Robotics 1

Trajectory planning in Cartesian space

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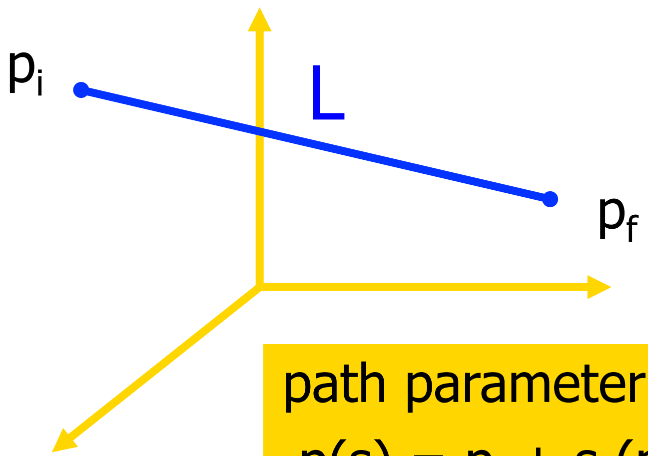
Trajectories in Cartesian space

- in general, the trajectory planning methods proposed in the joint space can be applied also in the Cartesian space
  - consider independently each component of the task vector (i.e., a position or an angle of a minimal representation of orientation)
- however, when planning a trajectory for the three orientation angles, the resulting global motion cannot be intuitively visualized in advance
- if possible, we still prefer to plan Cartesian trajectories separately for position and orientation
- the number of knots to be interpolated in the Cartesian space is typically low (e.g., 2 knots for a PTP motion, 3 if a "via point" is added) ⇒ use simple interpolating paths, such as straight lines, arc of circles, …

Planning a linear Cartesian path
(position only)



GIVEN
p_i, p_f, v_{max}, a_{max}
v_i, v_f (typically = 0)

$$L = \| p_f - p_i \|$$

$$\frac{p_f - p_i}{\| p_f - p_i \|} = \text{unit vector of directional cosines of the line}$$

path parameterization

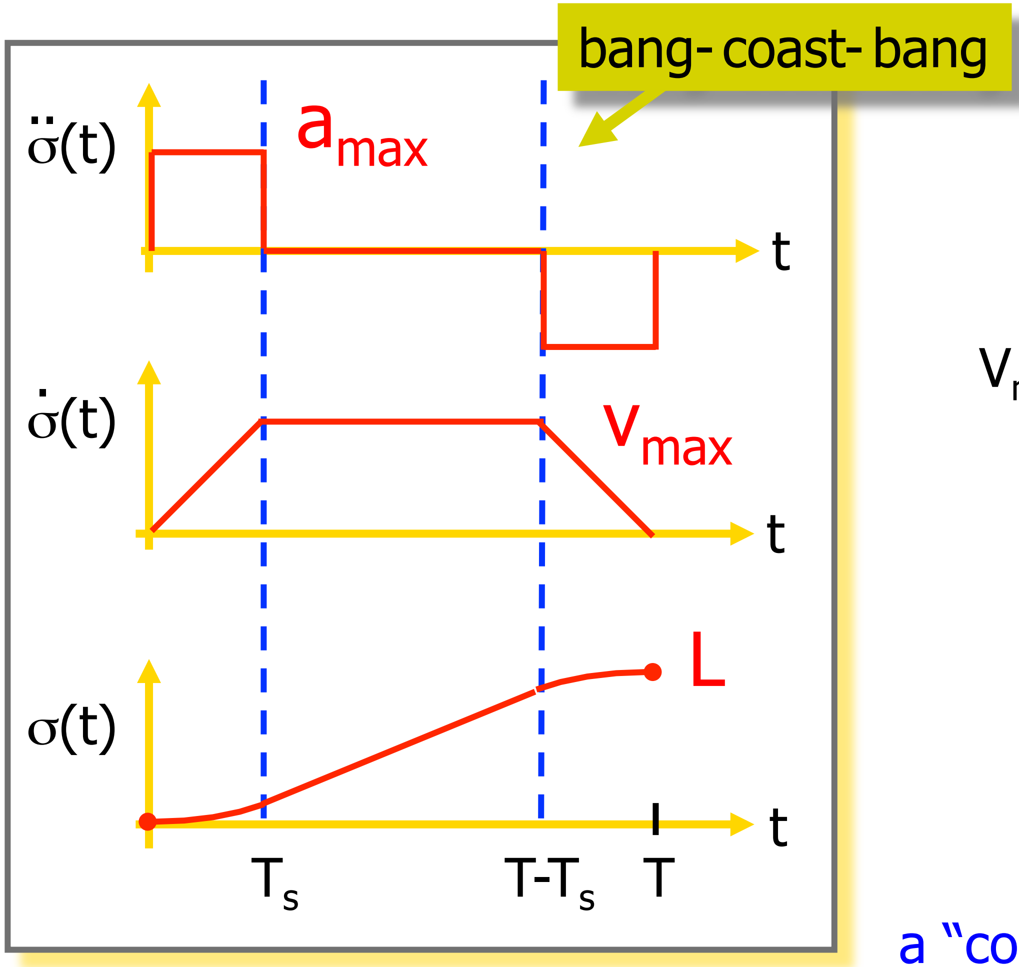
$$p(s) = p_i + s\,(p_f - p_i)$$

$s \in [0,1]$ ← setting $s = \sigma/L$, $\sigma \in [0,L]$ is the arc length (gives the current length of the path)

$$\dot{p}(s) = \frac{dp}{ds}\,\dot{s} = (p_f - p_i)\,\dot{s} = \frac{p_f - p_i}{L}\,\dot{\sigma}$$

$$\ddot{p}(s) = \cancel{\frac{d^2p}{ds^2}}\,\dot{s}^2 + \frac{dp}{ds}\,\ddot{s} = (p_f - p_i)\,\ddot{s} = \frac{p_f - p_i}{L}\,\ddot{\sigma}$$

Timing law with trapezoidal speed - 1



given*: L, v_{max}, a_{max}
find: T_s, T

$$V_{max}\,(T - T_s) = L$$

= area of the speed profile

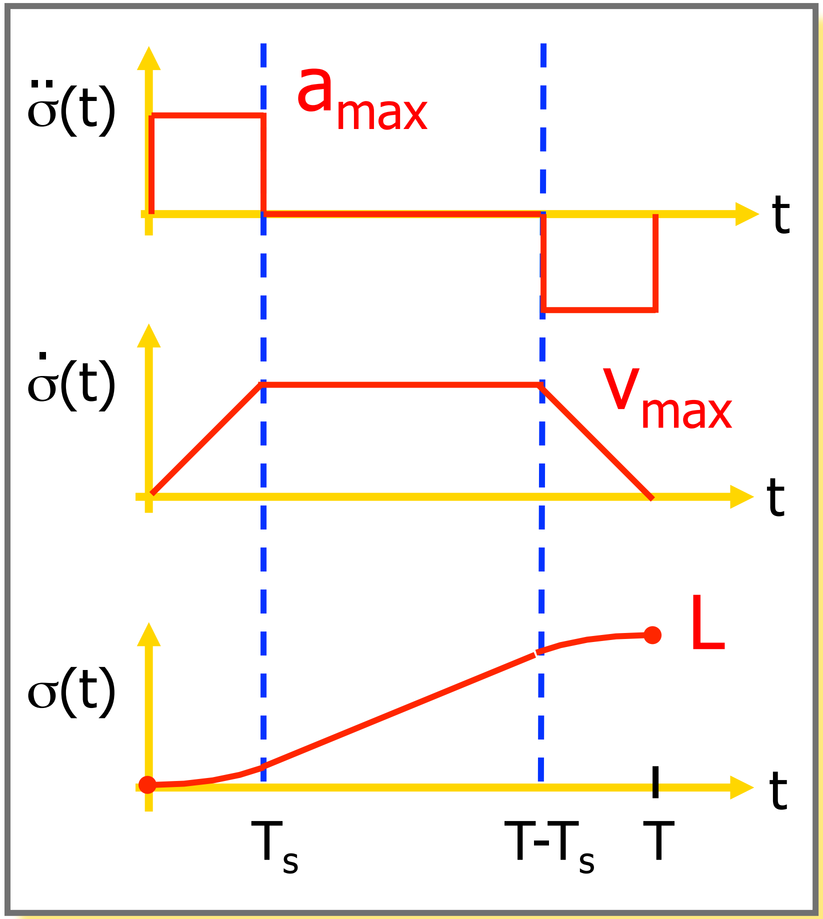
$$T_s = \frac{v_{max}}{a_{max}}$$

$$T = \frac{L\,a_{max} + v_{max}^2}{a_{max} v_{max}}$$

a "coast" phase exists iff: $L > v_{max}^2/a_{max}$

* = other input data combinations are possible (see textbook)

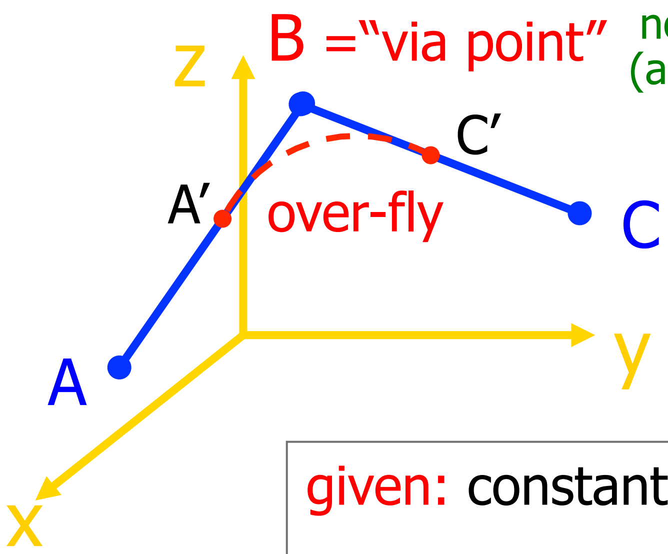
Timing law with trapezoidal speed - 2



$$\sigma(t) = \begin{cases} a_{max}\, t^2/2 & t \in [0, T_s] \\ v_{max}\, t - \dfrac{v_{max}^2}{2\, a_{max}} & t \in [T_s, T-T_s] \\ -\, a_{max}\, (t-T)^2/2 + v_{max}\, T - \dfrac{v_{max}^2}{a_{max}} & t \in [T-T_s, T] \end{cases}$$

can be used also in the joint space!

Concatenation of linear paths



no need to pass (and stop!) there

$$\frac{B - A}{\|B - A\|} = K_{AB}$$

$$\frac{C - B}{\|C - B\|} = K_{BC}$$

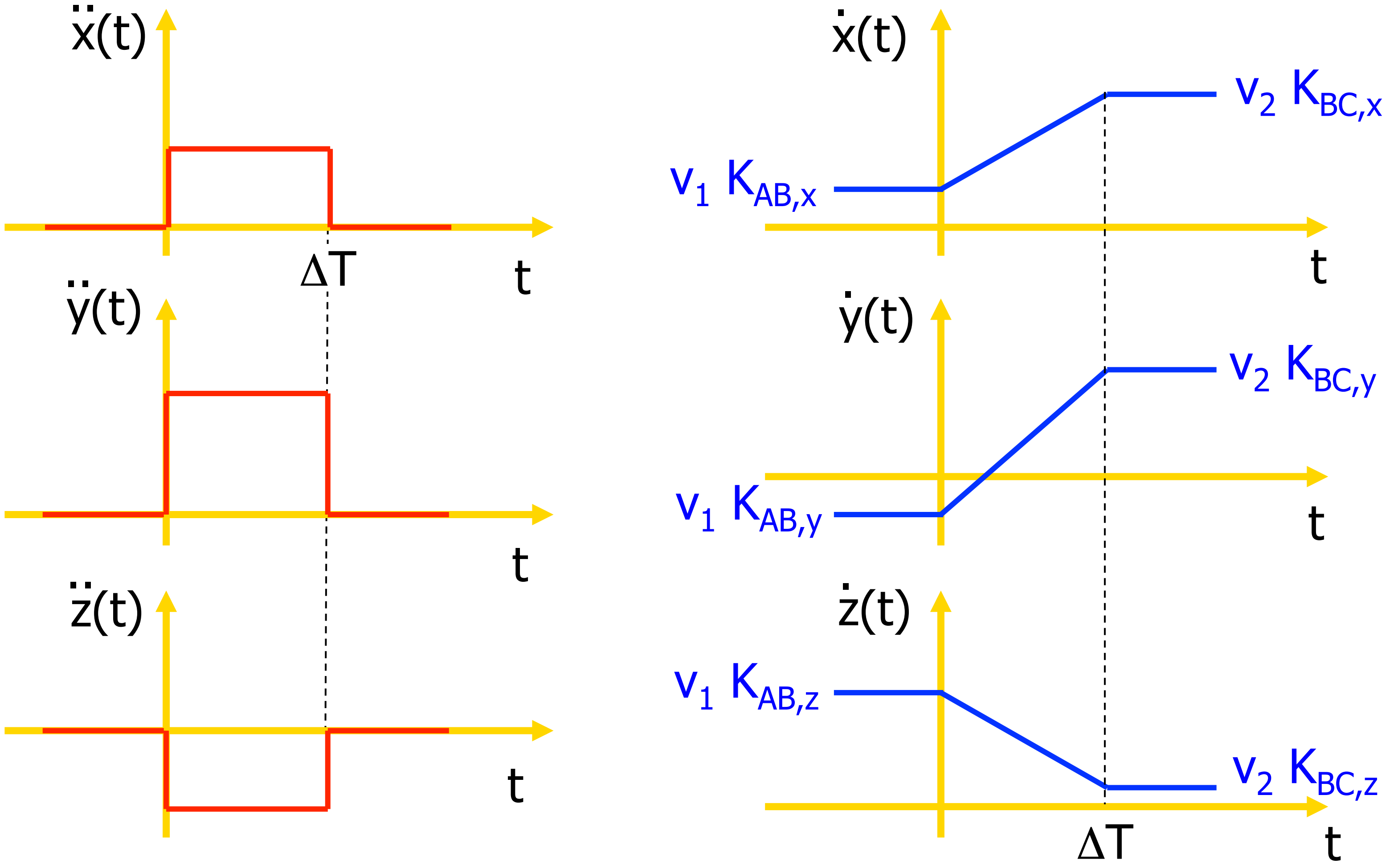
unit vectors of direction cosines

given: constant speeds v_1 on linear path AB
v_2 on linear path BC
desired transition: with constant acceleration for a time ΔT

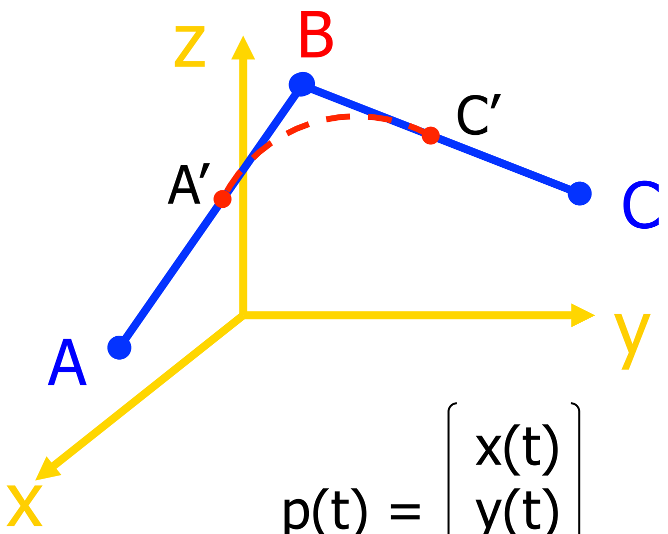
$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad t \in [0, \Delta T] \text{ (transition starts at } t = 0\text{)}$$

note: during over-fly, the path remains always in the plane specified by the two lines intersecting at B (in essence, it is a planar problem)

Time profiles on components



Timing law during transition



$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\frac{B - A}{\|B - A\|} = K_{AB}$$

$$\frac{C - B}{\|C - B\|} = K_{BC}$$

unit vectors of direction cosines

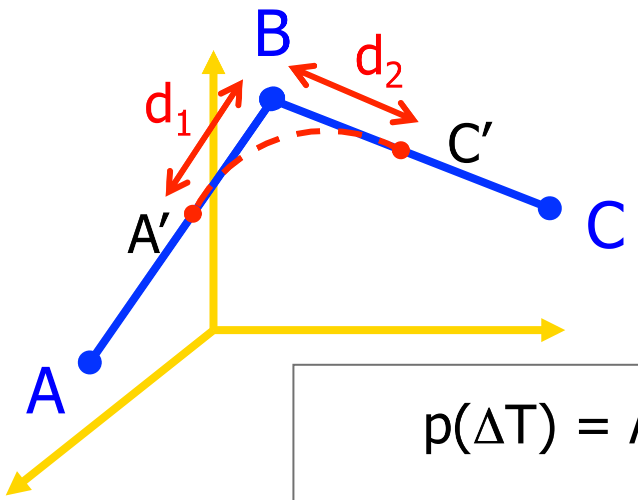
$t \in [0, \Delta T]$ (transition starts at $t = 0$)

$$\boxed{\ddot{p}(t) = 1/\Delta T\,(v_2 K_{BC} - v_1 K_{AB})} \xrightarrow{\int} \boxed{\dot{p}(t) = v_1 K_{AB} + t/\Delta T\,(v_2 K_{BC} - v_1 K_{AB})}$$

$$\xrightarrow{\int} \boxed{p(t) = A' + v_1 K_{AB}\, t + t^2/2\Delta T\,(v_2 K_{BC} - v_1 K_{AB})}$$

thus, we obtain a parabolic blending
(see textbook for this same approach in the joint space)

Solution
(various options)



$$B - A' = d_1 K_{AB}$$
$$C' - B = d_2 K_{BC} \qquad (1)$$

$$p(t) = A' + v_1 K_{AB} t + t^2/2\Delta T \, (v_2 K_{BC} - v_1 K_{AB})$$

$$p(\Delta T) = A' + \Delta T/2 \, (v_1 K_{AB} + v_2 K_{BC}) = C'$$

$$\Rightarrow \quad -B + A' + \Delta T/2 \, (v_1 K_{AB} + v_2 K_{BC}) = C' - B$$

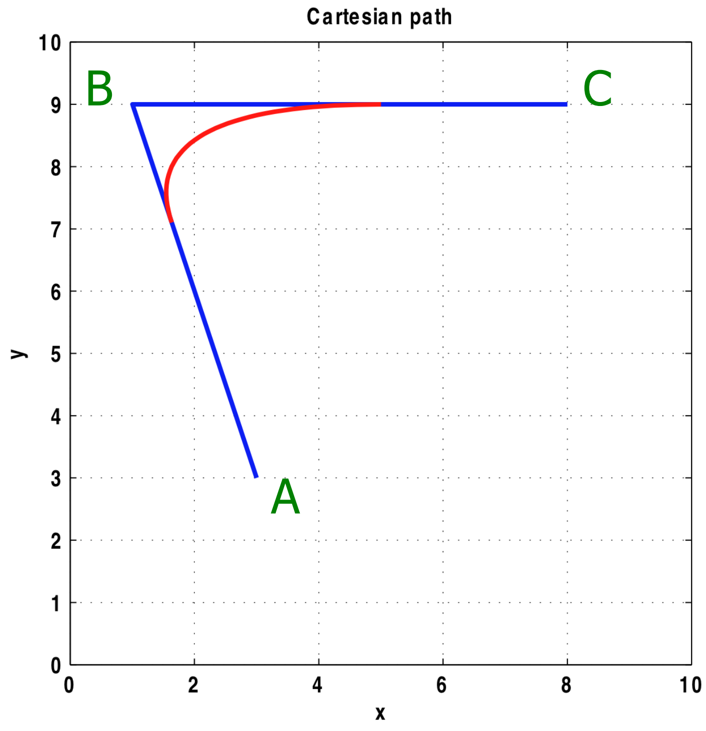
$$\overset{(1)}{\Rightarrow} \quad d_1 K_{AB} + d_2 K_{BC} = \Delta T/2 \, (v_1 K_{AB} + v_2 K_{BC})$$

$$\Rightarrow \quad \boxed{d_1 = v_1 \Delta T/2} \quad \boxed{d_2 = v_2 \Delta T/2}$$

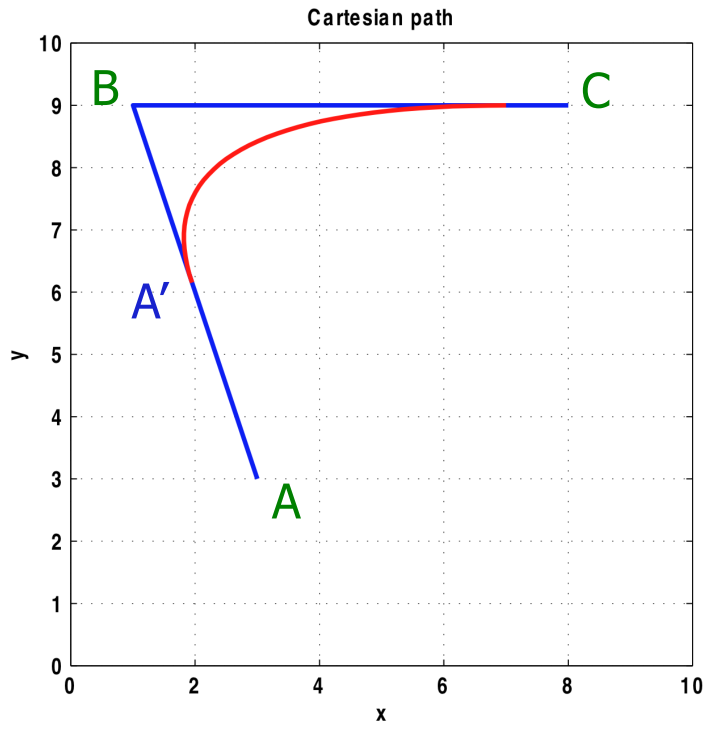
by choosing, e.g., d_1 (namely A′) $\rightarrow$ $\boxed{\Delta T = 2d_1/v_1 \Rightarrow d_2 = d_1 v_2/v_1}$

A numerical example

- transition from A=(3,3) to C=(8,9) via B=(1,9), with speed from $v_1=1$ to $v_2=2$
- exploiting two options for solution (resulting in different paths!)
  - assign transition time: $\Delta T=4$ (we re-center it here for $t \in [-\Delta T/2, \Delta T/2]$)
  - assign distance from B for departing: $d_1=3$ (assign d_2 for landing is handled similarly)



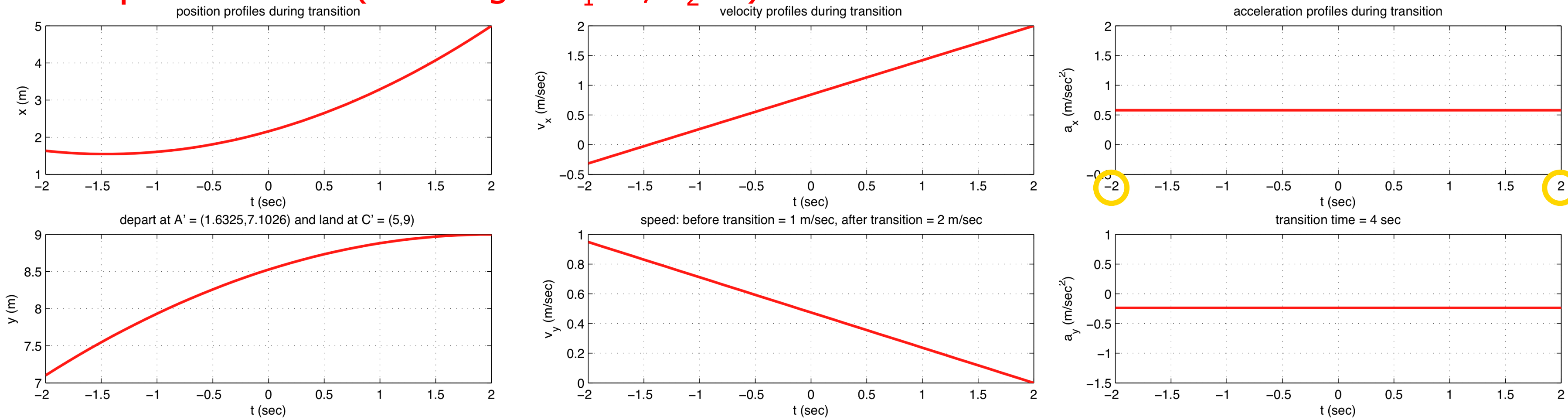
$\Delta T=4$



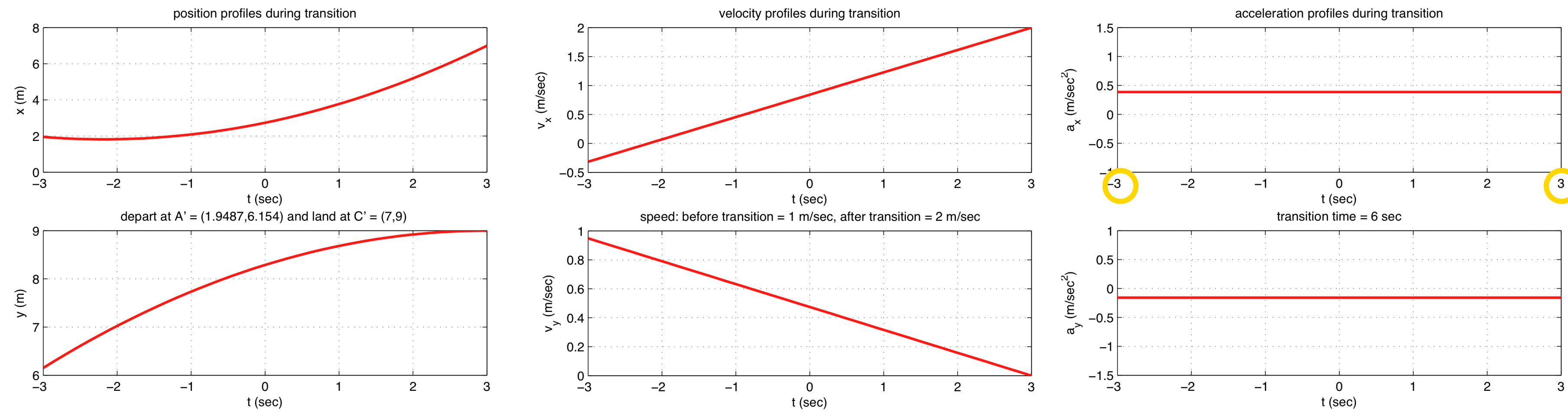
$d_1=3$

A numerical example (cont'd)

first option: $\Delta T=4$ (resulting in $d_1=2$, $d_2=4$)

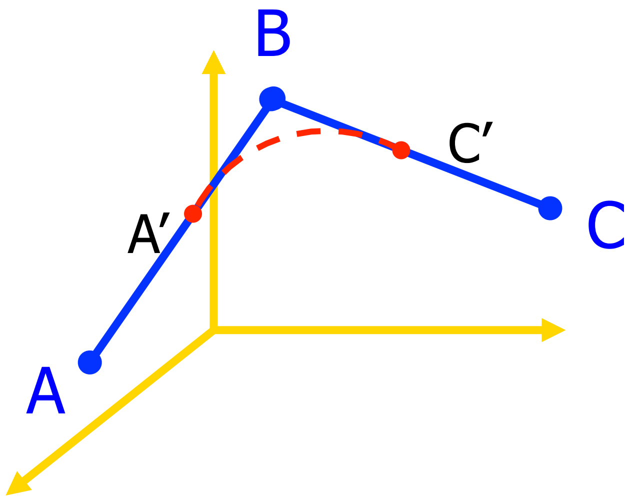


second option: $d_1=3$ (resulting in $\Delta T=6$, $d_2=6$)



actually, the same vel/acc profiles only with a different time scale!!

Alternative solution
(imposing acceleration)



$$\ddot{p}(t) = 1/\Delta T\ (v_2\, K_{BC} - v_1\, K_{AB})$$

$v_1 = v_2 = v_{max}$ (for simplicity)

$$\|\ddot{p}(t)\| = a_{max}$$

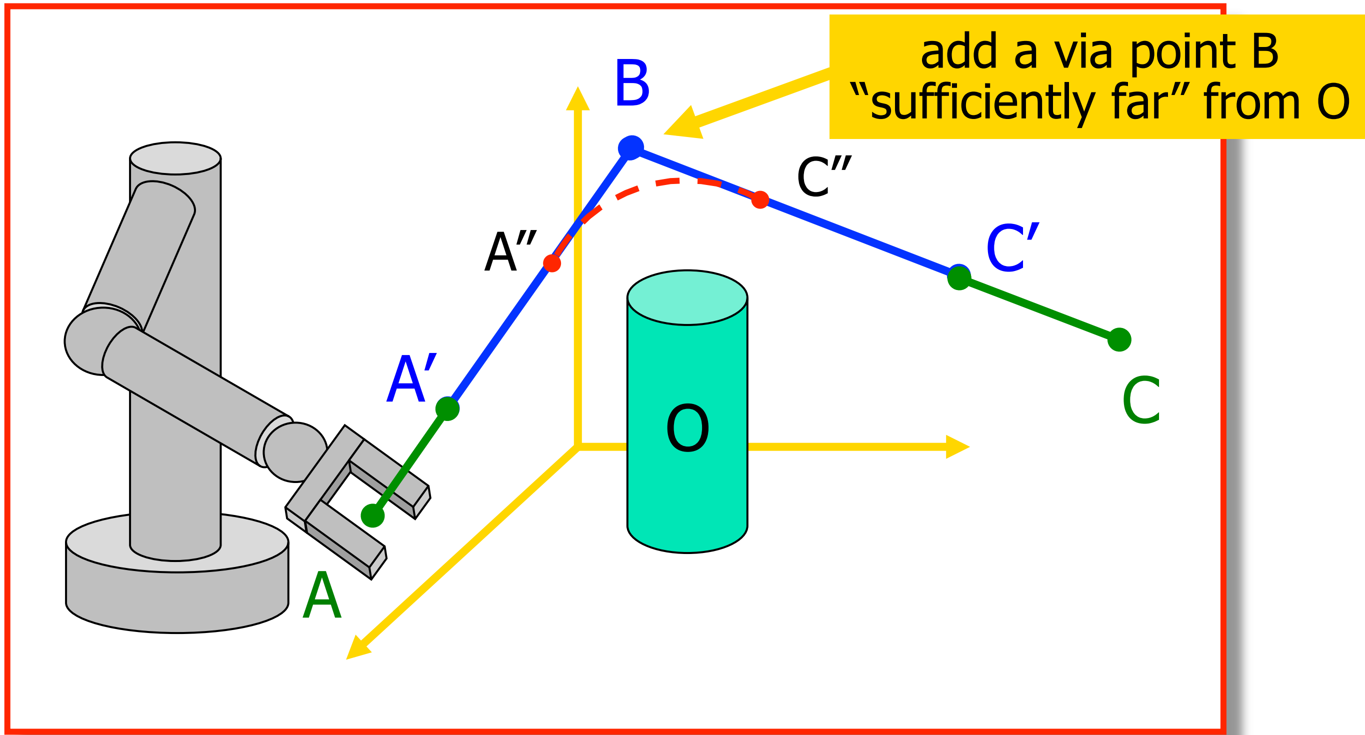
$$\Delta T = (v_{max}/a_{max})\ \|K_{BC} - K_{AB}\|$$

$$= (v_{max}/a_{max})\sqrt{2\,(1 - K_{BC,x}K_{AB,x} - K_{BC,y}K_{AB,y} - K_{BC,z}K_{AB,z})}$$

then, $d_1 = d_2 = v_{max}\,\Delta T/2$

Application example

plan a Cartesian trajectory from A to C (rest-to-rest)
that avoids the obstacle O, with $a \leq a_{max}$ and $v \leq v_{max}$



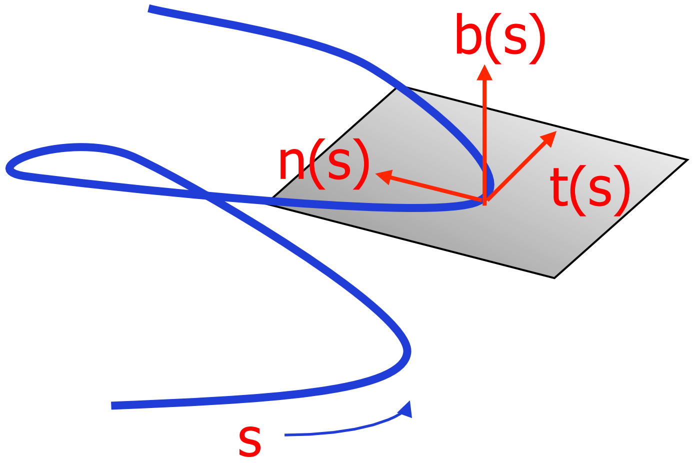
on $\overline{AA'}$ → a_{max} on $\overline{A'B}$ and $\overline{BC'}$ → v_{max} on $\overline{C'C}$ → $-a_{max}$
+ over-fly between A″ e C″

Other Cartesian paths

- circular path through 3 points in 3D (often built-in feature)
- linear path for the end-effector with constant orientation
- in robots with spherical wrist: planning can be decomposed into a path for wrist center and one for E-E orientation, with a common timing law
- though more complex in general, it is often convenient to parameterize the Cartesian geometric path $p(s)$ in terms of its arc length (e.g., with $s = R\theta$ for circular paths), so that
- velocity: $dp/dt = dp/ds \cdot ds/dt$
  - dp/ds = unit vector ($\|\cdot\|=1$) tangent to the path: tangent direction $t(s)$
  - ds/dt = absolute value of tangential velocity (= speed)
- acceleration: $d^2p/dt^2 = d^2p/ds^2 \cdot (ds/dt)^2 + dp/ds \cdot d^2s/dt^2$
  - $\|d^2p/ds^2\|$ = curvature $\kappa(s)$ (= 1/radius of curvature)
  - $d^2p/ds^2 \cdot (ds/dt)^2$ = centripetal acceleration: normal direction $n(s) \perp$ to the path, on the osculating plane; binormal direction $b(s) = t(s) \times n(s)$
  - d^2s/dt^2 = scalar value (with any sign) of tangential acceleration

Definition of Frenet frame

- For a generic (smooth) path $p(s)$ in R^3, parameterized by s (not necessarily its arc length), one can define a reference frame as in figure



$p' = dp/ds \qquad p'' = d^2p/ds^2$

derivatives w.r.t. the parameter

$t(s) = p'(s)/\|p'(s)\|$

unit tangent vector

$n(s) = p''(s)/\|p''(s)\|$

unit normal vector

($\in$ osculating plane)

$b(s) = t(s) \times n(s)$

unit binormal vector

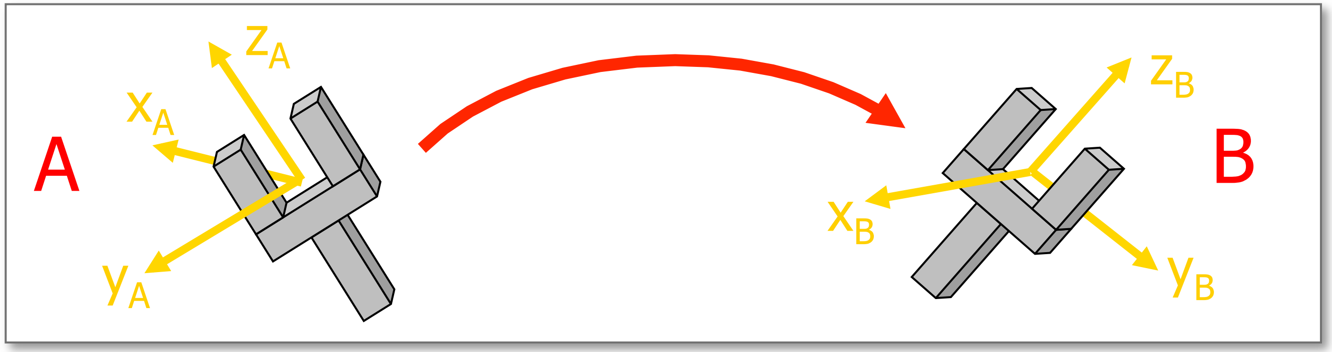
- general expression of path curvature (at a path point $p(s)$)

$$\kappa(s) = \|p'(s) \times p''(s)\| / \|p'(s)\|^3$$

Optimal trajectories

- for Cartesian robots (e.g., PPP joints)
  1. the straight line joining two position points in the Cartesian space is one path that can be executed in minimum time under velocity/acceleration constraints (but other such paths may exist, if (joint) motion can also be not coordinated)
  2. the optimal timing law is of the bang-coast-bang type in acceleration (in this special case, also in terms of actuator torques)
- for articulated robots (with at least a R joint)
  - 1. e 2. are no longer true in general in the Cartesian space, but time-optimality still holds in the joint space when assuming bounds on joint velocity/acceleration
    - straight line paths in the joint space do not correspond to straight line paths in the Cartesian space, and vice-versa
  - bounds on joint acceleration are conservative (though kinematically tractable) w.r.t. actual ones on actuator torques, which involve the robot dynamics
    - when changing robot configuration/state, different torque values are needed to impose the same joint accelerations

Planning orientation trajectories



- using minimal representations of orientation (e.g., ZXZ Euler angles ϕ,θ,ψ), we can plan independently a trajectory for each component
  - e.g., a linear path in space $\phi\ \theta\ \psi$, with a cubic timing law

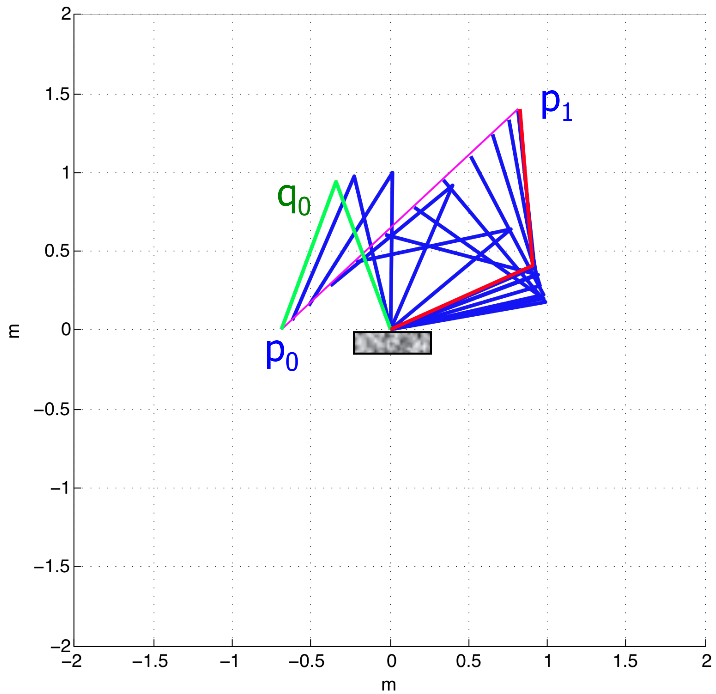
    $\Rightarrow$ but poor prediction/understanding of the resulting intermediate orientations
- alternative method: based on the axis/angle representation
  - determine the (neutral) axis r and the angle θ_{AB}: $R(r,\theta_{AB}) = R_A^T R_B$ (rotation matrix changing the orientation from A to B $\Rightarrow$ inverse axis-angle problem)
  - plan a timing law $\theta(t)$ for the (scalar) angle θ interpolating 0 with θ_{AB} (with possible constraints/boundary conditions on its time derivatives)
  - $\forall t$, $R_A R(r,\theta(t))$ specifies then the actual end-effector orientation at time t

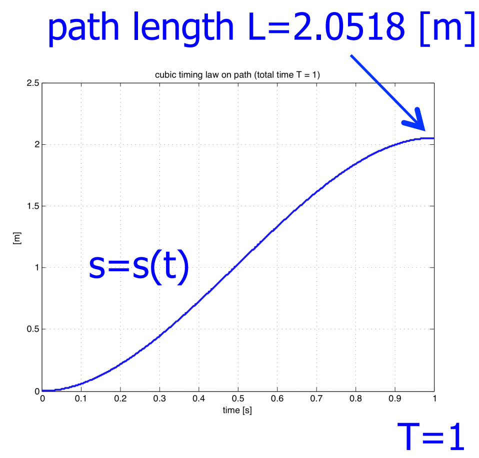
Uniform time scaling

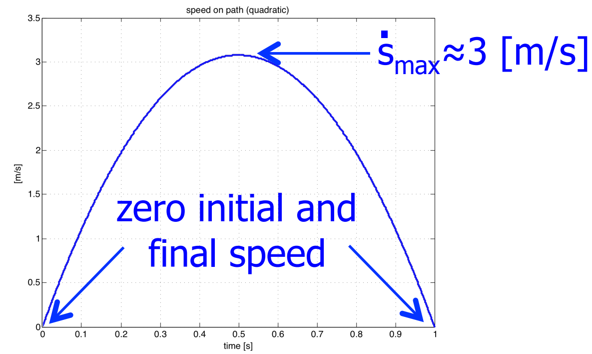
- for a given path $p(s)$ (in joint or Cartesian space) and a given timing law $s(\tau)$ ($\tau=t/T$, T="motion time"), we need to check if existing bounds v_{max} on (joint) velocity and/or a_{max} on (joint) acceleration are violated or not
  - … unless such constraints have already been taken into account during the trajectory planning, e.g., by using a bang-coast-bang acceleration timing law
- velocity scales linearly with motion time
  - $dp/dt = dp/ds \cdot ds/d\tau \cdot 1/T$
- acceleration scales quadratically with motion time
  - $d^2p/dt^2 = (d^2p/ds^2 \cdot (ds/d\tau)^2 + dp/ds \cdot d^2s/d\tau^2) \cdot 1/T^2$
- if motion is unfeasible, scale (increase) time $T \rightarrow kT$ ($k>1$), based on the "most violated" constraint (max of the ratios $|v|/v_{max}$ and $|a|/a_{max}$)
- if motion is "too slow" w.r.t. the robot capabilities, decrease T ($k<1$)
  - in both cases, after scaling, there will be (at least) one instant of saturation (for at least one variable)
  - no need to re-compute motion profiles from scratch after the scaling!

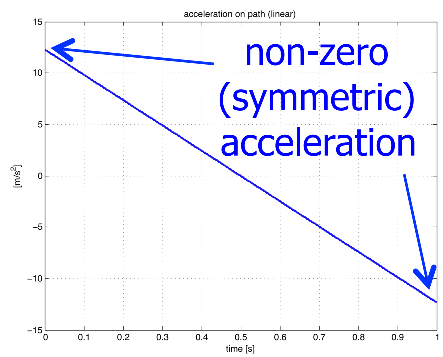
Numerical example - 1

- 2R planar robot with links of unitary length (1 [m])
- linear Cartesian path $p(s)$ from $q_0=(110°, 140°) \Rightarrow p_0=f(q_0)=(-.684, 0)$ [m] to $p_1=(0.816, 1.4)$, with rest-to-rest cubic timing law $s(t)$, $T=1$ [s]
- bounds in joint space: max (absolute) velocity $v_{max,1}= 2$, $v_{max,2}= 2.5$ [rad/s], max (absolute) acceleration $a_{max,1}= 5$, $a_{max,2}= 7$ [rad/s^2]





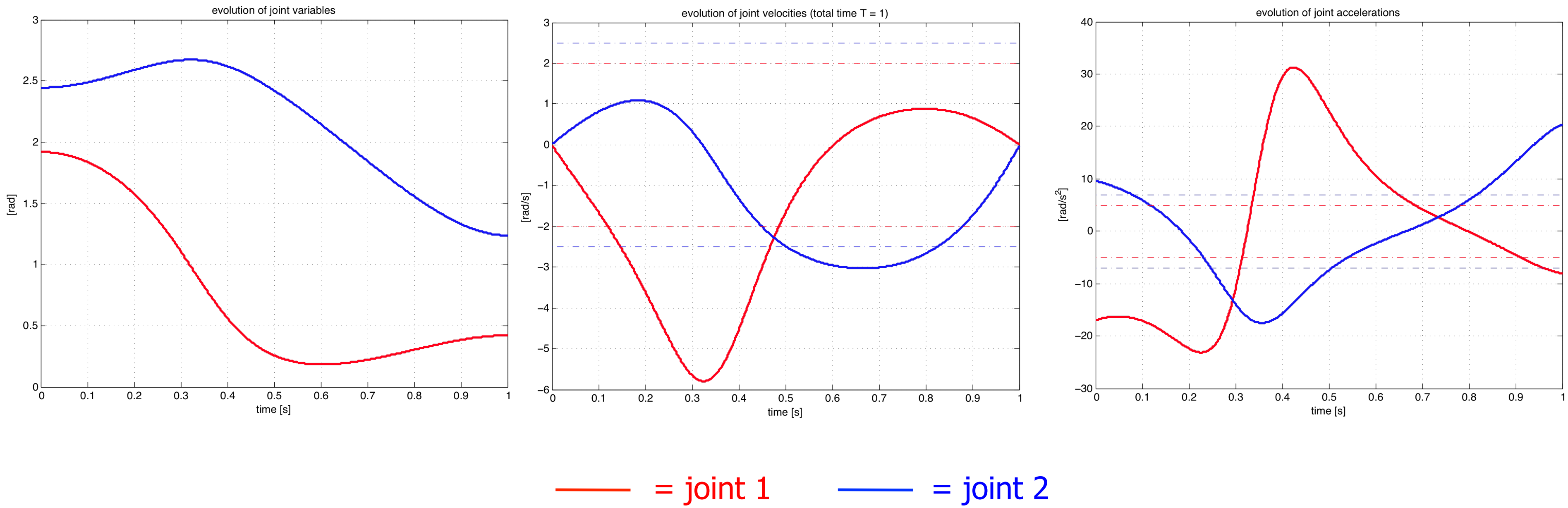




Numerical example - 2

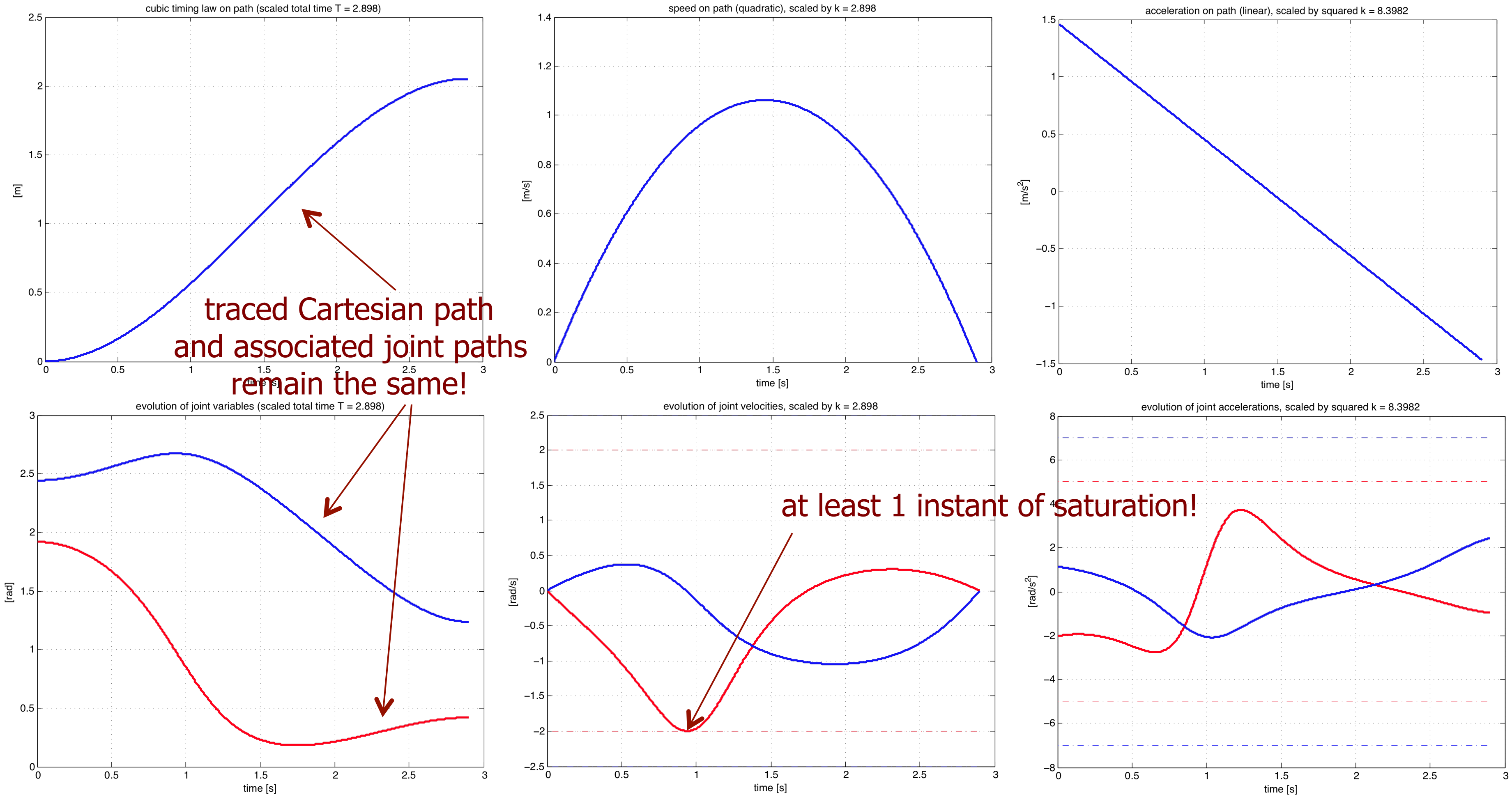
- violation of both joint velocity and acceleration bounds with T=1 [s]
  - max relative violation of joint velocities: $k_{vel} = 2.898$
  - max relative violation of joint accelerations: $k_{acc} = 6.2567$
- minimum uniform time scaling of Cartesian trajectory to recover feasibility

$$k = \max\{1, k_{vel}, \sqrt{k_{acc}}\} = 2.898 \quad \Rightarrow \quad T_{scaled} = kT = 2.898 > T$$



Numerical example - 3

- scaled trajectory with T_{scaled} = 2.898 [s]
  - speed [acceleration] on path and joint velocities [accelerations] scale linearly [quadratically]



= joint 1 = joint 2