sin q_2^*| > 0 \qquad \Rightarrow \qquad |\dot{q}_1| > V_1.$$

Under the condition of question b), the robot is not in a singularity at the initial time $t = 0$. Thus, using the problem data and eq. (2), the initial control velocity can be computed as

$$\begin{aligned} \dot{\boldsymbol{q}}(0) &= \boldsymbol{J}^{-1}(\boldsymbol{q}(0))\Big(\dot{\boldsymbol{r}}_d(0) + \boldsymbol{K}_P(\boldsymbol{r}_d(0) - \boldsymbol{f}(\boldsymbol{q}(0))\Big) \\ &= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.5\,K_{r_1} \\ 0.3\,K_{r_2} \\ -\frac{\pi}{4}K_{r_3} \end{pmatrix}\right) = \begin{pmatrix} 1 + 0.3\,K_{r_2} + \frac{\pi}{4}K_{r_3} \\ -\frac{\pi}{4}K_{r_3} \\ 0.5\,K_{r_1} \end{pmatrix}, \end{aligned}$$

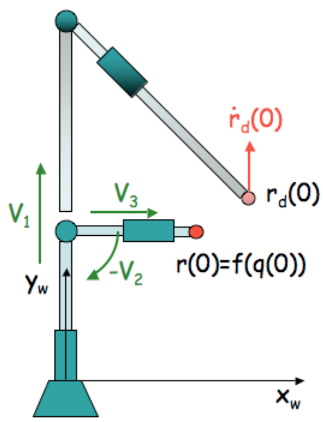
having set $\boldsymbol{K}_P = \text{diag}\{K_{r_1}, K_{r_2}, K_{r_3}\}$. From these expressions, one can directly choose two out of the three control gains:

$$\begin{aligned} |\dot{q}_3(0)| \leq V_3 \quad &\Rightarrow \quad 0.5\,K_{r_1} \leq V_3 = 4 \quad \Rightarrow \quad K_{r_1} = 8 > 0, \\ |\dot{q}_2(0)| \leq V_2 \quad &\Rightarrow \quad \frac{\pi}{4}K_{r_3} \leq V_2 = \pi \quad \Rightarrow \quad K_{r_3} = 4 > 0. \end{aligned}$$

Finally, using this definition, also the remaining gain is chosen:

$$|\dot{q}_1(0)| \leq V_1 \quad \Rightarrow \quad 1 + 0.3\,K_{r_2} + \frac{\pi}{4}K_{r_3} = 1 + 0.3\,K_{r_2} + \pi \leq V_1 = 5 \quad \Rightarrow \quad K_{r_2} = \frac{10}{3}(4 - \pi) > 0.$$

With the selected gains, all joint velocities will saturate at time $t = 0$ (the second joint velocity being at its negative limit $-V_2 = -\pi$) and, as a result, the *fastest* decrease of the initial task error $\boldsymbol{e}(0) = \boldsymbol{r}_d(0) - \boldsymbol{f}(\boldsymbol{q}(0))$ will be realized (with the task error converging anyway exponentially to zero, in a decoupled way for each task component). The situation at time $t = 0$ is depicted in the following figure, where the desired initial robot configuration is the lightly shaded one.



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