Video Segment

DARPA Robotics Challenge 2013 - A Woodstock for Robots, A. Aden-Buie and J. Markoff, 2013.

Dynamics

- Rigid Body Dynamics
- Newton-Euler Formulation
- Articulated Multi-Body Dynamics
- Recursive Algorithm
- Lagrange Formulation
- Explicit Form

Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

Lagrangian $\rightarrow$ $L = K - U$ (Kinetic Energy: K; Potential Energy: U)

Since $U = U(q)$

$$\Rightarrow \underbrace{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q}}_{\text{Inertial forces}} + \underbrace{\frac{\partial U}{\partial q}}_{\text{Gravity vector}} = \tau$$

Lagrange Equations

$$\underbrace{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q}}_{\text{Inertial forces}} = \tau - G; \quad G = \frac{\partial U}{\partial q}$$

$$\Downarrow$$

$$M(q)\ddot{q} + V(q,\dot{q}) = \tau - G(q)$$

Inertial forces

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q}\right)-\frac{\partial K}{\partial q}=\tau-G \qquad K=\frac{1}{2}\dot q^T M(q)\dot q$$

$$\frac{\partial K}{\partial \dot q}=\frac{\partial}{\partial \dot q}\left[\frac{1}{2}\dot q^T M(q)\dot q\right]=M(q)\dot q$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q}\right)=\frac{d}{dt}(M\dot q)=M\ddot q+\dot M\dot q$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q}\right)-\frac{\partial K}{\partial q}=M\ddot q+\dot M\dot q-\frac{1}{2}\begin{bmatrix}\dot q^T\frac{\partial M}{\partial q_1}\dot q\\ \vdots\\ \dot q^T\frac{\partial M}{\partial q_n}\dot q\end{bmatrix}=M\ddot q+V(q,\dot q)$$

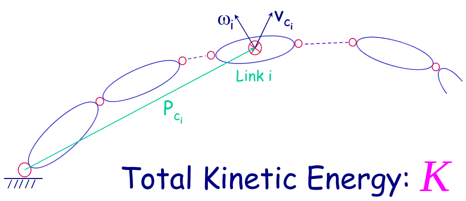
Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q}\right)-\frac{\partial K}{\partial q}=M\ddot q+\dot M\dot q-\frac{1}{2}\begin{bmatrix}\dot q^T\frac{\partial M}{\partial q_1}\dot q\\ \vdots\\ \dot q^T\frac{\partial M}{\partial q_n}\dot q\end{bmatrix}=M\ddot q+V(q,\dot q)$$

$$M(q)\ddot q \quad V(q,\dot q) \quad G(q)$$

$$M(q): K \quad \frac{1}{2}\dot q^T M\dot q \qquad M(q) \Rightarrow V(q,\dot q)$$

Equations of Motion

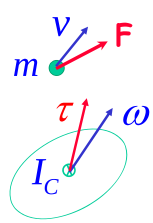


Total Kinetic Energy: K

$$K=\sum K_{Link\,i}\equiv\frac{1}{2}\dot q^T M\dot q$$

Kinetic Energy

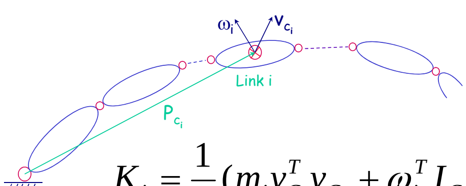
Work done by external forces to bring the system from rest to its current state.



$$K=\frac{1}{2}mv^2$$

$$K=\frac{1}{2}\omega^T I_C\,\omega$$

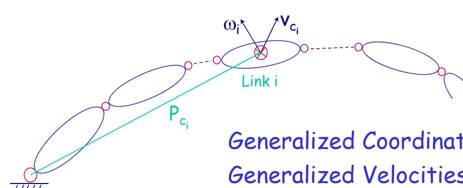
Equations of Motion — Explicit Form



$$K_i=\frac{1}{2}(m_i v_{C_i}^T v_{C_i}+\omega_i^T I_{Ci}\omega_i)$$

Total Kinetic Energy ⇨ $K=\sum_{i=1}^{n}K_i$

Equations of Motion — Explicit Form



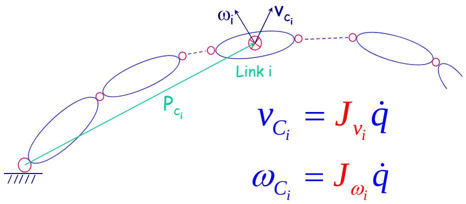
Generalized Coordinates q

Generalized Velocities $\dot q$

Kinetic Energy Quadratic Form of Generalized Velocities $K=\frac{1}{2}\dot q^T M\dot q$

$$\frac{1}{2}\dot q^T M\dot q\equiv\frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i}+\omega_i^T I_{Ci}\omega_i)$$

Equations of Motion — Explicit Form



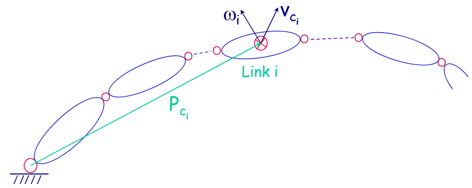
$$v_{C_i} = J_{v_i}\dot{q}$$

$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

$$\frac{1}{2}\dot{q}^T M\,\dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i}\omega_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n}(m_i \dot{q}^T J_{v_i}^T J_{v_i}\dot{q} + \dot{q}^T J_{\omega_i}^T I_{Ci} J_{\omega_i}\dot{q})$$

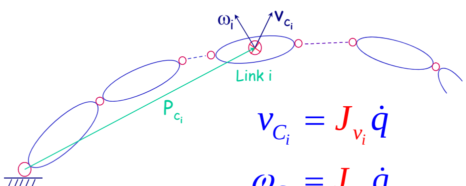
Equations of Motion — Explicit Form



$$\frac{1}{2}\dot{q}^T M\,\dot{q} = \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{Ci} J_{\omega_i})\right]\dot{q}$$

$$M = \sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{Ci} J_{\omega_i})$$

Equations of Motion — Explicit Form

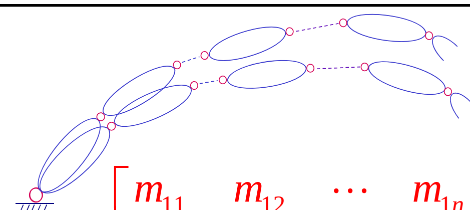


$$v_{C_i} = J_{v_i}\dot{q}$$

$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

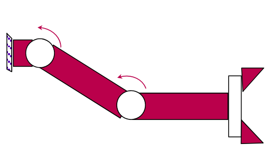
$$J_{v_i} = [\frac{\partial p_{C_i}}{\partial q_1} \quad \frac{\partial p_{C_i}}{\partial q_2} \quad \cdots \quad \frac{\partial p_{C_i}}{\partial q_i} \quad 0 \quad 0 \quad \cdots \quad 0]$$

$$J_{\omega_i} = [\bar{\varepsilon}_1 z_1 \quad \bar{\varepsilon}_2 z_2 \quad \cdots \quad \bar{\varepsilon}_i z_i \quad 0 \quad 0 \quad \cdots \quad 0]$$



$$M(q)_{(n\times n)} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

Vector $V(\mathbf{q},\dot{\mathbf{q}})$ — Centrifugal & Coriolis Forces



$$\begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}\begin{bmatrix}\ddot{q}_1 \\ \ddot{q}_2\end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_n}\dot{q} \end{bmatrix} = M\ddot{q} + V(q,\dot{q})$$

$$M(q)\ddot{q} \quad V(q,\dot{q}) \quad G(q)$$

$$M(q): K \quad \frac{1}{2}\dot{q}^T M\dot{q} \qquad M(q) \Rightarrow V(q,\dot{q})$$

Vector $V(\mathbf{q},\dot{\mathbf{q}})$

$$\frac{M}{q_1} \qquad \frac{\partial m_{11}}{\partial q_1}$$

$$V \quad \dot{M}\dot{\mathbf{q}} \quad \frac{1}{2}\begin{bmatrix}\dot{\mathbf{q}}^T M_{q_1}\dot{\mathbf{q}}\\ \dot{\mathbf{q}}^T M_{q_2}\dot{\mathbf{q}}\end{bmatrix} \quad \begin{pmatrix}\dot{m}_{11} & \dot{m}_{12}\\ \dot{m}_{12} & \dot{m}_{22}\end{pmatrix}\dot{\mathbf{q}} \quad \frac{1}{2}\begin{bmatrix}\dot{\mathbf{q}}^T\begin{pmatrix}m_{111} & m_{121}\\ m_{121} & m_{221}\end{pmatrix}\dot{\mathbf{q}}\\ \dot{\mathbf{q}}^T\begin{pmatrix}m_{112} & m_{122}\\ m_{122} & m_{222}\end{pmatrix}\dot{\mathbf{q}}\end{bmatrix}$$

$$\dot{m}_{ij} = m_{ij1}\dot{q}_1 + m_{ij2}\dot{q}_2 \qquad \frac{\partial m_{22}}{\partial q_2}$$

$$V(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix}\frac{1}{2}(m_{111}+m_{111}-m_{111}) & \frac{1}{2}(m_{122}+m_{122}-m_{221})\\ \frac{1}{2}(m_{211}+m_{211}-m_{112}) & \frac{1}{2}(m_{222}+m_{222}-m_{222})\end{bmatrix}\begin{bmatrix}\dot{q}_1^2\\ \dot{q}_2^2\end{bmatrix}$$

$$\begin{bmatrix}m_{112} & m_{121} & m_{121}\\ m_{212} & m_{221} & m_{122}\end{bmatrix}[\dot{q}_1\dot{q}_2]$$

Christoffel Symbols

$$b_{ijk} \quad \frac{1}{2}(m_{ijk} \quad m_{ikj} \quad m_{jki}) \qquad \frac{m_{ij}}{q_k}$$

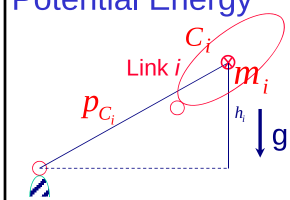
$$V \quad \begin{bmatrix}b_{111} & b_{122}\\ b_{211} & b_{222}\end{bmatrix}\begin{bmatrix}\dot{q}_1^2\\ \dot{q}_2^2\end{bmatrix} \quad \begin{bmatrix}2b_{112}\\ 2b_{212}\end{bmatrix}[\dot{q}_1\dot{q}_2]$$

$C(\mathbf{q})$ $\quad$ $B(\mathbf{q})$

$$\underset{(n\ n)}{C(\mathbf{q})}\underset{(n\ 1)}{[\dot{\mathbf{q}}^2]} \quad \begin{bmatrix}b_{1,11} & b_{1,22} & \cdots & b_{1,nn}\\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn}\\ \vdots & \vdots & \vdots & \vdots\\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn}\end{bmatrix}\begin{bmatrix}\dot{q}_1^2\\ \dot{q}_2^2\\ \vdots\\ \dot{q}_n^2\end{bmatrix}$$

$$\underset{(n\ \frac{(n\ 1)n}{2})}{B(\mathbf{q})}\ \underset{(\frac{(n\ 1)n}{2}\ 1)}{[\dot{\mathbf{q}}\dot{\mathbf{q}}]} \quad \begin{bmatrix}2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n\ 1)n}\\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n\ 1)n}\\ \vdots & \vdots & \vdots & \vdots\\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n\ 1)n}\end{bmatrix}\begin{bmatrix}\dot{q}_1\dot{q}_2\\ \dot{q}_1\dot{q}_3\\ \vdots\\ \dot{q}_{(n\ 1)}\dot{q}_n\end{bmatrix}$$

Potential Energy



Gravity Vector

$$U_i \quad m_i g_0 h_i \quad U_0$$

$$U_i = m_i(-g^T p_{C_i}) + U_0$$

$$U = \sum U_i$$

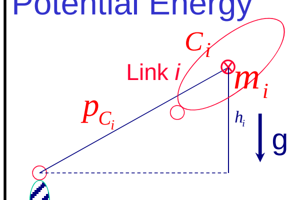
Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = \tau - G; \qquad G = \frac{\partial U}{\partial q}$$

Inertial forces

$$M(q)\ddot{q} + V(q,\dot{q}) = \tau - G(q)$$

Potential Energy



Gravity Vector

$$U_i \quad m_i g_0 h_i \quad U_0$$

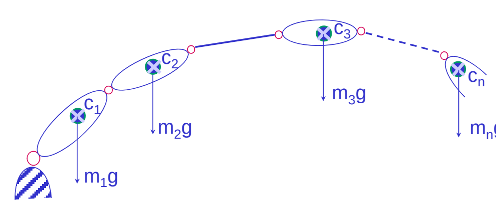
$$U_i = m_i(-g^T p_{C_i}) + U_0$$

$$U = \sum U_i$$

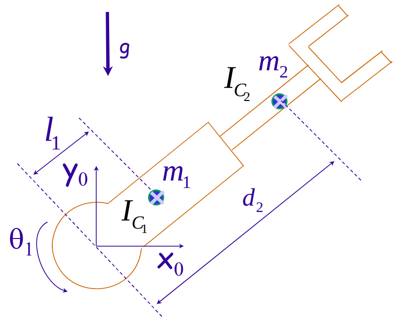
$$G_j \quad \frac{U}{q_j} \quad \sum_{i\ 1}^{n} (m_i g^T \frac{\mathbf{p}_{C_i}}{q_j})$$

$$G = -\begin{pmatrix}J_{v_1}^T & J_{v_2}^T & \cdots & J_{v_n}^T\end{pmatrix}\begin{pmatrix}m_1 g\\ m_2 g\\ \vdots\\ m_n g\end{pmatrix}$$

Gravity Vector



$$G \quad (J_{v_1}^T(m_1 g) \quad J_{v_2}^T(m_2 g) \quad \cdots \quad J_{v_n}^T(m_n g))$$

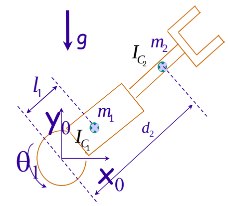


Matrix M

$$M = m_1 J_{v_1}^T J_{v_1} + J_{\omega_1}^T I_{C_1} J_{\omega_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T I_{C_2} J_{\omega_2}$$

J_{v_1} and J_{v_2} : direct differentiation of the vectors:

$${}^0\mathbf{p}_{C_1} \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}; \text{ and } {}^0\mathbf{p}_{C_2} \begin{bmatrix} d_2 c_1 \\ d_2 s_1 \\ 0 \end{bmatrix}$$



In frame {0}, these matrices are:

$${}^0J_{v_1} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } {}^0J_{v_2} = \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix}$$

This yields

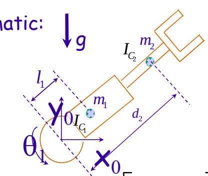
$$m_1({}^0J_{v_1}^T\, {}^0J_{v_1}) \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } m_2({}^0J_{v_2}^T\, {}^0J_{v_2}) \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

The matrices $J_{\,1}$ and $J_{\,2}$ are given by

$$J_{\,1}\ [\ _1\ \mathbf{z}_1\ \ 0] \quad \text{and } J_{\,2}\ [\ _1\ \mathbf{z}_1\ \ \ _2\ \mathbf{z}_2]$$

Joint 1 is revolute and joint 2 is prismatic:

$${}^1J_{\omega_1} = {}^1J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$



And

$$({}^1J_{\omega_1}^T\, {}^1I_{C_1}\, {}^1J_{\omega_1}) = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } ({}^1J_{\omega_2}^T\, {}^1I_{C_2}\, {}^1J_{\omega_2}) = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}$$

Finally,

$$M \begin{bmatrix} m_1 l_1^2 \quad I_{zz1} \quad m_2 d_2^2 \quad I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$

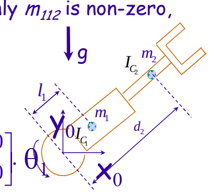
Centrifugal and Coriolis Vector V

$$b_{i,jk} \quad \frac{1}{2}(m_{ijk} \quad m_{ikj} \quad m_{jki})$$

where $m_{ijk} \quad \frac{m_{ij}}{q_k}$; with $b_{iii} \quad 0$ and $b_{ijj} \quad 0$ for $i \quad j$

For this manipulator, only m_{11} is configuration dependent - function of d_2. This implies that only m_{112} is non-zero,

$$m_{112} \quad 2m_2 d_2.$$



Matrix B $\quad B \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}.$

Matrix C $\quad C \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ m_2 d_2 & 0 \end{bmatrix}.$

Matrix V $\quad V \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} [\dot{\ }_1 \dot{d}_2] \begin{bmatrix} 0 & 0 \\ m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\ }_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$

Vector V

$$V \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} [\dot{\ }_1 \dot{d}_2] \begin{bmatrix} 0 & 0 \\ m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\ }_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$$

The Gravity Vector G

$$\mathbf{G} \quad [J_{v_1}^T m_1 \mathbf{g} \quad J_{v_2}^T m_2 \mathbf{g}].$$

In frame {0}, ${}^0\mathbf{g} \quad (0 \quad g \quad 0)^T$ and the gravity vector is

$${}^0\mathbf{G} \quad \begin{bmatrix} l_1 s_1 & l_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ m_1 g \\ 0 \end{bmatrix} \begin{bmatrix} d_2 s_1 & d_2 c_1 & 0 \\ c_1 & s_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ m_2 g \\ 0 \end{bmatrix}$$

and

$${}^0\mathbf{G} \quad \begin{bmatrix} (m_1 l_1 \quad m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix}$$

Equations of Motion

$$\begin{bmatrix} m_1 l_1^2 \quad I_{zz1} \quad m_2 d_2^2 \quad I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\ }_1 \\ \ddot{d}_2 \end{bmatrix}$$

$$\begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} [\dot{\ }_1 \dot{d}_2] \begin{bmatrix} 0 & 0 \\ m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\ }_1^2 \\ \dot{d}_2^2 \end{bmatrix}$$

$$\begin{bmatrix} (m_1 l_1 \quad m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$