Video Segment

From Compliant Balancing to Dynamic Walking, ATR/NICT, Japan, ICRA 2010

Dynamics

- Rigid Body Dynamics
- Newton-Euler Formulation
- Articulated Multi-Body Dynamics
- Recursive Algorithm
- Lagrange Formulation
- Explicit Form

MA23

Joint Space Dynamics

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \Gamma$$

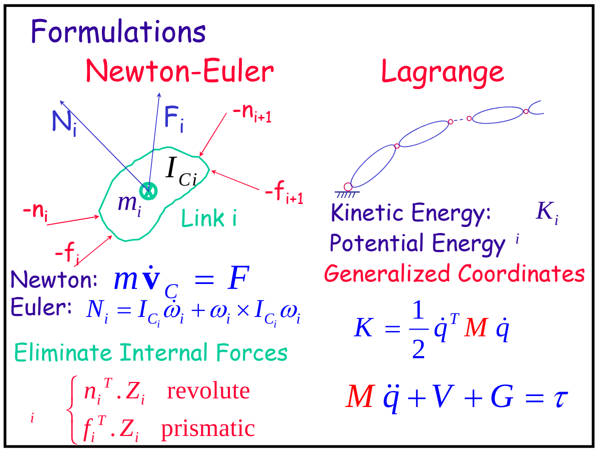
q: Generalized Joint Coordinates

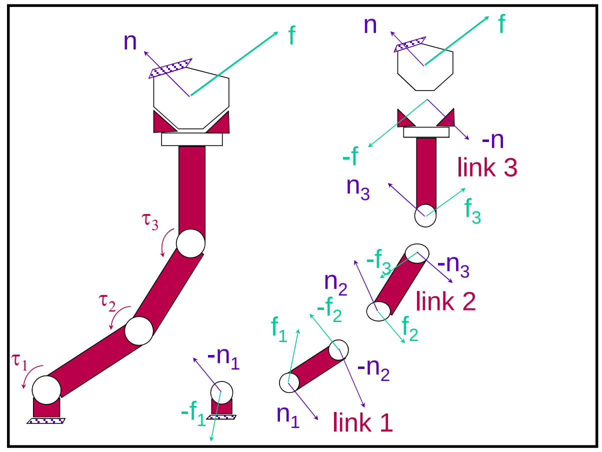
$M(q)$: Mass Matrix - Kinetic Energy Matrix

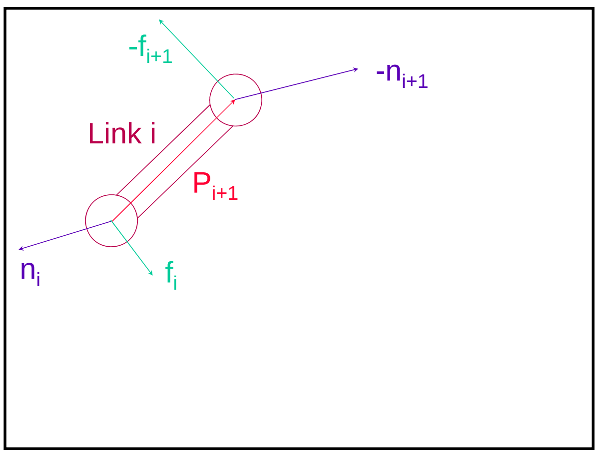
$V(q,\dot{q})$: Centrifugal and Coriolis forces

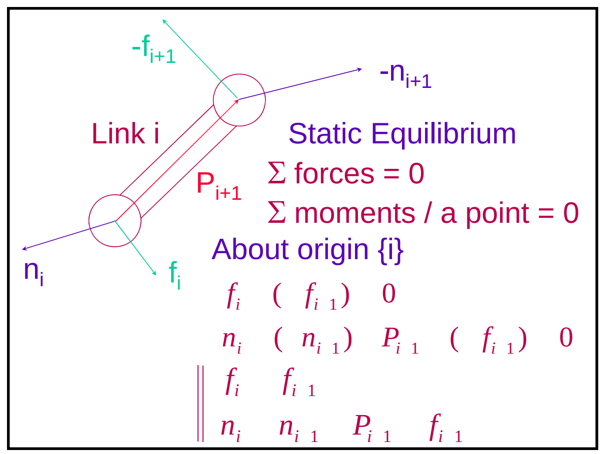
$G(q)$: Gravity forces

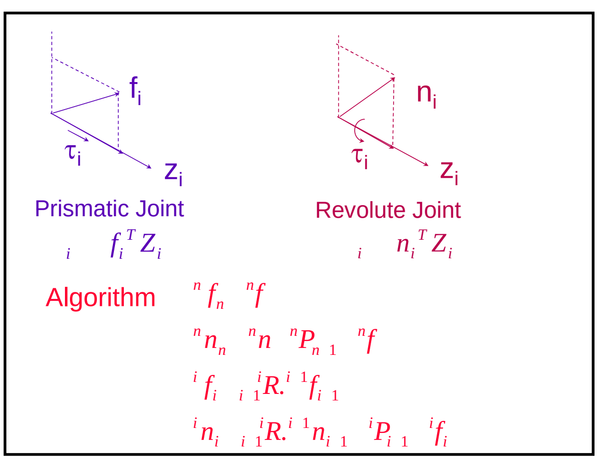
Γ: Generalized forces

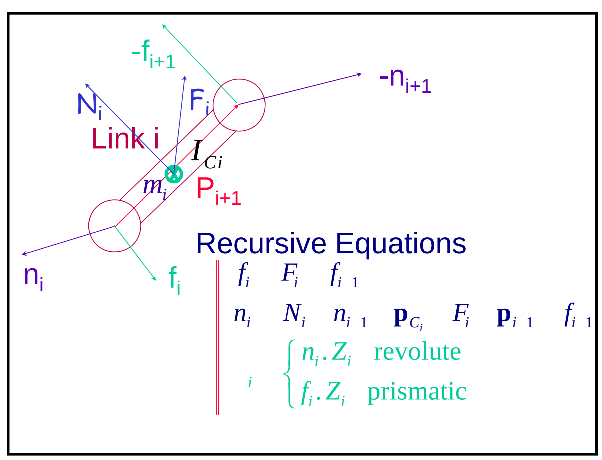

Formulations
Newton-Euler
Lagrange
N_i
F_i
$-n_{i+1}$
I_{Ci}
$-f_{i+1}$
$-n_i$
m_i
Link i
$-f_i$
Kinetic Energy: K_i
Potential Energy
Generalized Coordinates
Newton: $m\dot{v}_C = F$
Euler: $N_i = I_{C_i}\dot{\omega}_i + \omega_i \times I_{C_i}\omega_i$
Eliminate Internal Forces
$K = \frac{1}{2}\dot{q}^T M \dot{q}$
$n_i^T.Z_i$ revolute
$f_i^T.Z_i$ prismatic
$M\ddot{q} + V + G = \tau$


n
f
n
f
-f
-n
link 3
n_3
f_3
τ_3
$-f_3$
$-n_3$
n_2
$-f_2$
link 2
τ_2
f_1
f_2
τ_1
$-n_1$
$-n_2$
$-f_1$
n_1
link 1

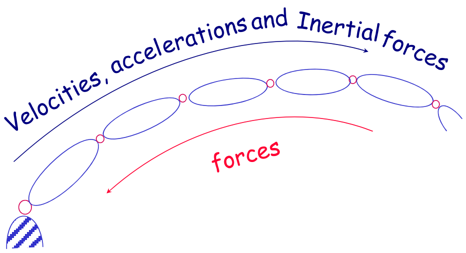

$-f_{i+1}$
$-n_{i+1}$
Link i
P_{i+1}
n_i
f_i


$-f_{i+1}$
$-n_{i+1}$
Link i
Static Equilibrium
P_{i+1}
Σ forces = 0
Σ moments / a point = 0
n_i
About origin {i}
f_i
f_i ($f_{i\ 1}$) 0
n_i ($n_{i\ 1}$) $P_{i\ 1}$ ($f_{i\ 1}$) 0
f_i $f_{i\ 1}$
n_i $n_{i\ 1}$ $P_{i\ 1}$ $f_{i\ 1}$


f_i
n_i
τ_i
z_i
τ_i
z_i
Prismatic Joint
Revolute Joint
$f_i^T Z_i$
$n_i^T Z_i$
Algorithm
${}^n f_n$ ${}^n f$
${}^n n_n$ ${}^n n$ ${}^n P_{n\ 1}$ ${}^n f$
${}^i f_i$ ${}_{i\ 1}^{i}R.{}^{i\ 1}f_{i\ 1}$
${}^i n_i$ ${}_{i\ 1}^{i}R.{}^{i\ 1}n_{i\ 1}$ ${}^i P_{i\ 1}$ ${}^i f_i$

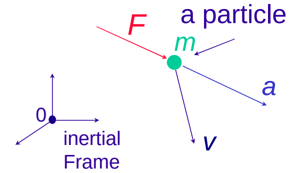

$-f_{i+1}$
$-n_{i+1}$
N_i
F_i
Link i
I_{Ci}
m_i
P_{i+1}
Recursive Equations
n_i
f_i
f_i F_i $f_{i\ 1}$
n_i N_i $n_{i\ 1}$ $\mathbf{p}_{C_i}$ F_i $\mathbf{p}_{i\ 1}$ $f_{i\ 1}$
$n_i.Z_i$ revolute
$f_i.Z_i$ prismatic

Newton-Euler Algorithm



Newton's Law

$\underline{F} \quad m\underline{a}$



$\frac{d}{dt}(mv) \quad F$

rate of change of the linear momentum is equal to the applied force

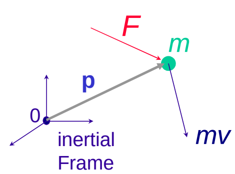
Linear Momentum

$\underline{\varphi = mv}$

Angular Momentum

$m\dot{\mathbf{v}} \quad F$

take the moment /0



$\mathbf{p} \quad m\dot{\mathbf{v}} \quad \mathbf{p} \quad F$

N

$\frac{d}{dt}(\mathbf{p} \quad m\mathbf{v}) \quad \mathbf{p} \quad m\dot{\mathbf{v}} \quad \mathbf{v} \quad m\mathbf{v} \quad \mathbf{p} \quad m\dot{\mathbf{v}}$ (=0)

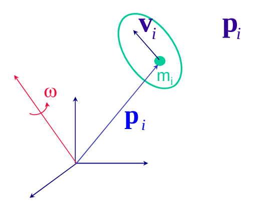
$\boxed{\frac{d}{dt}(\mathbf{p} \times m\,\mathbf{v}) = N}$

applied moment

angular momentum $\phi = p \times mv$

Rigid Body

Rotational Motion



$\mathbf{p}_i$

Angular Momentum $\quad \mathbf{p}_i \quad m_i\mathbf{v}_i$

$\sum_i m_i\mathbf{p}_i \quad (\quad \mathbf{p}_i)$

$\sum_i m_i \quad dv \quad (\;:density)$

$\phi = \int_V p \times (\omega \times p)\rho dv$

$\phi = \int_V p \times (\omega \times p)\rho dv$

$\mathbf{p} \quad (\quad \mathbf{p}) \quad \hat{\mathbf{p}}(\quad \hat{\mathbf{p}})$

$\phi = [\int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\,\rho\, dv\,]\omega$

Inertia Tensor

$\underline{I}$

$I \quad \int_V \hat{\mathbf{p}}\hat{\mathbf{p}} \quad dv$

Linear Momentum	Angular Momentum
$\underline{\varphi = mv}$	$\underline{I}$
Newton Equation	Euler Equation
$\frac{d}{dt}(mv) \quad F$	$\frac{d}{dt}(I\omega) = N$
$\boxed{\dot{\varphi} = F}$	$\boxed{\dot{} \quad N}$
$ma = F$	$I\,\dot{} \quad I \quad N$

Inertia Tensor

$$I \quad \int_V \hat{\mathbf{p}}\hat{\mathbf{p}}\ dv \qquad (\ \hat{\mathbf{p}}\hat{\mathbf{p}})\quad (\mathbf{p}^T\mathbf{p})I_3 \quad \mathbf{p}\mathbf{p}^T$$

$$I = \int_V [(\mathbf{p}^T\mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T]\rho dv$$

$$\mathbf{p} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad ;\mathbf{p}^T\mathbf{p} \quad x^2 \quad y^2 \quad z^2$$

$$(\mathbf{p}^T\mathbf{p})I_3 \quad (x^2 \quad y^2 \quad z^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{p}\mathbf{p}^T \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}(x \quad y \quad z) \quad \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}$$

$$(-\hat{\mathbf{p}}\hat{\mathbf{p}}) = \begin{bmatrix} y^2 \quad z^2 & xy & xz \\ xy & z^2 \quad x^2 & yz \\ xz & yz & x^2 \quad y^2 \end{bmatrix}$$

Inertia Tensor

$$I \quad \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

Moments of Inertia

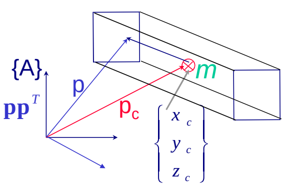
$$I_{xx} \quad \iiint (y^2 \quad z^2)\ dxdydz$$
$$I_{yy} \quad \iiint (z^2 \quad x^2)\ dxdydz$$
$$I_{zz} \quad \iiint (x^2 \quad y^2)\ dxdydz$$

Products of Inertia

$$I_{xy} \quad \iiint xy\ \ dxdydz$$
$$I_{xz} \quad \iiint xz\ \ dxdydz$$
$$I_{yz} \quad \iiint yz\ \ dxdydz$$

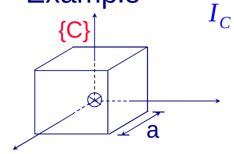
Parallel Axis theorem

$$I \quad \int_V \hat{\mathbf{p}}\hat{\mathbf{p}}\ dv$$
$$(\ \hat{\mathbf{p}}\hat{\mathbf{p}})\quad (\mathbf{p}^T\mathbf{p})I_3 \quad \mathbf{p}\mathbf{p}^T$$



$$I_A = I_C + m[(\mathbf{p}_C{}^T\mathbf{p}_C)I_3 - \mathbf{p}_C\mathbf{p}_C{}^T]$$
$$I_{Azz} = I_{Czz} + m(x_C^2 + y_C^2)$$
$$I_{Axy} = I_{Cxy} + m x_C y_C$$

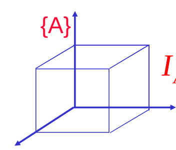
Example



$$I_{Czz} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \iint \rho(x^2 + y^2)dxdydz$$

$$I_{Czz} = \frac{1}{6}\rho a^5; \qquad m \qquad a^3$$

$$I_{Cxx} = I_{Cyy} = I_{Czz} = \frac{ma^2}{6}$$

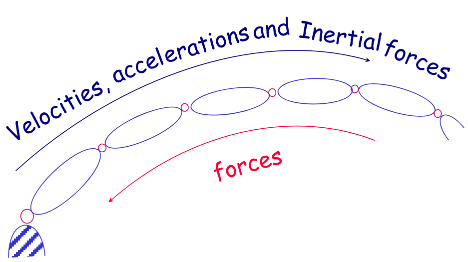


$${}^Ax_c \quad {}^Ay_c \quad {}^Az_c \quad \frac{a}{2}$$

$$I_{Axx} = I_{Ayy} = I_{Azz} = I_{Czz} + \frac{ma^2}{2} = \frac{2}{3}ma^2$$

$$I_{Axy} = I_{Axz} = I_{Ayz} = \frac{m a^2}{4}$$

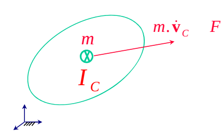
Newton-Euler Algorithm



Newton-Euler Equations

Translational Motion

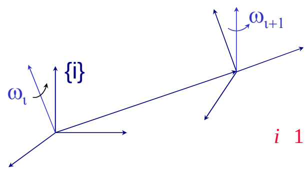
$$m\dot{\mathbf{v}}_C = F$$



Rotational Motion

$$I_C\dot{\omega} + \omega \times I_C\omega = N$$

Angular Acceleration

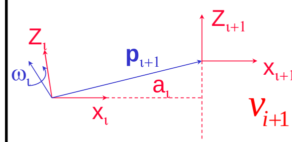


$i\ 1 \qquad i \qquad i\ 1$

$i\ 1 \qquad \dot{} \quad i\ 1 Z_{i\ 1}$

$\dot{}\ _{i\ 1} \quad \dot{}\ _{i} \quad \dot{}\ _{i\ 1}(\ _i \ Z_{i\ 1}) \quad \ddot{}\ _{i\ 1} Z_{i\ 1}$

Linear Acceleration



$v_{i+1} = v_i + \omega_i \times p_{i+1} + V_{i+1}$

$V_{i\ 1} \qquad \dot{d}_{i\ 1} Z_{i\ 1}$

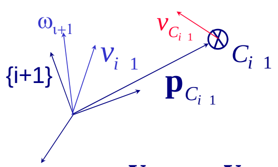
$P_{i\ 1} \quad a_i x_i \quad d_{i\ 1} Z_{i\ 1}$

$\dot{\mathbf{v}}_{i\ 1} \quad \dot{\mathbf{v}}_i \quad \dot{}\ _i \quad \mathbf{p}_{i\ 1} \quad _i \quad \dot{\mathbf{p}}_{i\ 1} \quad \dot{V}_{i\ 1}$

$\dot{\mathbf{v}}_{i\ 1} \quad \dot{\mathbf{v}}_i \quad \dot{}\ _i \quad \mathbf{p}_{i\ 1} \quad _i \ (\ _i \quad \mathbf{p}_{i\ 1})$

$2\dot{d}_{i\ 1} \ _i \quad Z_{i\ 1} \quad \ddot{d}_{i\ 1} Z_{i\ 1}$

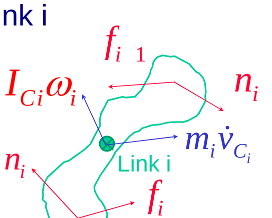
Velocity and Acceleration at center of mass



$\mathbf{v}_{C_{i\ 1}} \quad \mathbf{v}_{i\ 1} \quad _{i\ 1} \quad \mathbf{p}_{C_{i\ 1}}$

$\dot{\mathbf{v}}_{C_{i\ 1}} \quad \dot{\mathbf{v}}_{i\ 1} \quad \dot{}\ _{i\ 1} \quad \mathbf{p}_{C_{i\ 1}} \quad _{i\ 1} \ (\ _{i\ 1} \quad \mathbf{p}_{C_{i\ 1}})$

Dynamic forces on Link i



$I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$

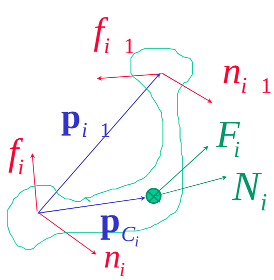
$m_i \dot{\mathbf{v}}_{C_i} \qquad forces$

$I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i = \sum moments / c_i$

Inertial forces/moments

$F_i \quad m_i \dot{\mathbf{v}}_{C_i}$

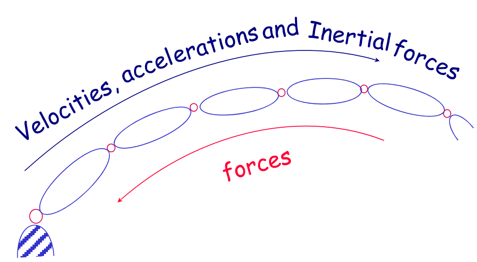
$N_i = I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$



$F_i \quad f_i \quad f_{i\ 1}$

$N_i \quad n_i \quad n_{i\ 1} \quad (\quad \mathbf{p}_{C_i}) \quad f_i \quad (\mathbf{p}_{i\ 1} \quad \mathbf{p}_{C_i}) \quad (\quad f_{i\ 1})$

Newton-Euler Algorithm



Recursive Equations

$f_i \quad F_i \quad f_{i\ 1}$

$n_i \quad N_i \quad n_{i\ 1} \quad \mathbf{p}_{C_i} \quad F_i \quad \mathbf{p}_{i\ 1} \quad f_{i\ 1}$

$_i \begin{cases} n_i . Z_i & \text{revolute} \\ f_i . Z_i & \text{prismatic} \end{cases}$

with $F_i \quad m_i \dot{\mathbf{v}}_{C_i}$

$N_i = I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$

where $_{i\ 1} \quad _i \quad _{i\ 1} \quad _i \quad \dot{}_{i\ 1} Z_{i\ 1}$

$\dot{}_{i\ 1} \quad \dot{}_i \quad _i \quad Z_{i\ 1} \dot{}_{i\ 1} \quad \ddot{}_{i\ 1} Z_{i\ 1}$

$\dot{\mathbf{v}}_{i\ 1} \quad \dot{\mathbf{v}}_i \quad \dot{}_i \quad \mathbf{p}_{i\ 1} \quad _i \quad (\quad _i \quad \mathbf{p}_{i\ 1}) \quad 2\dot{d}_{i\ 1} \quad _i \quad Z_{i\ 1} \quad \ddot{d}_{i\ 1} Z_{i\ 1}$

$\dot{\mathbf{v}}_{C_{i\ 1}} \quad \dot{\mathbf{v}}_{i\ 1} \quad \dot{}_{i\ 1} \quad \mathbf{p}_{C_{i\ 1}} \quad _{i\ 1} \quad (\quad _{i\ 1} \quad \mathbf{p}_{C_{i\ 1}})$

Outward iterations: i : 0 ⟶ 5

$^{i\ 1}{}_{i\ 1} \quad ^{i\ 1}_{i}R^i \quad _i \quad \dot{}_{i\ 1} \quad ^{i\ 1}Z_{i\ 1}$

$^{i\ 1}\dot{}_{i\ 1} \quad ^{i\ 1}_{i}R^i \dot{}_i \quad ^{i\ 1}_{i}R^i \quad _i \quad ^{i\ 1}Z_{i\ 1} \quad \dot{}_{i\ 1} \quad \ddot{}_{i\ 1} \quad ^{i\ 1}Z_{i\ 1}$

$^{i\ 1}\dot{\mathbf{v}}_{i\ 1} \quad ^{i\ 1}_{i}R(^i \dot{}_i \quad ^i\mathbf{p}_{i\ 1} \quad ^i{}_i \quad (^i{}_i \quad ^i\mathbf{p}_{i\ 1}) \quad ^i\dot{\mathbf{v}}_i)$

$^{i\ 1}\dot{\mathbf{v}}_{C_{i\ 1}} \quad ^{i\ 1}\dot{}_{i\ 1} \quad ^{i\ 1}\mathbf{p}_{C_{i\ 1}} \quad ^{i\ 1}{}_{i\ 1} \quad (^{i\ 1}{}_{i\ 1} \quad ^{i\ 1}\mathbf{p}_{C_{i\ 1}}) \quad ^{i\ 1}\dot{\mathbf{v}}_{i\ 1}$

$^{i\ 1}F_{i\ 1} \quad m_{i\ 1} \quad ^{i\ 1}\dot{\mathbf{v}}_{C_{i\ 1}}$

$^{i\ 1}N_{i\ 1} \quad ^{C_{i\ 1}}I_{i\ 1} \quad ^{i\ 1}\dot{}_{i\ 1} \quad ^{i\ 1}{}_{i\ 1} \quad ^{C_{i\ 1}}I_{i\ 1} \quad ^{i\ 1}{}_{i\ 1}$

Inward iterations: i : 6 ⟶ 1

$^i f_i \quad ^i_{i\ 1}R \quad ^{i\ 1}f_{i\ 1} \quad ^iF_i$

$^i n_i \quad ^iN_i \quad ^i_{i\ 1}R \quad ^{i\ 1}n_{i\ 1} \quad ^i\mathbf{p}_{C_i} \quad ^iF_i \quad ^i\mathbf{p}_{i\ 1} \quad ^i_{i\ 1}R \quad ^{i\ 1}f_{i\ 1}$

$_i \quad ^i n_i^T \, ^iZ_i$

Gravity: set $^0\dot{\mathbf{v}}_0 \quad 1G$