Robotics 1

Inverse differential kinematics
Statics and force transformations

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Inversion of differential kinematics

- find the joint velocity vector that realizes a desired end-effector "generalized" velocity (linear and angular)

generalized velocity

$$v = J(q)\,\dot{q} \quad \xrightarrow[\text{non-singular}]{\text{J square and}} \quad \dot{q} = J^{-1}(q)\,v$$

- problems
  - near a singularity of the Jacobian matrix (high $\dot{q}$)
  - for redundant robots (no standard "inverse" of a rectangular matrix)

in these cases, "more robust" inversion methods are needed

Incremental solution
to inverse kinematics problems

- joint velocity inversion can be used also to solve on-line and incrementally a "sequence" of inverse kinematics problems
- each problem differs by a small amount dr from previous one

$$r = f_r(q)$$

direct kinematics

$$dr = \frac{\partial f_r(q)}{\partial q} dq = J_r(q)\, dq$$

differential kinematics

$$r \rightarrow r + dr$$

$$r + dr = f_r(q) \quad \Rightarrow \quad q = f_r^{-1}(r + dr)$$

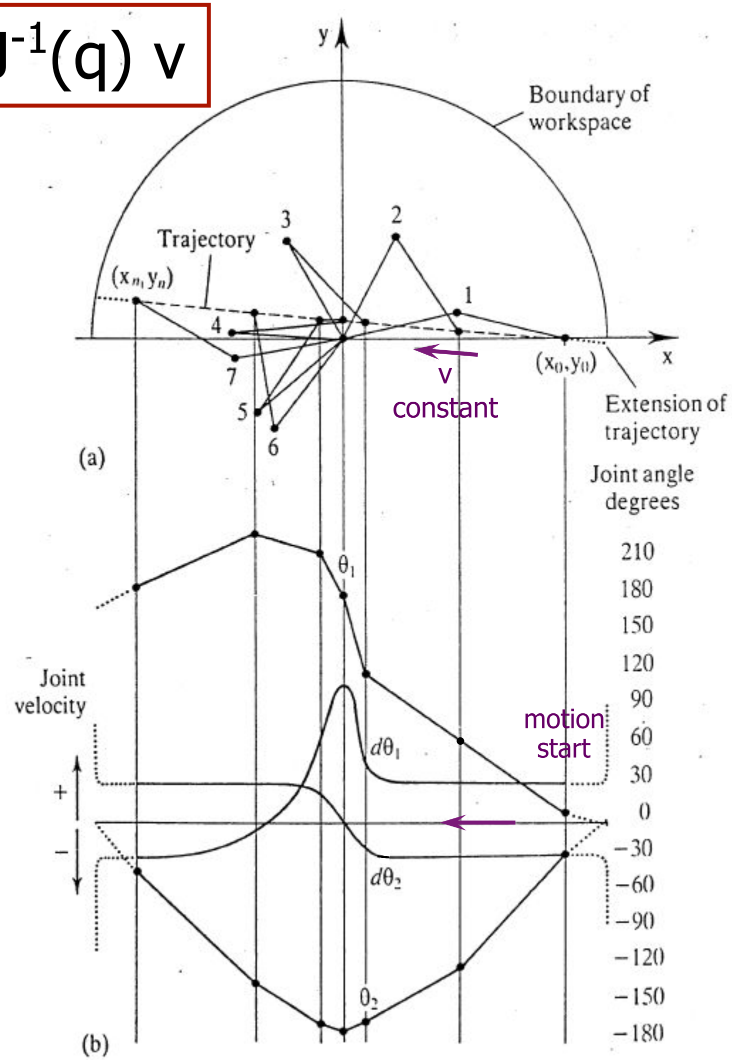
first, increment the desired task variables — then, solve the inverse kinematics problem

$$dq = J_r^{-1}(q)\, dr \quad \Rightarrow \quad q \rightarrow q + dq$$

first, solve the inverse differential kinematics problem — then, increment the original joint variables

Behavior near a singularity

$$\boxed{\dot{q} = J^{-1}(q)\, v}$$



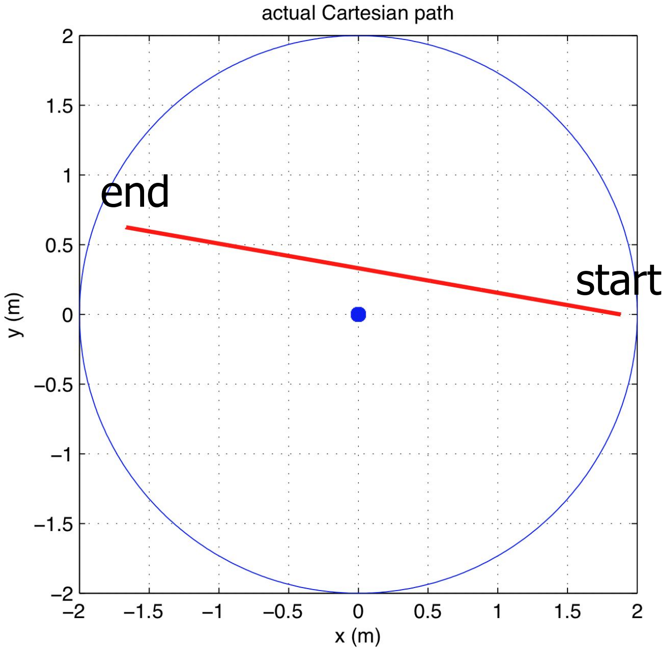
- problems arise only when commanding joint motion by inversion of a given Cartesian motion task
- here, a linear Cartesian trajectory for a planar 2R robot
- there is a sudden increase of the displacement/velocity of the first joint near $\theta_2 = -\pi$ (end-effector close to the origin), despite the required Cartesian displacement is small

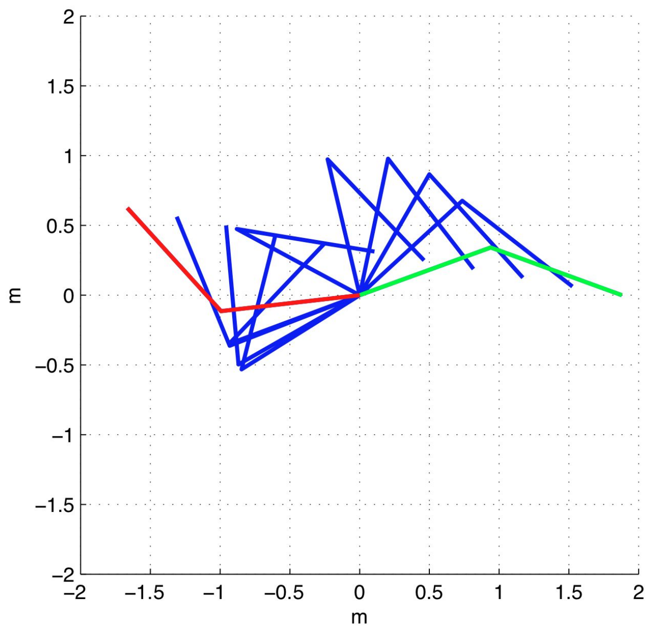
Simulation results

planar 2R robot in straight line Cartesian motion

$$\boxed{\dot{q} = J^{-1}(q)\, v}$$

regular case



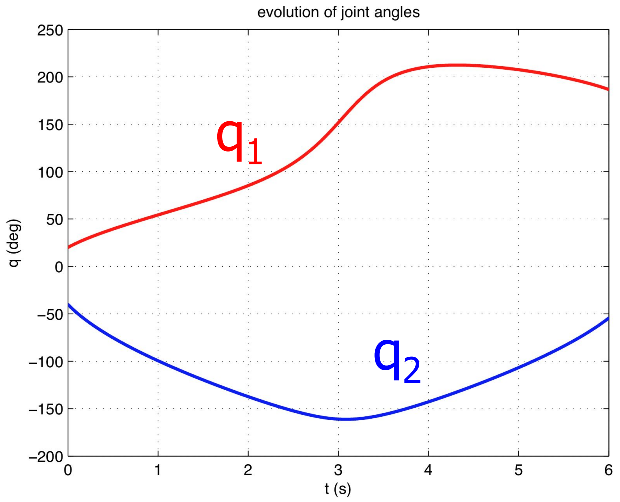


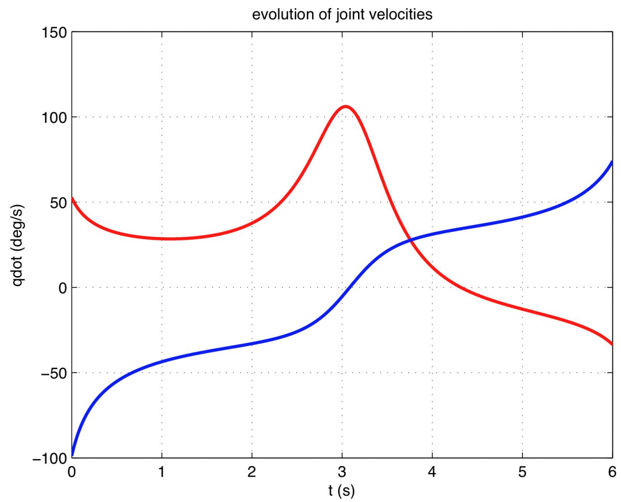
a line from right to left, at α=170° angle with x-axis,
executed at constant speed v=0.6 m/s for T=6 s

Simulation results

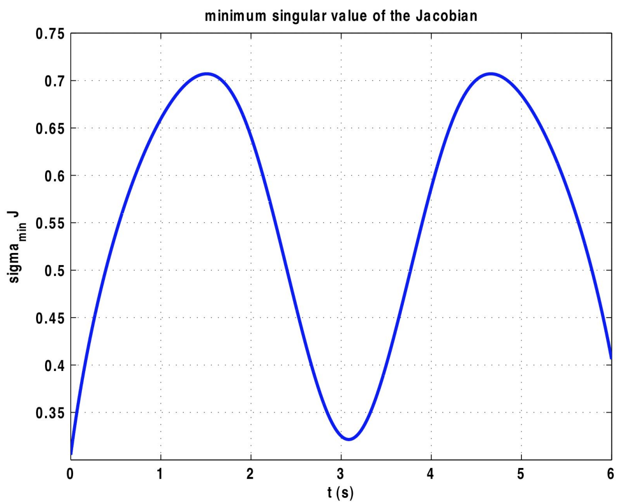
planar 2R robot in straight line Cartesian motion

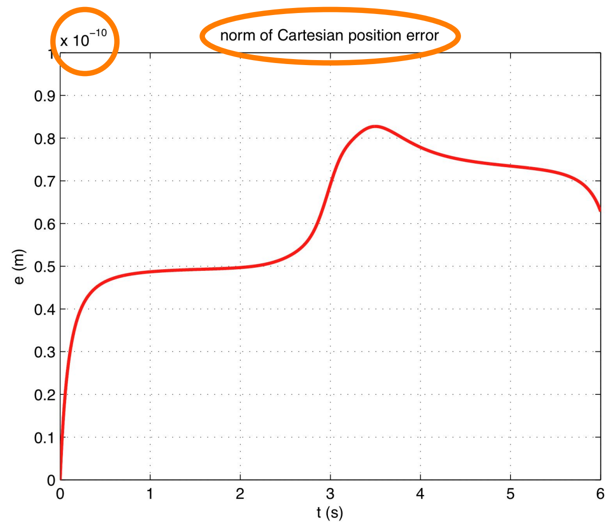
path at α=170°





regular case





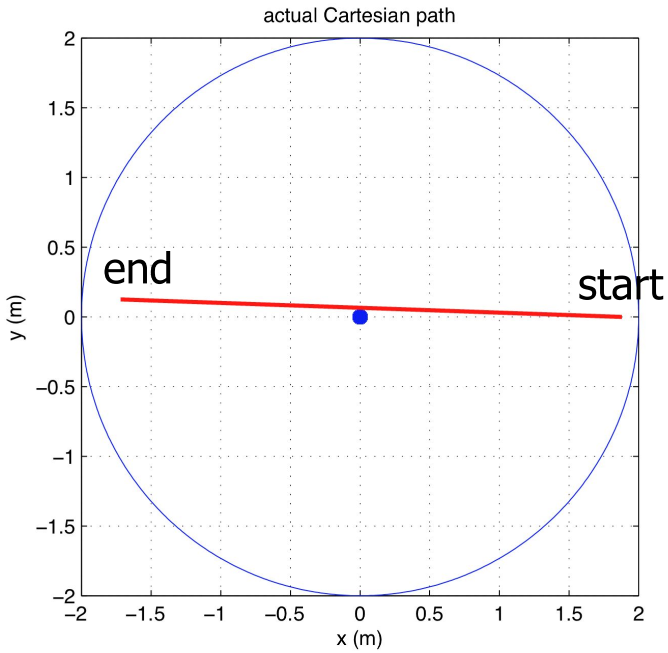
error due only to numerical integration (10^{-10})

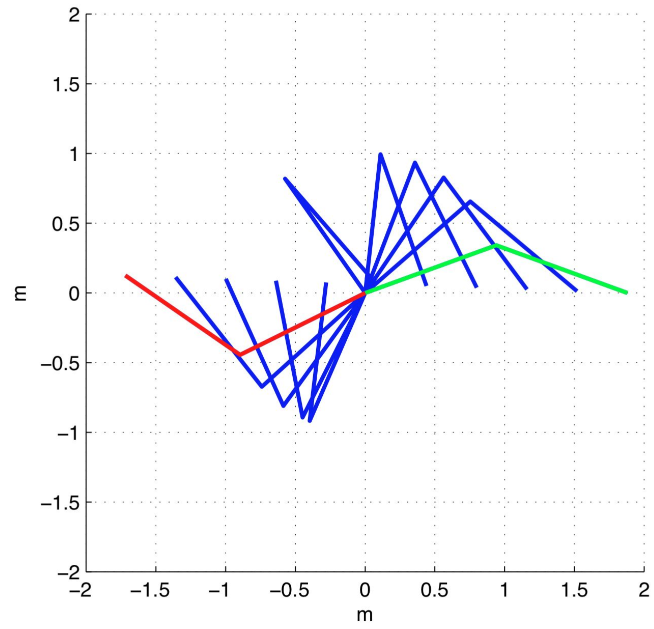
Simulation results

planar 2R robot in straight line Cartesian motion

$$\boxed{\dot{q} = J^{-1}(q)\, v}$$

close to singular case



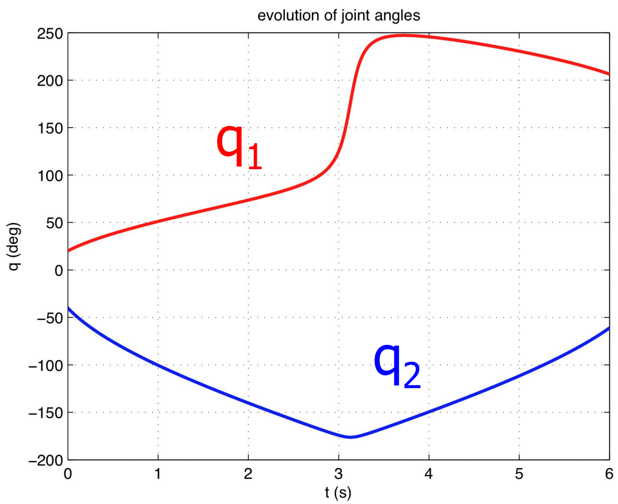


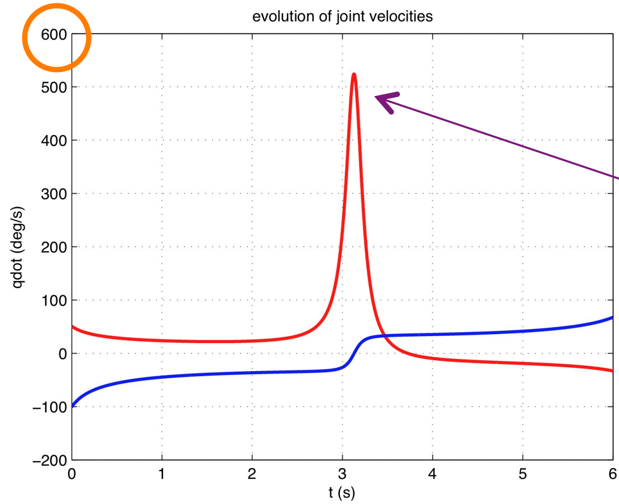
a line from right to left, at α=178° angle with x-axis, executed at constant speed v=0.6 m/s for T=6 s

Simulation results

planar 2R robot in straight line Cartesian motion

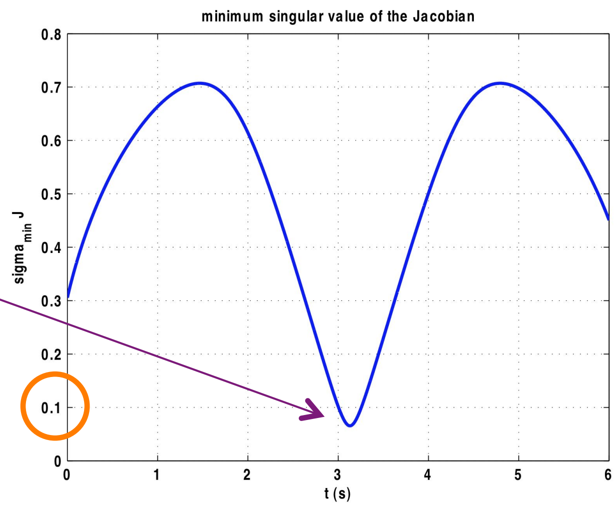
path at α=178°

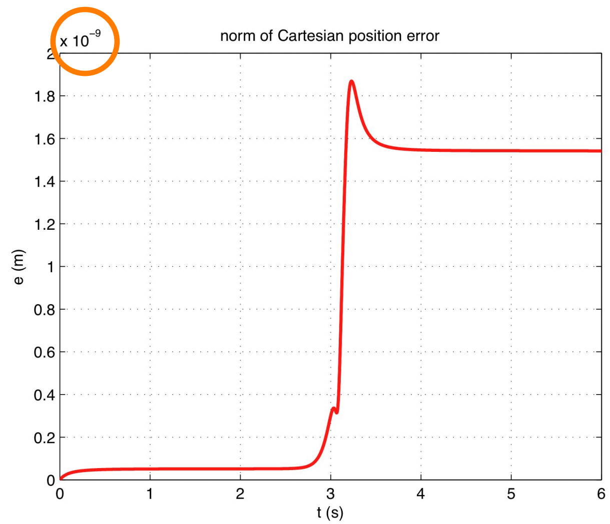




large peak of $\dot{q}_1$

close to singular case





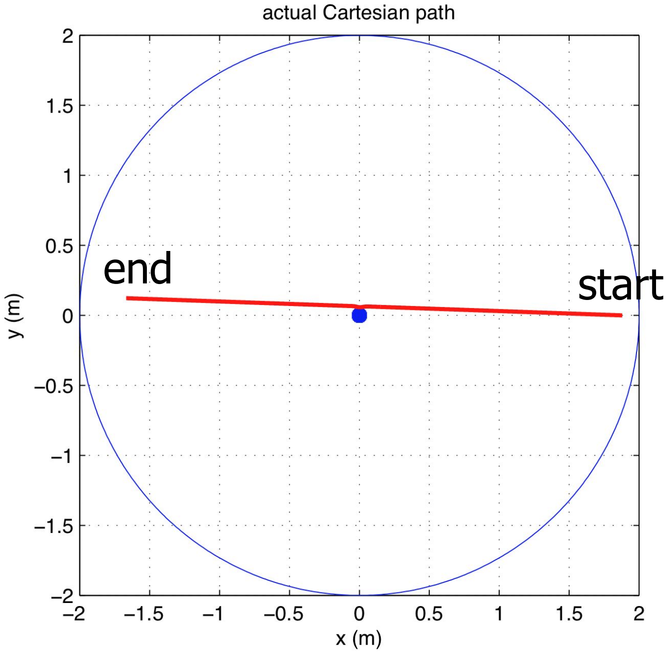
still very small, but increased numerical integration error ($2 \cdot 10^{-9}$)

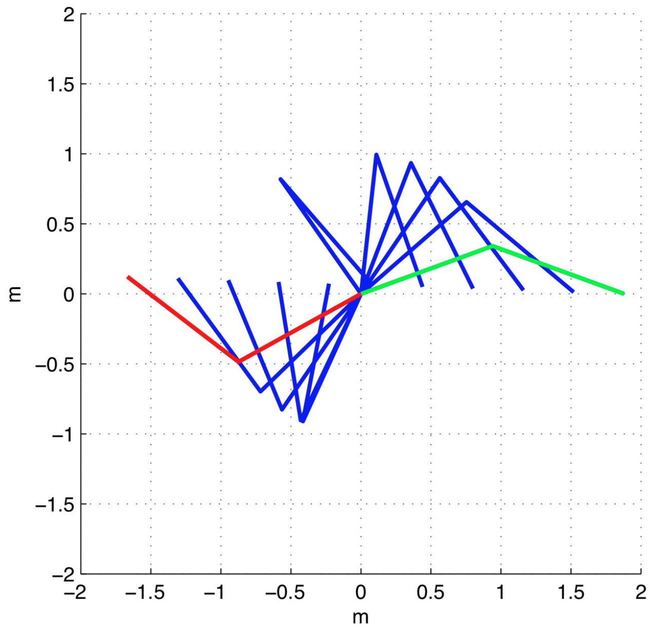
Simulation results

planar 2R robot in straight line Cartesian motion

$$\boxed{\dot{q} = J^{-1}(q)\, v}$$

close to singular case
with joint velocity saturation at V_i=300°/s



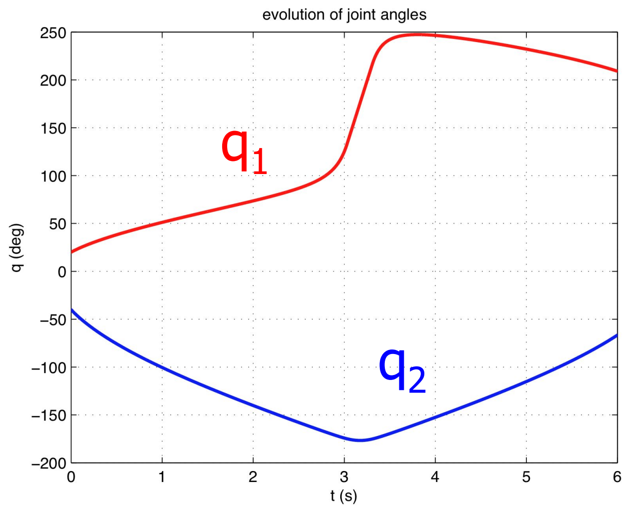


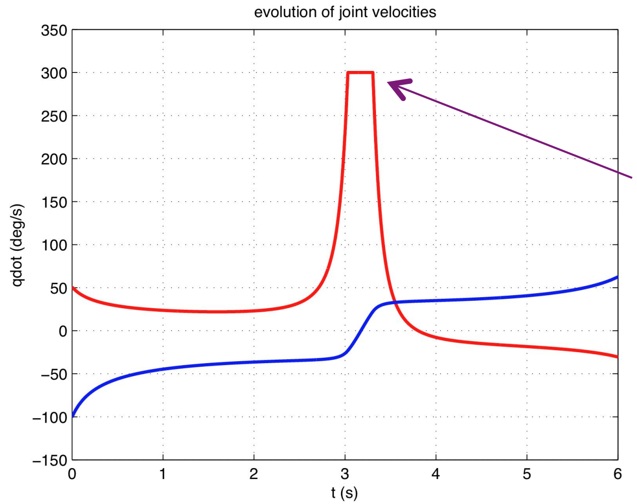
a line from right to left, at α=178° angle with x-axis,
executed at constant speed v=0.6 m/s for T=6 s

Simulation results

planar 2R robot in straight line Cartesian motion

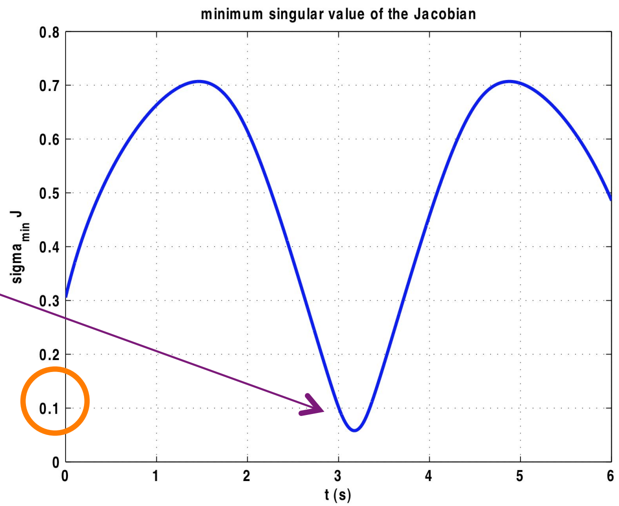
path at α=178°

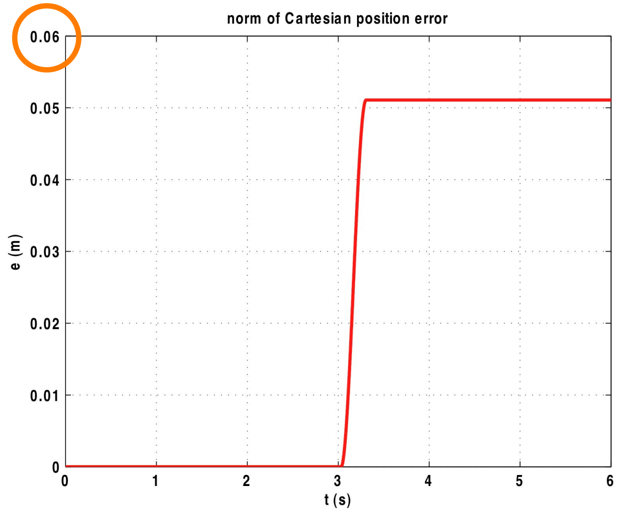




saturated value of $\dot{q}_1$

close to singular case





actual position error!! (6 cm)

Damped Least Squares method

$$\min_{\dot{q}} H = \frac{\lambda}{2}\|\dot{q}\|^2 + \frac{1}{2}\|J\dot{q} - v\|^2, \qquad \lambda \geq 0$$

J_{DLS}

$$\dot{q} = (\lambda I_n + J^T J)^{-1} J^T v = J^T(\lambda I_m + J J^T)^{-1} v$$

equivalent expressions, but this one is more convenient in redundant robots!

- inversion of differential kinematics as an optimization problem
- function H = weighted sum of two objectives (minimum error norm on achieved end-effector velocity and minimum norm of joint velocity)
- $\lambda = 0$ when "far enough" from a singularity
- with $\lambda > 0$, there is a (vector) error ε ($= v - J\dot{q}$) in executing the desired end-effector velocity v (check that $\varepsilon = \lambda \left(\lambda I_m + J J^T\right)^{-1} v$!), but the joint velocities are always reduced ("damped")
- J_{DLS} can be used for both m = n and m < n cases

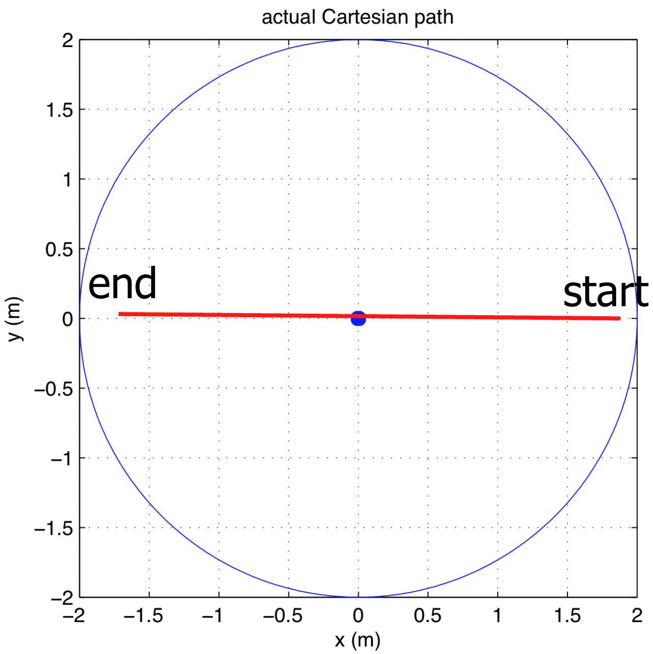
Simulation results

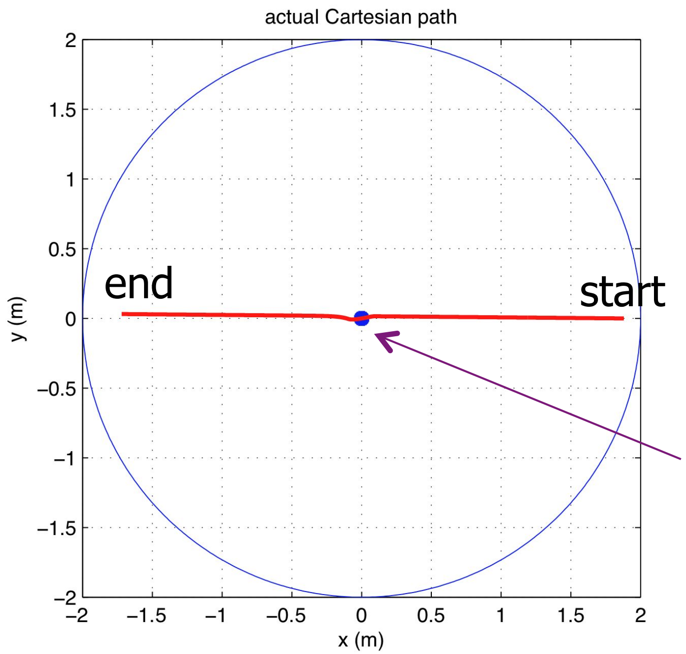
planar 2R robot in straight line Cartesian motion

a comparison of inverse and damped inverse Jacobian methods
even closer to singular case

$\boxed{\dot{q} = J^{-1}(q)\, v}$

$\boxed{\dot{q} = J_{DLS}(q)\, v}$





a line from right to left, at α=179.5° angle with x-axis,
executed at constant speed v=0.6 m/s for T=6 s

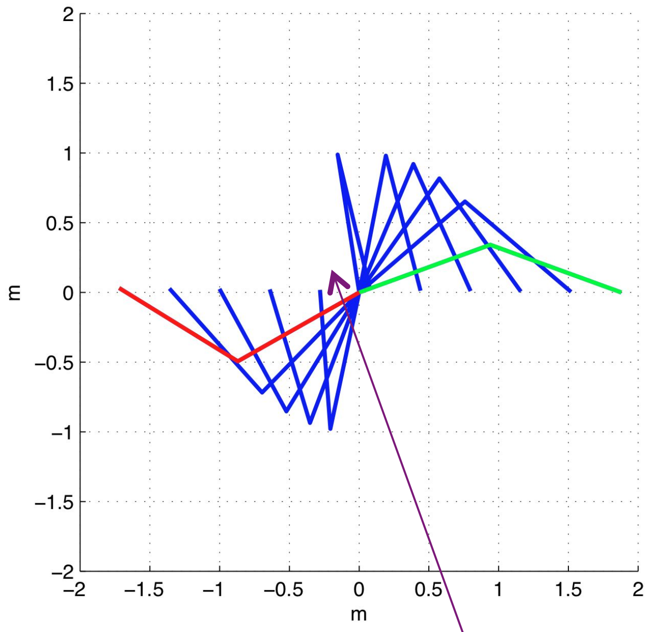
Simulation results

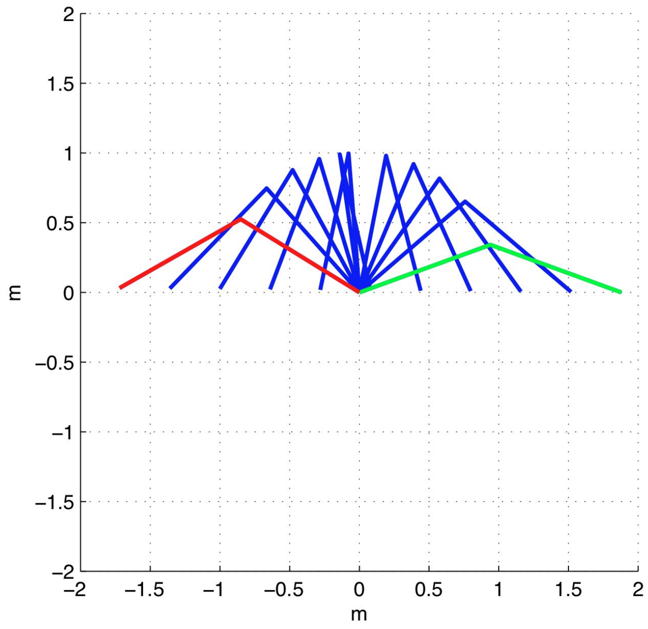
planar 2R robot in straight line Cartesian motion

$\boxed{\dot{q} = J^{-1}(q)\, v}$

path at
α=179.5°

$\boxed{\dot{q} = J_{DLS}(q)\, v}$





here, a very fast
reconfiguration of
first joint ...

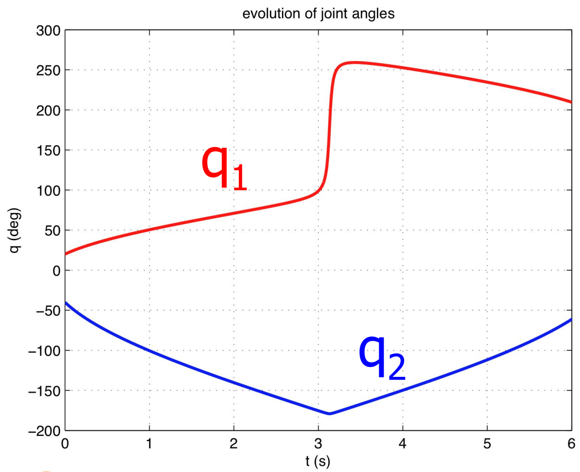
a completely different inverse solution,
around/after crossing the region
close to the folded singularity

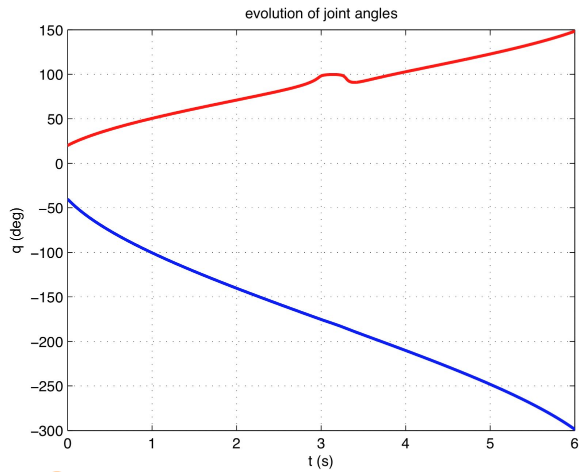
Simulation results

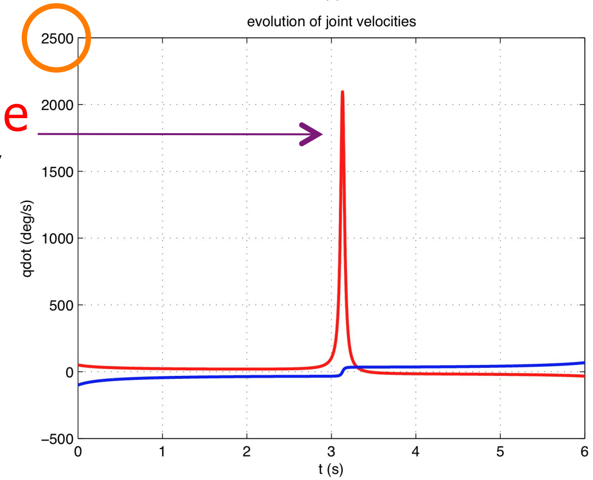
planar 2R robot in straight line Cartesian motion

$\dot{q} = J^{-1}(q)\, v$

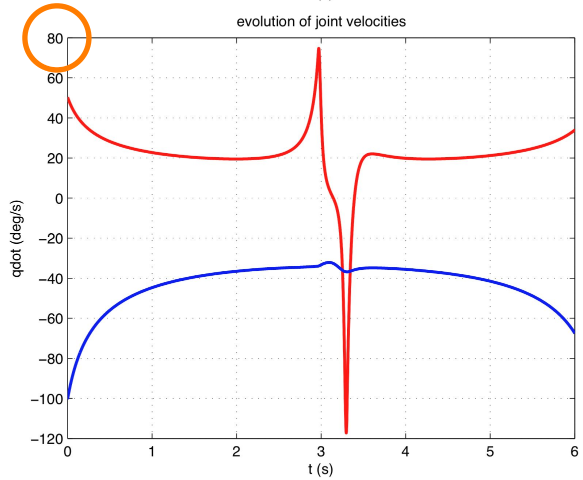
$\dot{q} = J_{DLS}(q)\, v$







extremely large peak velocity of first joint!!

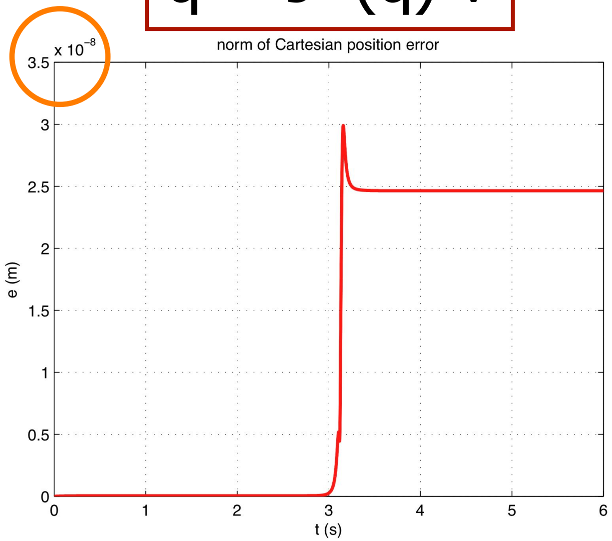


smooth joint motion with limited joint velocities!

Simulation results

planar 2R robot in straight line Cartesian motion

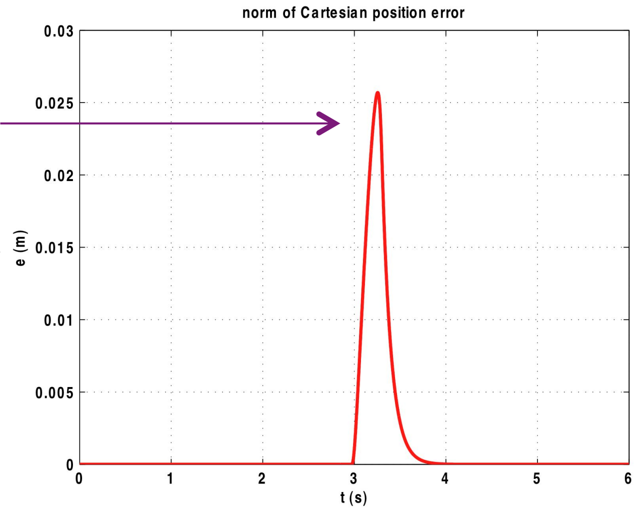
$$\boxed{\dot{q} = J^{-1}(q)\, v}$$



increased numerical integration error ($3 \cdot 10^{-8}$)

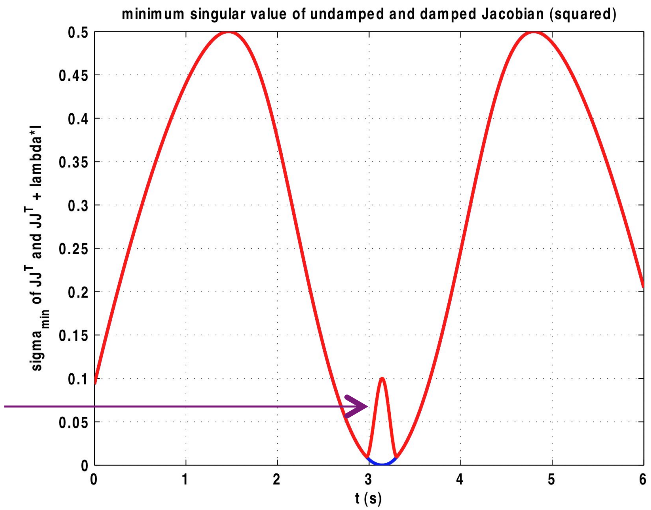
$$\boxed{\dot{q} = J_{DLS}(q)\, v}$$

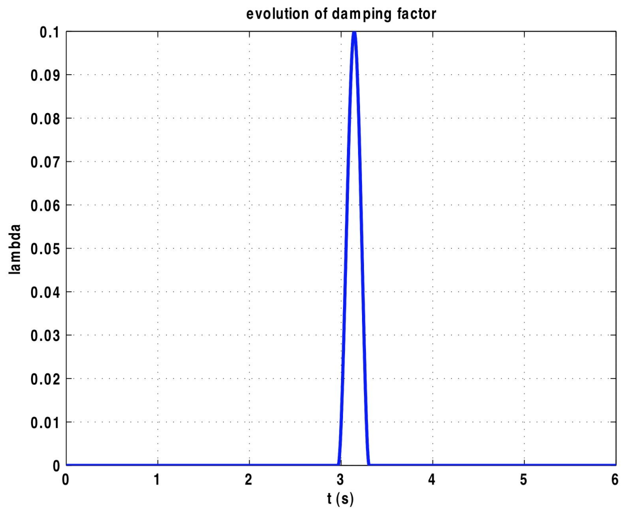
error (25 mm) when crossing the singularity, later recovered by feedback action ($v \Rightarrow v+Ke$)



minimum singular value of JJ^T and $\lambda I+JJ^T$

they differ only when damping factor is non-zero





damping factor λ is chosen non-zero only close to singularity!

Use of the pseudo-inverse

a constrained optimization (minimum norm) problem

$$\min_{\dot q} H = \frac{1}{2}\|\dot q\|^2 \qquad \text{such that} \qquad J\dot q - v = 0$$

solution $\boxed{\dot q = J^{\#} v}$ pseudo-inverse of J

- if $v \in \mathcal{R}(J)$, the constraint is satisfied (v is feasible)
- else $J\dot q = v^{\perp}$ where $v^{\perp}$ minimizes the error $\|J\dot q - v\|$

orthogonal projection of v on $\mathcal{R}(J)$

Properties of the pseudo-inverse

it is the unique matrix that satisfies the four relationships

- $JJ^\sharp J = J \quad J^\sharp J J^\sharp = J^\sharp$

  $(J^\sharp J)^T = J^\sharp J \quad (JJ^\sharp)^T = JJ^\sharp$

- if rank $\rho = m = n$: $\quad J^\sharp = J^{-1}$

- if $\rho = m < n$: $\quad J^\sharp = J^T(JJ^T)^{-1}$

it always exists and is computed in general numerically using the SVD = Singular Value Decomposition of J

(e.g., with the MATLAB function **pinv**)

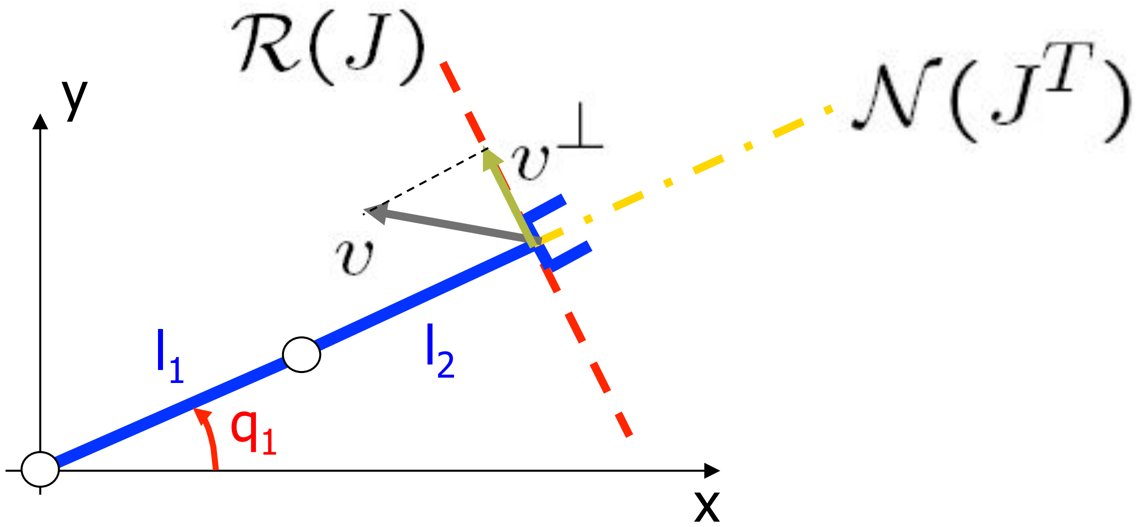
Numerical example

Jacobian of 2R arm with $l_1 = l_2 = 1$ and $q_2 = 0$ (rank $\rho = 1$)

$$J = \begin{bmatrix} -2s_1 & -s_1 \\ 2c_1 & c_1 \end{bmatrix} \qquad J^\sharp = \frac{1}{5}\begin{bmatrix} -2s_1 & 2c_1 \\ -s_1 & c_1 \end{bmatrix}$$

$$\dot{q} = J^\sharp v$$

is the minimum norm joint velocity vector that realizes $v^\perp$



General solution for m<n

all solutions (an infinite number) of the inverse differential kinematics problem can be written as

$$\dot{q} = J^{\#} v + (I - J^{\#} J)\xi$$

any joint velocity...

"projection" matrix in the kernel of J

this is also the solution to a slightly modified constrained optimization problem (biased toward the joint velocity ξ, chosen to avoid obstacles, joint limits, etc.)

$$\min H = \frac{1}{2}\|\dot{q} - \xi\|^2 \qquad \text{such that} \qquad J\dot{q} - v = 0$$

verification of which actual task velocity is going to be obtained

$$v_{actual} = J\dot{q} = J\left(J^{\#} v + (I - J^{\#} J)\xi\right) = JJ^{\#} v + (\cancel{J - JJ^{\#} J})\xi = JJ^{\#}(Jw) = (JJ^{\#} J)w = Jw = v$$

if $v \in \Re(J) \Rightarrow v = Jw,$ for some w

Higher-order differential inversion

- inversion of motion from task to joint space can be performed also at a higher differential level
- acceleration-level: given q, $\dot{q}$

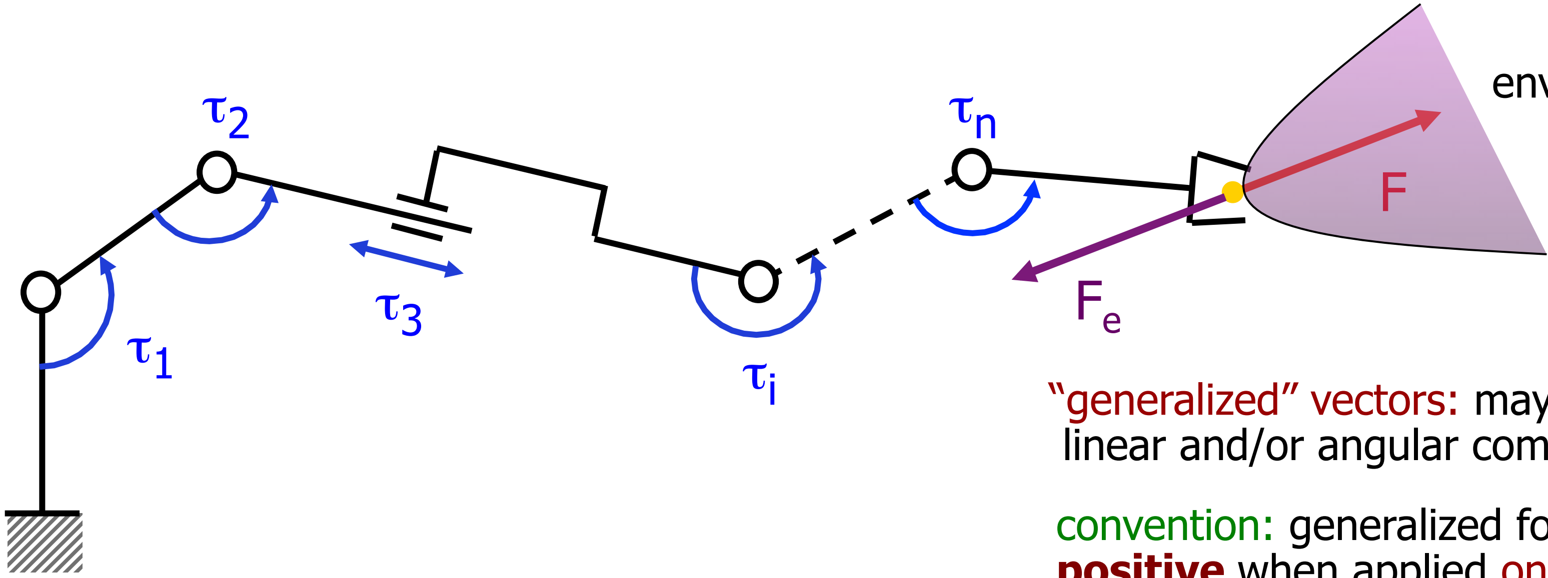
$$\ddot{q} = J_r^{-1}(q)\left(\ddot{r} - \dot{J}_r(q)\dot{q}\right)$$

- jerk-level: given q, $\dot{q}$, $\ddot{q}$

$$\dddot{q} = J_r^{-1}(q)\left(\dddot{r} - \dot{J}_r(q)\ddot{q} - 2\ddot{J}_r(q)\dot{q}\right)$$

- the (inverse) of the Jacobian is always the leading term
- smoother joint motions are expected (at least, due to the existence of higher-order time derivatives $\ddot{r}$, $\dddot{r}$, ...)

Generalized forces and torques



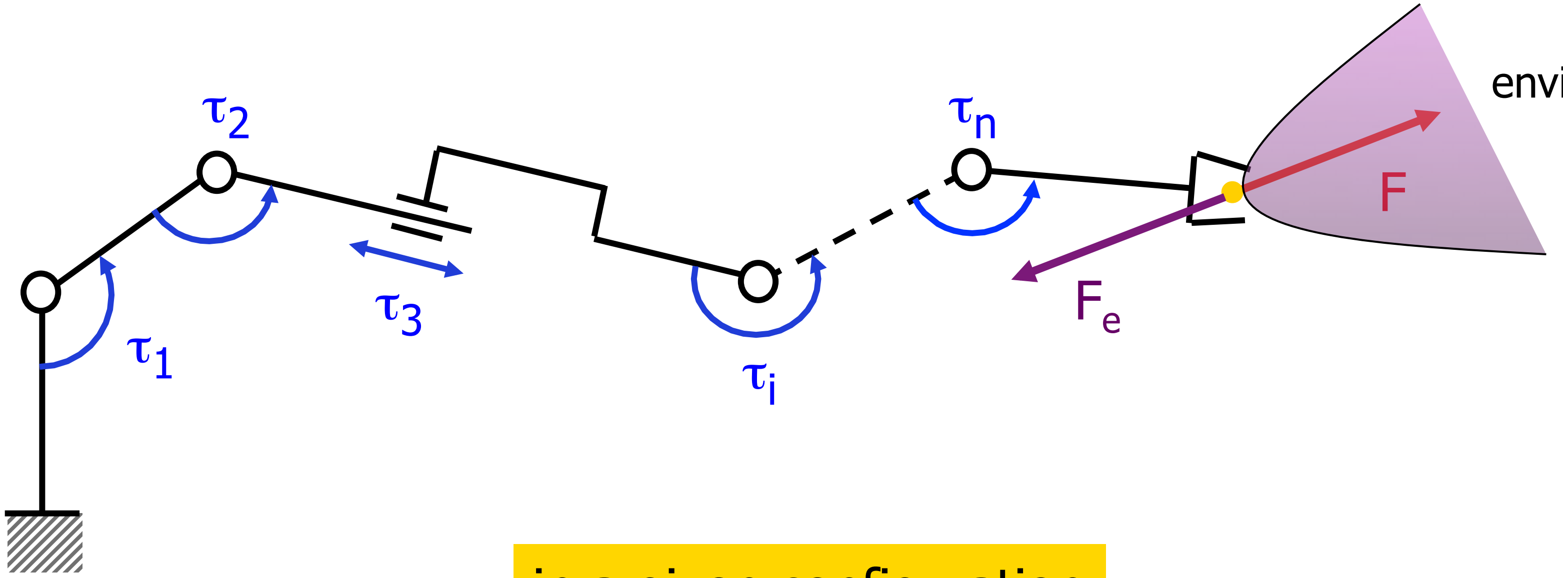
"generalized" vectors: may contain linear and/or angular components

convention: generalized forces are **positive** when applied on the robot

- τ = forces/torques exerted by the motors at the robot joints
- F = equivalent forces/torques exerted at the robot end-effector
- F_e = forces/torques exerted by the environment at the end-effector
- principle of action and reaction: $\boxed{F_e = -F}$

reaction from environment is equal and opposite to the robot action on it

Transformation of forces – Statics



in a given configuration

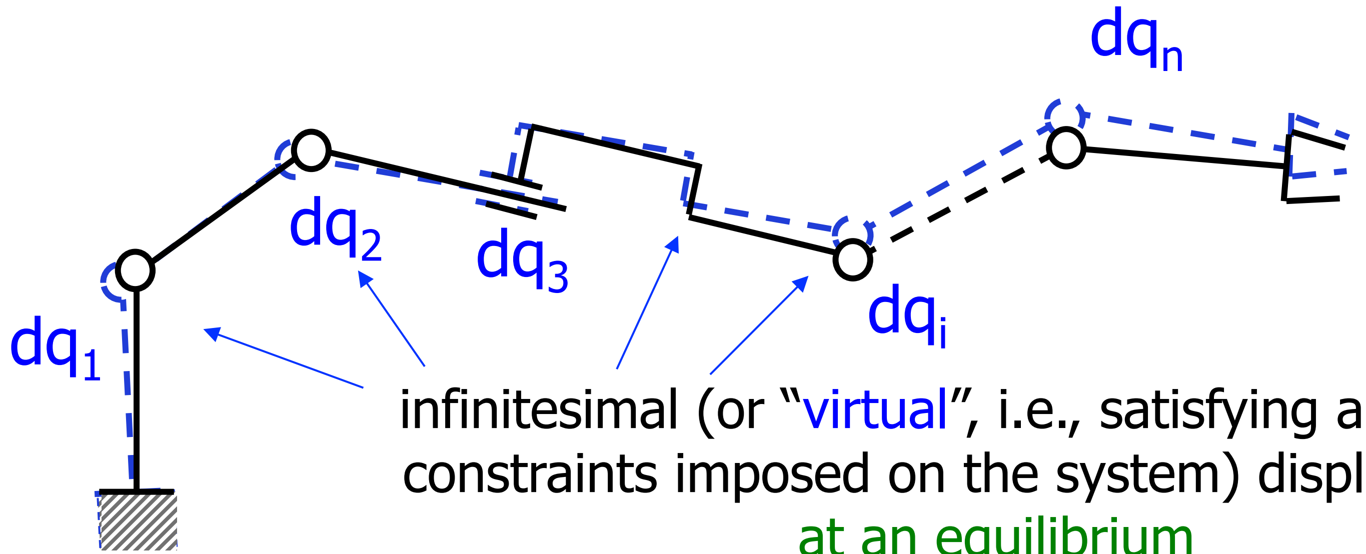
- what is the transformation between F at robot end-effector and τ at joints?

in **static equilibrium** conditions (i.e., **no motion**):

- what F will be exerted on environment by a τ applied at the robot joints?
- what τ at the joints will balance a F_e (= -F) exerted by the environment?

all equivalent formulations

Virtual displacements and works



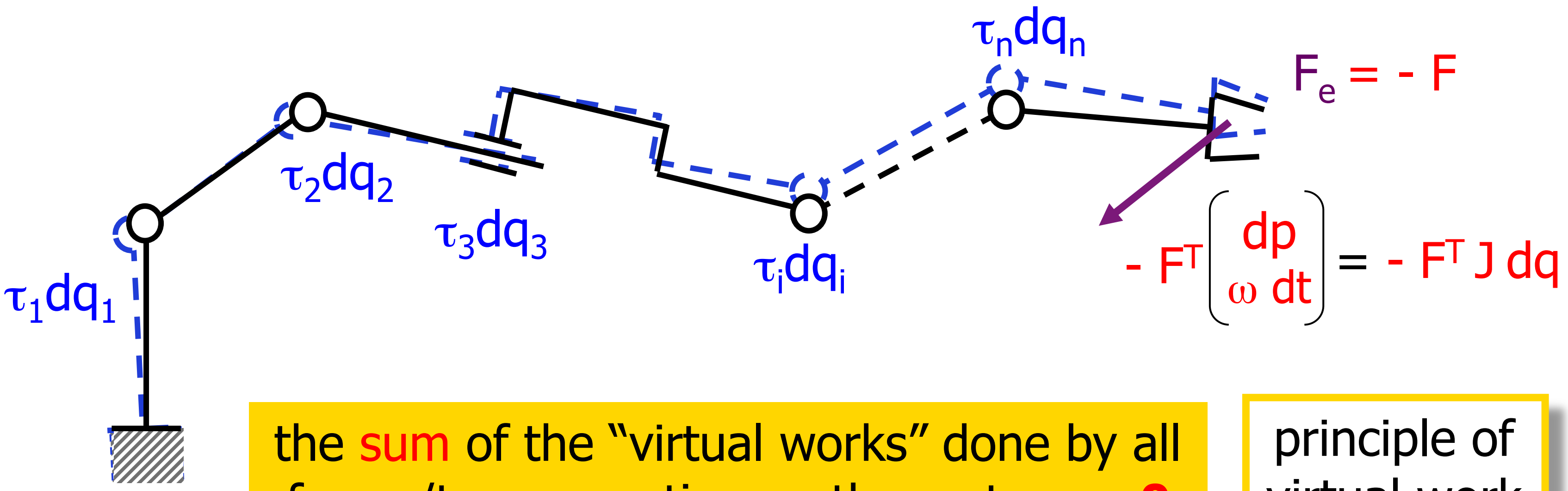
$$\begin{pmatrix} dp \\ \omega\, dt \end{pmatrix} = J\, dq$$

infinitesimal (or "virtual", i.e., satisfying all possible constraints imposed on the system) displacements at an equilibrium

- without kinetic energy variation (zero acceleration)
- without dissipative effects (zero velocity)

the "virtual work" is the work done by all forces/torques acting on the system for a given virtual displacement

Principle of virtual work



the sum of the "virtual works" done by all forces/torques acting on the system **= 0**

principle of virtual work

$$\tau^T dq - F^T \begin{bmatrix} dp \\ \omega dt \end{bmatrix} = \tau^T dq - F^T J dq = 0 \qquad \boxed{\forall dq}$$

$$\Rightarrow \quad \boxed{\tau = J^T(q) F}$$

Duality between velocity and force

$J(q)$

velocity $\dot{q}$
(or displacement dq)
in the joint space

generalized velocity v
(or e-e displacement $\begin{bmatrix} dp \\ \omega dt \end{bmatrix}$)
in the Cartesian space

---

forces/torques τ
at the joints

generalized forces F
at the Cartesian e-e

$J^T(q)$

the singular configurations
for the velocity map are the same
as those for the force map

$\rho(J) = \rho(J^T)$

Dual subspaces of velocity and force

summary of definitions

$$\mathcal{R}(J) = \{v \in I\!R^m : \exists \dot{q} \in I\!R^n, J\dot{q} = v\}$$

$$\mathcal{N}(J^T) = \{v \in I\!R^m : \nexists \dot{q} \in I\!R^n, J\dot{q} = v\}$$

$$= \{F \in I\!R^m : J^T F = 0\}$$

$$\mathcal{R}(J) + \mathcal{N}(J^T) = I\!R^m$$

$$\mathcal{R}(J^T) = \{\tau \in I\!R^n : \exists F \in I\!R^m, J^T F = \tau\}$$

$$\mathcal{N}(J) = \{\tau \in I\!R^n : \nexists F \in I\!R^m, J^T F = \tau\}$$

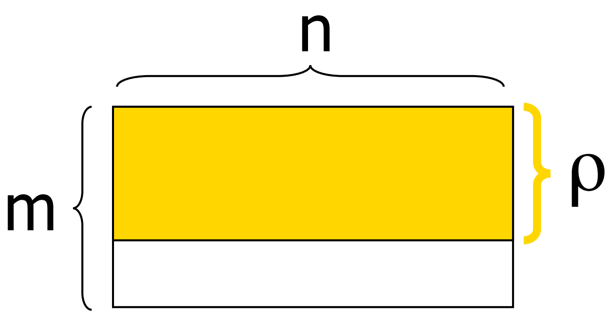
$$= \{\dot{q} \in I\!R^n : J\dot{q} = 0\}$$

$$\mathcal{R}(J^T) + \mathcal{N}(J) = I\!R^n$$

Velocity and force singularities

list of possible cases

ρ = rank(J) = rank(J^T) $\leq$ min(m,n)

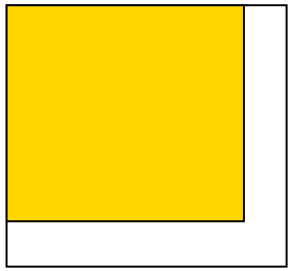


1. $\rho < m$

$$\exists \dot{q} \neq 0 : \quad J\dot{q} = 0$$
$$\exists F \neq 0 : \quad J^T F = 0$$

2. $\rho = m$

$$\exists \dot{q} \neq 0 : \quad J\dot{q} = 0$$
$$\mathcal{N}(J^T) = \{0\}$$

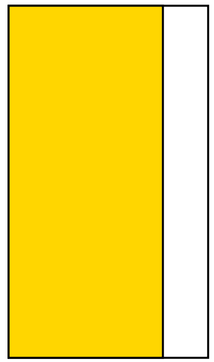


1. det J = 0

$$\exists \dot{q} \neq 0 : \quad J\dot{q} = 0$$
$$\exists F \neq 0 : \quad J^T F = 0$$

2. det J $\neq$ 0

$$\mathcal{N}(J) = \{0\}$$
$$\mathcal{N}(J^T) = \{0\}$$



1. $\rho < n$

$$\exists \dot{q} \neq 0 : \quad J\dot{q} = 0$$
$$\exists F \neq 0 : \quad J^T F = 0$$

2. $\rho = n$

$$\exists F \neq 0 : \quad J^T F = 0$$
$$\mathcal{N}(J) = \{0\}$$

Example of singularity analysis

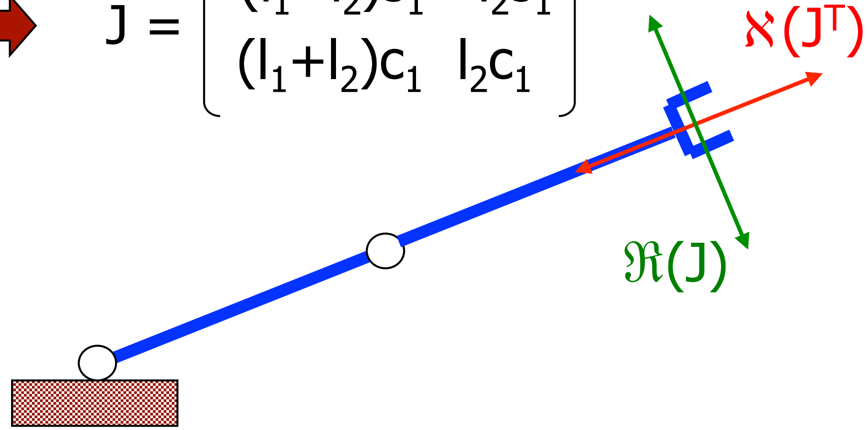
planar 2R arm with generic link lengths l_1 and l_2

$$J(q) = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix} \qquad \det J(q) = l_1 l_2 s_2$$

singularity at $q_2 = 0$ (arm straight) $\Rightarrow$ $J = \begin{pmatrix} -(l_1+l_2)s_1 & -l_2 s_1 \\ (l_1+l_2)c_1 & l_2 c_1 \end{pmatrix}$

$$\Re(J) = \alpha \begin{pmatrix} -s_1 \\ c_1 \end{pmatrix} \qquad \aleph(J^T) = \alpha \begin{pmatrix} c_1 \\ s_1 \end{pmatrix}$$

$$\Re(J^T) = \beta \begin{pmatrix} l_1+l_2 \\ l_2 \end{pmatrix} \qquad \aleph(J) = \beta \begin{pmatrix} l_2 \\ -(l_1+l_2) \end{pmatrix}$$



singularity at $q_2 = \pi$ (arm folded) $\Rightarrow$ $J = \begin{pmatrix} (l_2-l_1)s_1 & l_2 s_1 \\ -(l_2-l_1)c_1 & -l_2 c_1 \end{pmatrix}$

$\Re(J)$ and $\aleph(J^T)$ as above

$$\Re(J^T) = \beta \begin{pmatrix} l_2-l_1 \\ l_2 \end{pmatrix} \left(\text{for } l_1 = l_2,\ \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \qquad \aleph(J) = \beta \begin{pmatrix} l_2 \\ -(l_2-l_1) \end{pmatrix} \left(\text{for } l_1 = l_2,\ \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

Velocity manipulability

- in a given configuration, we wish to evaluate how "effective" is the mechanical transformation between joint velocities and end-effector velocities
  - "how easily" can the end-effector be moved in the various directions of the task space
  - equivalently, "how far" is the robot from a singular condition
- we consider all end-effector velocities that can be obtained by choosing joint velocity vectors of unit norm

$$\dot{q}^T \dot{q} = 1 \quad \Rightarrow \quad \boxed{v^T J^{\#T} J^{\#} v = 1}$$

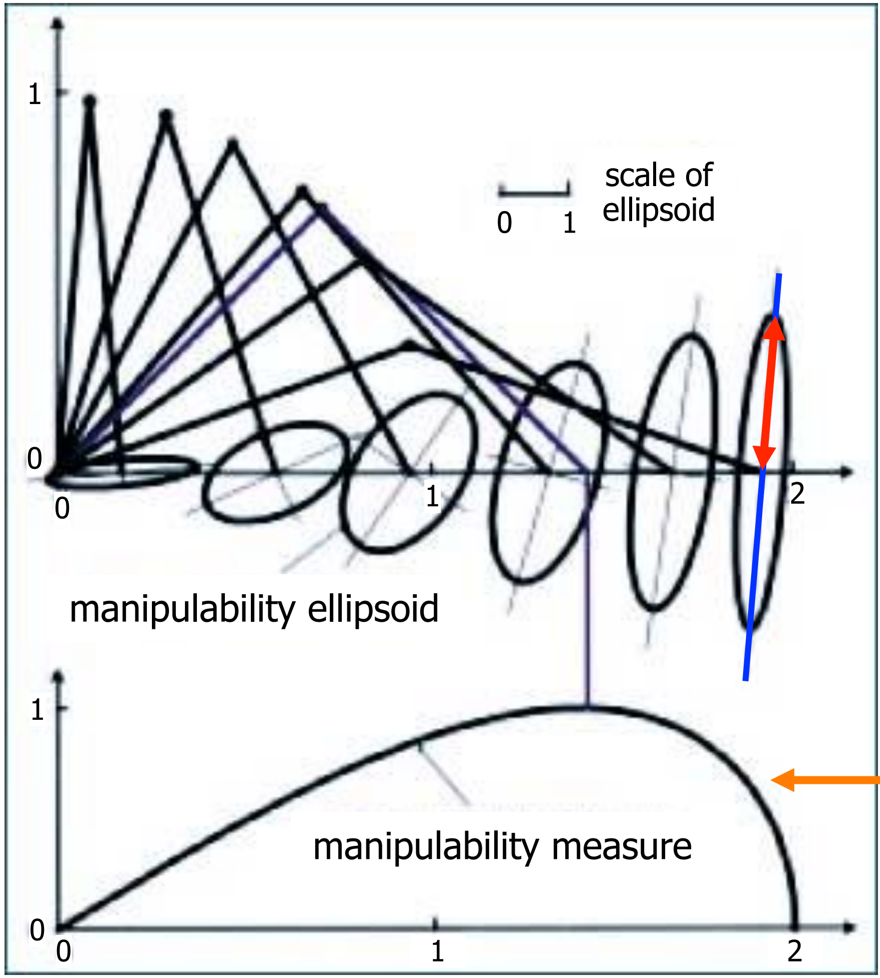
task velocity manipulability ellipsoid

$$\boxed{(JJ^T)^{-1}} \quad \text{if } \rho = m$$

note: the "core" matrix of the ellipsoid equation $v^T A^{-1} v=1$ is the matrix A!

Manipulability ellipsoid
in velocity

planar 2R arm with unitary links



length of principal (semi-)axes:
singular values of J (in its SVD)

$$\sigma_i\{J\} = \sqrt{\lambda_i\{JJ^T\}} \geq 0$$

in a singularity, the ellipsoid loses a dimension
(for m=2, it becomes a segment)

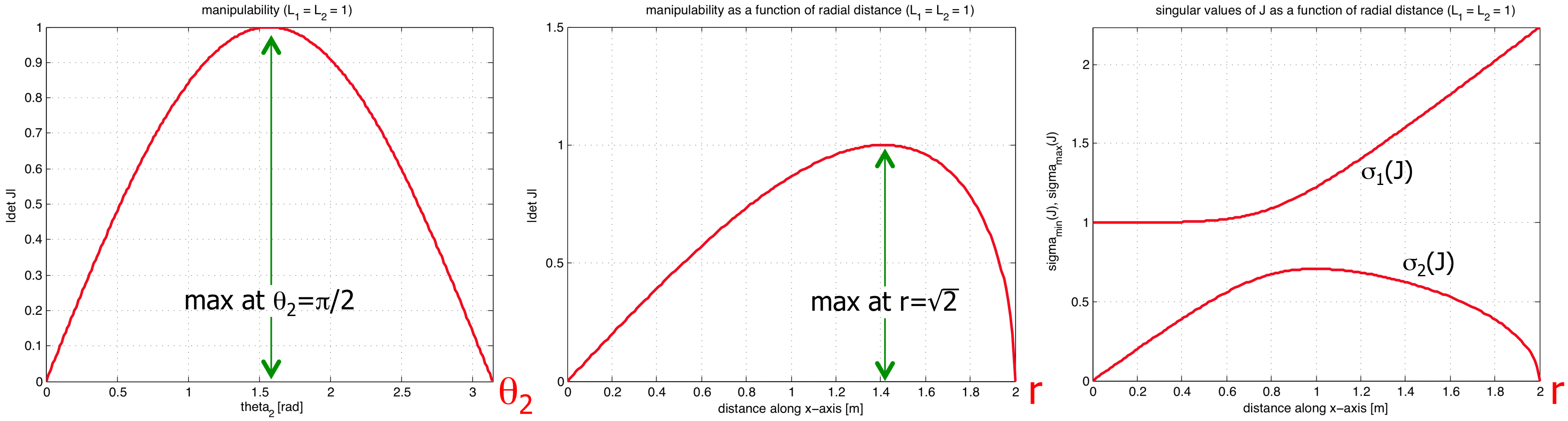
direction of principal axes:
(orthogonal) eigenvectors associated to λ_i

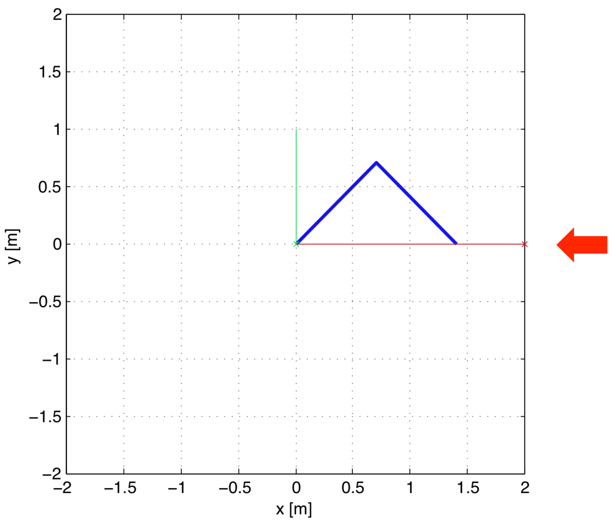
$$w = \sqrt{\det JJ^T} = \prod_{i=1}^{m} \sigma_i \geq 0$$

proportional to the volume of the ellipsoid (for m=2, to its area)

Manipulability measure

planar 2R arm with unitary links: Jacobian J is square ➡ $\sqrt{\det\left(JJ^T\right)} = \sqrt{\det J \cdot \det J^T} = \left|\det J\right| = \prod_{i=1}^{2}\sigma_i$





best posture for manipulation
(similar to a human arm!)

full isotropy is never obtained
in this case, since it always $\sigma_1 \neq \sigma_2$

Force manipulability

- in a given configuration, evaluate how "effective" is the transformation between joint torques and end-effector forces
  - "how easily" can the end-effector apply generalized forces (or balance applied ones) in the various directions of the task space
  - in singular configurations, there are directions in the task space where external forces/torques are balanced by the robot without the need of any joint torque
- we consider all end-effector forces that can be applied (or balanced) by choosing joint torque vectors of unit norm

$$\tau^T \tau = 1 \quad \Rightarrow \quad \boxed{F^T J J^T F = 1}$$

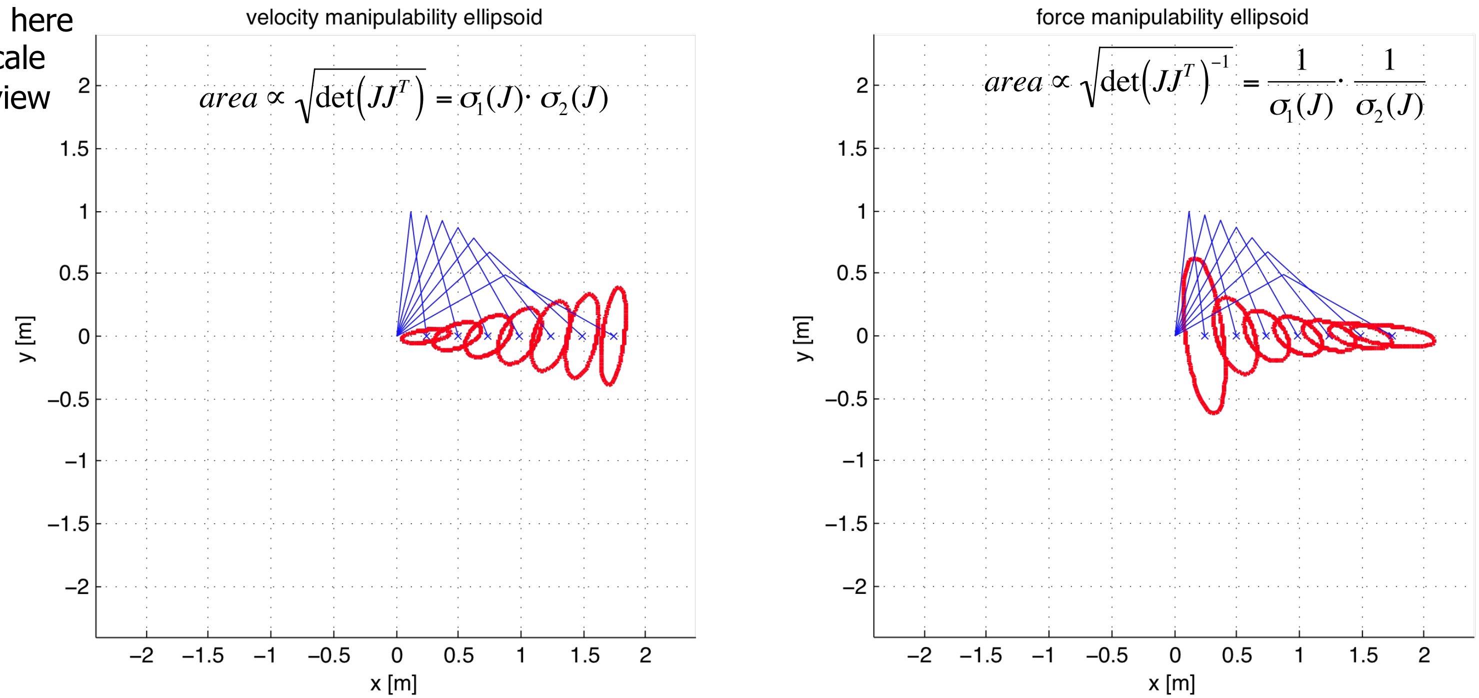
same directions of the principal axes of the velocity ellipsoid, but with semi-axes of inverse lengths → task force manipulability ellipsoid

Velocity and force manipulability

dual comparison of actuation vs. control

note:
velocity and force ellipsoids have here a different scale for a better view

planar 2R arm with unitary links



Cartesian **actuation** task (high joint-to-task transformation ratio):
preferred velocity (or force) directions are those where the ellipsoid *stretches*

⇕

Cartesian **control** task (low transformation ratio = high resolution):
preferred velocity (or force) directions are those where the ellipsoid *shrinks*

Velocity and force transformations

- the same reasoning made for relating end-effector to joint forces/torques (static equilibrium + principle of virtual work) is used also for relating forces and torques applied at different places of a rigid body and/or expressed in different reference frames

relation among generalized velocities

$$\begin{bmatrix} v_A \\ \omega_A \end{bmatrix} = \begin{bmatrix} {}^A R_B & -{}^A R_B S({}^B r_{BA}) \\ 0 & {}^A R_B \end{bmatrix} \begin{bmatrix} v_B \\ \omega_B \end{bmatrix} = J_{BA} \begin{bmatrix} v_B \\ \omega_B \end{bmatrix}$$

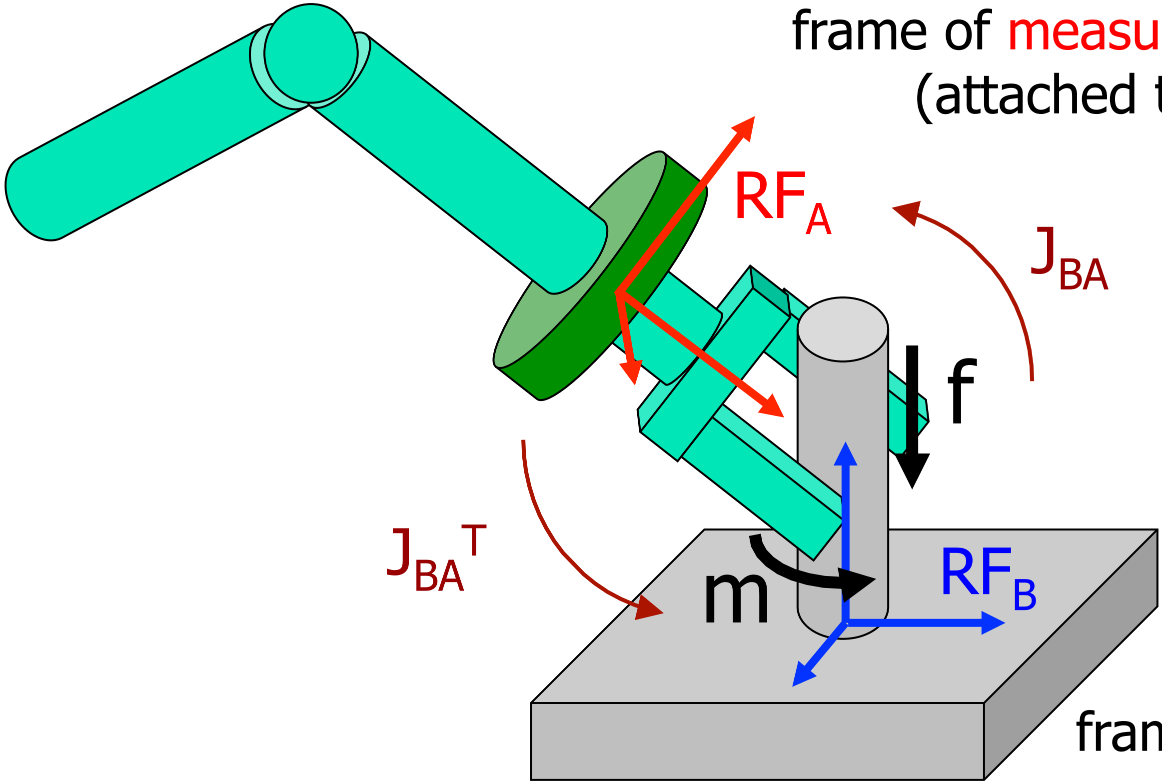
$$\Updownarrow$$

$$\begin{bmatrix} f_B \\ m_B \end{bmatrix} = J_{BA}^T \begin{bmatrix} f_A \\ m_A \end{bmatrix} = \begin{bmatrix} {}^B R_A & 0 \\ S({}^B r_{BA})\,{}^B R_A & {}^B R_A \end{bmatrix} \begin{bmatrix} f_A \\ m_A \end{bmatrix}$$

relation among generalized forces

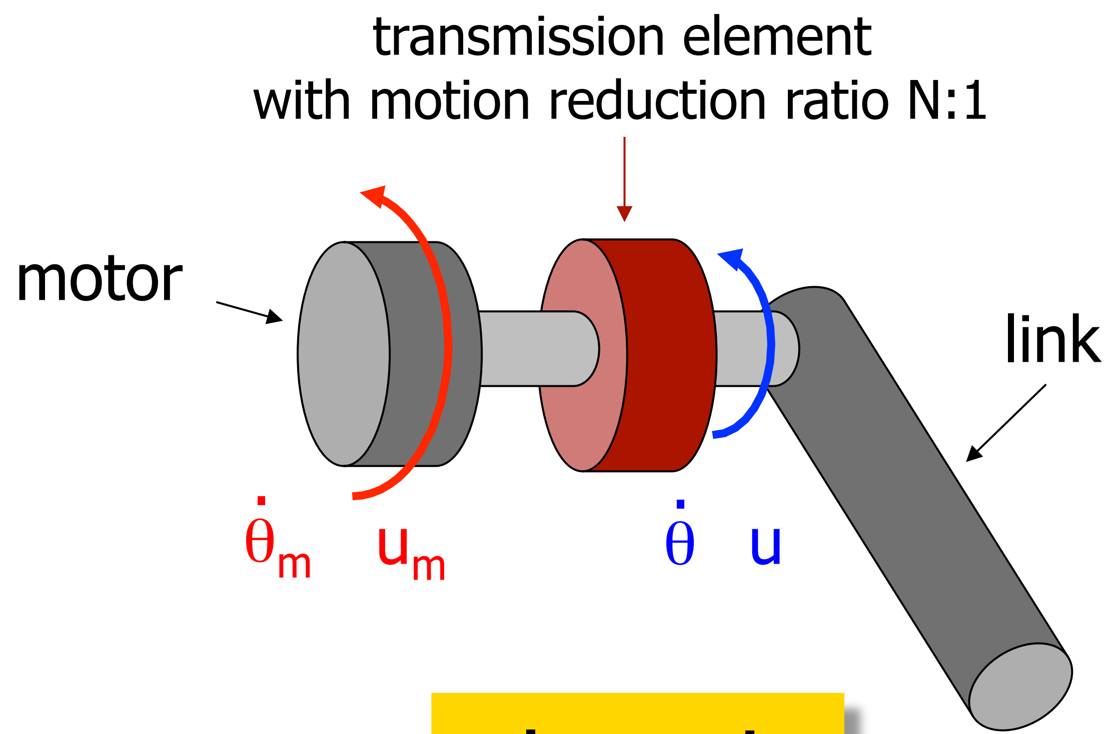
Example 1: 6D force/torque sensor

frame of measure for the forces/torques
(attached to the wrist sensor)



frame of interest for evaluating
forces/torques in a task
with environment contact

Example 2: Gear reduction at joints



one of the simplest applications
of the principle of virtual work!

$$\dot{\theta}_m = N\dot{\theta}$$

$$u = N u_m$$