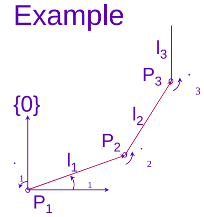
Movie Segment

The Flying Machine Lab, ETH Zurich, 2011.

J a c o b i a n

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

Example



$v_{i\ 1} \quad v_i \quad {}_i \quad P_{i\ 1}$

- $v_{P_1} \quad 0 \qquad {}^0{}_1 \quad \dot{}_1 . {}^0Z_1$
- $v_{P_2} \quad v_{P_1} \qquad {}_1 \quad P_2$
- $v_{P_3} \quad v_{P_2} \qquad {}_2 \quad P_3$

$$ {}^0v_{P_2} \quad 0 \quad \begin{bmatrix} 0 & \dot{}_1 & 0 \\ \dot{}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} l_1.c_1 \\ l_1.s_1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} . \dot{}_1 $$

$$ {}^0v_{P_3} \quad {}^0v_{P_2} \quad {}^0{}_2 \quad {}^0P_3 $$

$$ {}^0v_{P_3} \quad \begin{bmatrix} l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} . \dot{}_1 \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .(\dot{}_1 \quad \dot{}_2) . {}^0P_3 \qquad \begin{bmatrix} l_2.c_{12} \\ l_2.s_{12} \\ 0 \end{bmatrix} $$

$$ \begin{bmatrix} l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} . \dot{}_1 \quad \begin{bmatrix} l_2.s_{12} \\ l_2.c_{12} \\ 0 \end{bmatrix} .(\dot{}_1 \quad \dot{}_2) $$

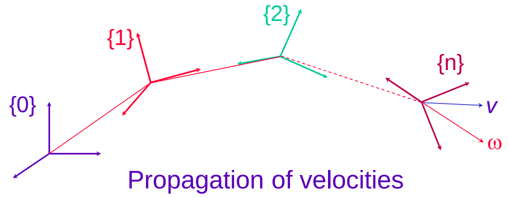
$$ {}^0{}_3 \quad (\dot{}_1 \quad \dot{}_2 \quad \dot{}_3) . {}^0Z_0 $$

$$ {}^0v_{P_3} \quad \underbrace{\begin{bmatrix} (l_1s_1 \quad l_2s_{12}) & l_2s_{12} & 0 \\ l_1c_1 \quad l_2c_{12} & l_2c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{J_v} . \begin{bmatrix} \dot{}_1 \\ \dot{}_2 \\ \dot{}_3 \end{bmatrix} $$

$$ {}^0{}_3 \quad \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{J_\omega} . \begin{bmatrix} \dot{}_1 \\ \dot{}_2 \\ \dot{}_3 \end{bmatrix} $$

$$ \begin{pmatrix} v \\ \ \end{pmatrix} \quad J. \begin{pmatrix} \dot{}_1 \\ \dot{}_2 \\ \dot{}_3 \end{pmatrix} $$

Spatial Mechanisms

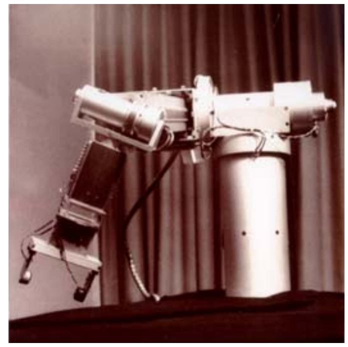


Propagation of velocities

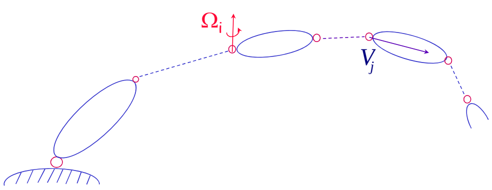
$\dot{x}$ < v : linear velocity; ω : angular velocity

$\dot{x}$ $J(\ ).$

Stanford Scheinman Arm



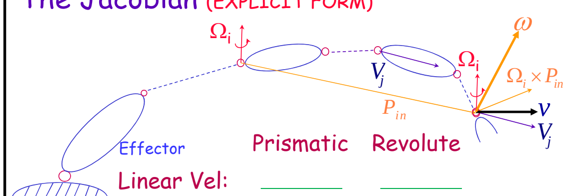
The Jacobian (EXPLICIT FORM)



Revolute Joint $\Omega_i = Z_i \dot{q}_i$

Prismatic Joint $V_i = Z_i \dot{q}_i$

The Jacobian (EXPLICIT FORM)



Effector	Prismatic	Revolute
Linear Vel:	______	______
Angular Vel:	______	______

Effector Linear Velocity

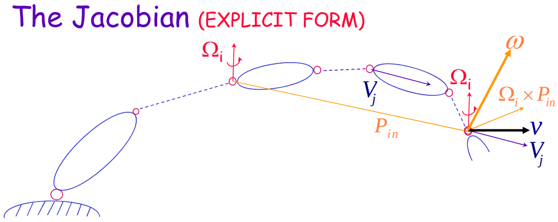
$$v = \sum_{i=1}^{n} [\epsilon_i V_i + \bar{\epsilon}_i (\Omega_i \times P_{in})] \quad \Leftarrow \quad V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

n — i i i 1

$$\Leftarrow \quad \Omega_i = Z_i \dot{q}_i$$

The Jacobian (EXPLICIT FORM)



Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i Z_i + \bar{\epsilon}_i (Z_i \times P_{in})] \dot{q}_i \quad \Leftarrow \quad V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} (\bar{\epsilon}_i Z_i) \dot{q}_i \quad \Leftarrow \quad \Omega_i = Z_i \dot{q}_i$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n})] \dot{q}_1 + \cdots$$
$$+ [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1} (Z_{n-1} \times P_{(n-1)n})] \dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = \begin{bmatrix} \epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) & \epsilon_2 Z_2 + \bar{\epsilon}_2 (Z_2 \times P_{2n}) & \cdots \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

$$\omega = \bar{\epsilon}_1 Z_1 \dot{q}_1 + \bar{\epsilon}_2 Z_2 \dot{q}_2 + \cdots + \bar{\epsilon}_n Z_n \dot{q}_n$$

$$\omega = \begin{bmatrix} \bar{\epsilon}_1 Z_1 & \bar{\epsilon}_2 Z_2 & \cdots & \bar{\epsilon}_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{\omega = J_\omega \dot{q}}$$

The Jacobian

$$J = \begin{pmatrix} J_v \\ J_w \end{pmatrix}$$

Matrix J_v (direct differentiation)

$$v \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \quad \dot{x}_P \quad \frac{x_P}{q_1}.\dot{q}_1 \quad \frac{x_P}{q_2}.\dot{q}_2 \quad \cdots \quad \frac{x_P}{q_n}.\dot{q}_n$$

$$J_v = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \end{pmatrix}$$

Jacobian in a Frame

Vector Representation

$$J \begin{pmatrix} \frac{x_P}{q_1} & \frac{x_P}{q_2} & \cdots & \frac{x_P}{q_n} \\ \_{1}.Z_1 & \_{2}.Z_2 & \cdots & \_{n}.Z_n \end{pmatrix}$$

In {0}

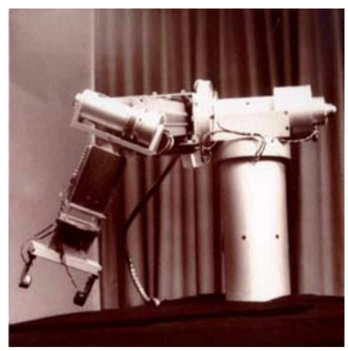
$$^0J \begin{pmatrix} \frac{^0x_P}{q_1} & \frac{^0x_P}{q_2} & \cdots & \frac{^0x_P}{q_n} \\ \_{1}.{}^0Z_1 & \_{2}.{}^0Z_2 & \cdots & \_{n}.{}^0Z_n \end{pmatrix}$$

J in Frame {0}

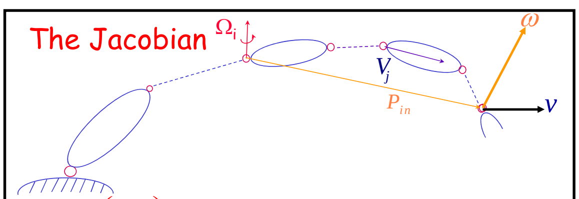
$$^0Z_i = {}^0_iR\ {}^iZ_i; \quad {}^iZ_i = Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$^0J \begin{pmatrix} \frac{}{q_1}({}^0x_P) & \frac{}{q_2}({}^0x_P) & \cdots & \frac{}{q_n}({}^0x_P) \\ \_{1}.({}^0_1R.Z) & \_{2}.({}^0_2R.Z) & \cdots & \_{n}.({}^0_nR.Z) \end{pmatrix}$$

Stanford Scheinman Arm



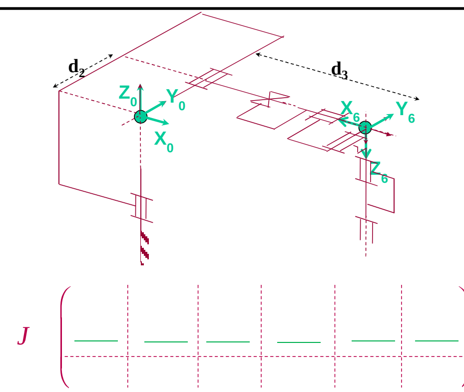
The Jacobian

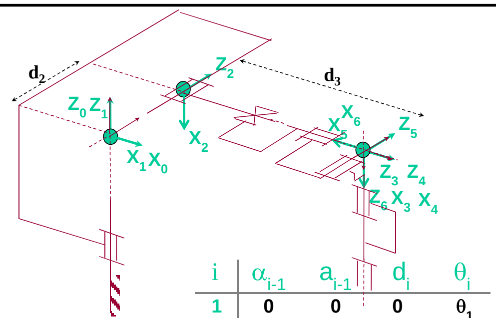


$$J = \begin{pmatrix} J_v \\ J_w \end{pmatrix} \qquad v = J_v\dot{q} \qquad \omega = J_\omega\dot{q}$$

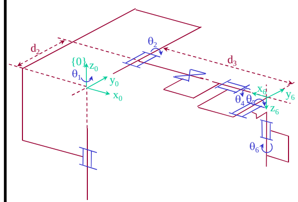
$$J_v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) \quad \epsilon_2 Z_2 + \bar{\epsilon}_2 (Z_2 \times P_{2n}) \quad \cdots]$$

$$J_\omega = [\bar{\epsilon}_1 Z_1 \quad \bar{\epsilon}_2 Z_2 \quad \cdots \quad \bar{\epsilon}_n Z_n]$$





i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

$$ {}^{i-1}_{\;\;i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Forward Kinematics: ${}^0_N T = {}^0_1 T \; {}^1_2 T \; \ldots \; {}^{N-1}_{\;\;N} T$

Stanford Scheinman Arm

$$ {}^0_1T = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} c_2 & s_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_4T = \begin{bmatrix} c_4 & s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^4_5T = \begin{bmatrix} c_5 & s_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^5_6T = \begin{bmatrix} c_6 & s_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_1T = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_2T = \begin{bmatrix} c_1c_2 & c_1s_2 & s_1 & s_1d_2 \\ s_1c_2 & s_1s_2 & c_1 & c_1d_2 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

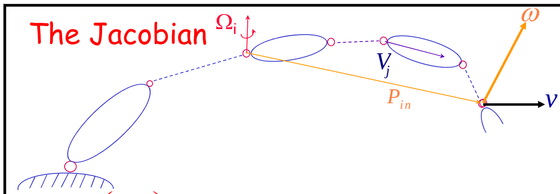
$$ {}^0_3T = \begin{bmatrix} c_1c_2 & s_1 & c_1s_2 & c_1d_3s_2 \quad s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1d_3s_2 \quad c_1d_2 \\ s_2 & 0 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_4T = \begin{bmatrix} c_1c_2c_4 - s_1s_4 & -c_1c_2s_4 - s_1c_4 & c_1s_2 & c_1d_3s_2 - s_1d_2 \\ s_1c_2c_4 + c_1s_4 & -s_1c_2s_4 + c_1c_4 & s_1s_2 & s_1d_3s_2 + c_1d_2 \\ -s_2c_4 & s_2s_4 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_5T = \begin{bmatrix} X & X & c_1c_2s_4 \quad s_1c_4 & c_1d_3s_2 \quad s_1d_2 \\ X & X & s_1c_2s_4 \quad c_1c_4 & s_1d_3s_2 \quad c_1d_2 \\ X & X & s_2s_4 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_6T = \begin{bmatrix} X & X & c_1c_2c_4s_5 \quad s_1s_4s_5 \quad c_1s_2s_5 & c_1d_3s_2 \quad s_1d_2 \\ X & X & s_1c_2c_4s_5 \quad c_1s_4s_5 \quad s_1s_2c_5 & s_1d_3s_2 \quad c_1d_2 \\ X & X & s_2c_4s_5 \quad c_5c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The Jacobian

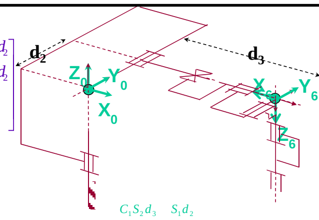


$$J = \begin{pmatrix} J_v \\ J_w \end{pmatrix} \qquad v = J_v \dot{q} \qquad \omega = J_\omega \dot{q}$$

$$J_v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) \quad \epsilon_2 Z_2 + \bar{\epsilon}_2 (Z_2 \times P_{2n}) \quad \cdots]$$

$$J_\omega = [\bar{\epsilon}_1 Z_1 \quad \bar{\epsilon}_2 Z_2 \quad \cdots \quad \bar{\epsilon}_n Z_n]$$

$$ {}^0_6T \begin{bmatrix} X & X & c_1c_2c_4s_5 & s_1s_4s_5 & c_1s_2s_5 & c_1d_3s_2 & s_1d_2 \\ X & X & s_1c_2c_4s_5 & c_1s_4s_5 & s_1s_2c_5 & s_1d_3s_2 & c_1d_2 \\ X & X & & s_2c_4s_5 & c_5c_2 & & d_3c_2 \\ 0 & 0 & & 0 & & & 1 \end{bmatrix}$$



$$x = \begin{pmatrix} x_P \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \left| \begin{matrix} C_1S_2d_3 \quad S_1d_2 \\ S_1S_2d_3 \quad C_1d_2 \\ C_2d_3 \\ C_1[C_2(C_4C_5C_6 \quad S_4S_6) \quad S_2S_5C_6] \quad S_1(S_4C_5C_6 \quad C_4S_6) \\ S_1[C_2(C_4C_5C_6 \quad S_4S_6) \quad S_2S_5C_6] \quad C_1(S_4C_5C_6 \quad C_4S_6) \\ S_2(C_4C_5C_6 \quad S_4S_6) \quad C_2S_5C_6 \\ C_1[\quad C_2(C_4C_5S_6 \quad S_4C_6) \quad S_2S_5S_6] \quad S_1(\quad S_4C_5S_6 \quad C_4C_6) \\ S_1[\quad C_2(C_4C_5S_6 \quad S_4C_6) \quad S_2S_5S_6] \quad C_1(\quad S_4C_5S_6 \quad C_4C_6) \\ S_2(C_4C_5S_6 \quad S_4C_6) \quad C_2S_5S_6 \\ C_1(C_2C_4S_5 \quad S_2C_5) \quad S_1S_4S_5 \\ S_1(C_2C_4S_5 \quad S_2C_5) \quad C_1S_4S_5 \\ S_2C_4S_5 \quad C_2C_5 \end{matrix} \right|$$

Stanford Scheinman Arm Jacobian

$$ {}^0J = \begin{pmatrix} \dfrac{\partial\, {}^0x_P}{\partial q_1} & \dfrac{\partial\, {}^0x_P}{\partial q_2} & \dfrac{\partial\, {}^0x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^0Z_1 & {}^0Z_2 & 0 & {}^0Z_4 & {}^0Z_5 & {}^0Z_6 \end{pmatrix}$$

$$x = \begin{pmatrix} x_P \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \left| \begin{matrix} C_1S_2d_3 \quad S_1d_2 \\ S_1S_2d_3 \quad C_1d_2 \\ C_2d_3 \\ C_1[C_2(C_4C_5C_6 \quad S_4S_6) \quad S_2S_5C_6] \quad S_1(S_4C_5C_6 \quad C_4S_6) \\ S_1[C_2(C_4C_5C_6 \quad S_4S_6) \quad S_2S_5C_6] \quad C_1(S_4C_5C_6 \quad C_4S_6) \\ S_2(C_4C_5C_6 \quad S_4S_6) \quad C_2S_5C_6 \\ C_1[\quad C_2(C_4C_5S_6 \quad S_4C_6) \quad S_2S_5S_6] \quad S_1(\quad S_4C_5S_6 \quad C_4C_6) \\ S_1[\quad C_2(C_4C_5S_6 \quad S_4C_6) \quad S_2S_5S_6] \quad C_1(\quad S_4C_5S_6 \quad C_4C_6) \\ S_2(C_4C_5S_6 \quad S_4C_6) \quad C_2S_5S_6 \\ C_1(C_2C_4S_5 \quad S_2C_5) \quad S_1S_4S_5 \\ S_1(C_2C_4S_5 \quad S_2C_5) \quad C_1S_4S_5 \\ S_2C_4S_5 \quad C_2C_5 \end{matrix} \right|$$

Stanford Scheinman Arm Jacobian

$$ {}^0J = \begin{pmatrix} \dfrac{\partial\, {}^0x_P}{\partial q_1} & \dfrac{\partial\, {}^0x_P}{\partial q_2} & \dfrac{\partial\, {}^0x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^0Z_1 & {}^0Z_2 & 0 & {}^0Z_4 & {}^0Z_5 & {}^0Z_6 \end{pmatrix}$$

$$\begin{bmatrix} c_1d_2 & s_1s_2d_3 & c_1c_2d_3 & c_1s_2 & 0 & 0 & 0 \\ s_1d_2 & c_1s_2d_3 & s_1c_2d_3 & s_1s_2 & 0 & 0 & 0 \\ 0 & s_2d_3 & c_2 & 0 & 0 & & 0 \\ 0 & s_1 & 0 & c_1s_2 & c_1c_2s_4 \quad s_1c_4 & c_1c_2c_4s_5 \quad s_1s_4s_5 & c_1s_2c_5 \\ 0 & c_1 & 0 & s_1s_2 & s_1c_2s_4 \quad c_1c_4 & s_1c_2c_4s_5 \quad c_1s_4s_5 & s_1s_2c_5 \\ 1 & 0 & 0 & c_2 & s_2s_4 & s_2c_4s_5 & c_5c_2 \end{bmatrix}$$

Kinematic Singularity

The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$J = \begin{pmatrix} J_1 & J_2 & \cdots & J_n \end{pmatrix}$$

$$\det(J) = 0$$

$$\det({}^iJ) = \det({}^jJ)$$

Kinematic Singularity

$$ {}^BJ = \begin{pmatrix} {}^B_AR & 0 \\ 0 & {}^B_AR \end{pmatrix} {}^AJ$$

$$\det[{}^BJ] \equiv \det[{}^AJ]$$

$$\boxed{\det({}^iJ) = \det({}^jJ)}$$

Singular Configurations

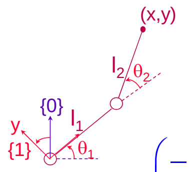
$$\det[J(q)] = 0$$

Singular Configurations

$$\det[J(q)] = S_1(q)S_2(q)\ldots S_s(q) = 0$$

$$\Rightarrow \quad \begin{array}{l} S_1(q)=0 \\ S_2(q)=0 \\ \vdots \\ S_s(q)=0 \end{array}$$

Example (Kinematic Singularities)



$$x = l_1C1 + l_2C12$$

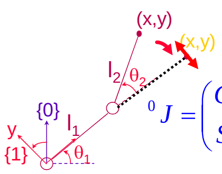
$$y = l_1S1 + l_2S12$$

$$J = \begin{pmatrix} -\left(l_1S1 + l_2S12\right) & -l_2S12 \\ l_1C1 + l_2C12 & l_2C12 \end{pmatrix}$$

$$\det\left(J\right) = l_1 l_2 S2$$

Singularity at $q_2 = k\pi$

Example (Kinematic Singularities)



$${}^1J = {}^1_0R\ {}^0J$$

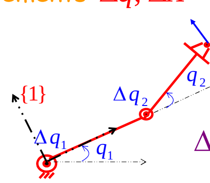
$${}^0J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix}\begin{pmatrix} -l_2S2 & -l_2S2 \\ l_1 + l_2C2 & l_2C2 \end{pmatrix}$$

At Singularity

$${}^1J = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$$

$$\begin{bmatrix} {}^1x & 0 \\ {}^1y & (l_1 \quad l_2) \quad {}_1 \quad l_2 \quad {}_2 \end{bmatrix}$$

Small Displacements $\Delta q, \Delta X$



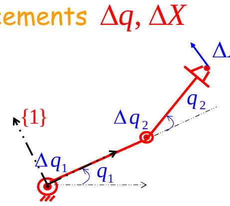
$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\Delta q = J^{-1}\Delta X$$

small θ_2

$$J^{-1}_{(1)} \cong \begin{pmatrix} \dfrac{1}{l_1\theta_2} & \dfrac{1}{l_1} \\ -\dfrac{l_1 + l_2}{l_1 l_2 \theta_2} & -\dfrac{1}{l_1} \end{pmatrix}$$

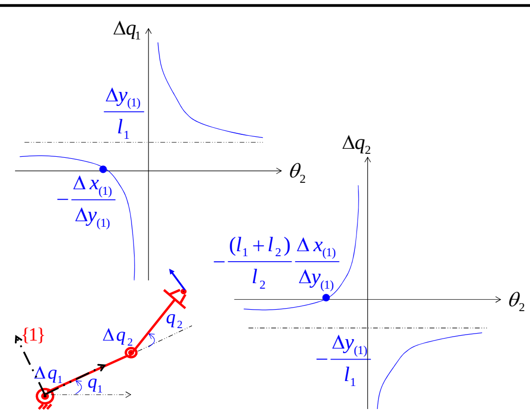
Small Displacements $\Delta q, \Delta X$



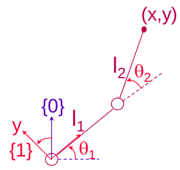
$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\Delta q_1 = \frac{\Delta x_{(1)}}{l_1} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$

$$\Delta q_2 = \frac{\left(l_1 + l_2\right)\Delta x_{(1)}}{l_1 l_2} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$



Kinematic Singularities (reduced matrix)

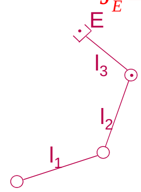


$$J=\begin{pmatrix} -(l_1S1+l_2S12) & -l_2S12 \\ l_1C1+l_2C12 & l_2C12 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\det(J)=l_1l_2S2 \qquad J=\begin{pmatrix} -(l_1S1+l_2S12) & -l_2S12 \\ l_1C1+l_2C12 & l_2C12 \end{pmatrix}$$

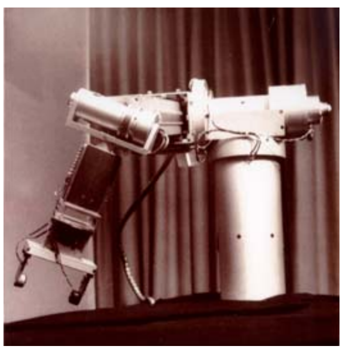
Singularity at $q_2=k\pi$

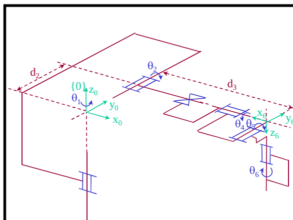
$${}^0J_E=\begin{pmatrix} -l_1s_1-l_2s_{12}-l_3s_{123} & -l_2s_{12}-l_3s_{123} & -l_3s_{123} \\ l_1c_1+l_2c_{12}+l_3c_{123} & l_2c_{12}+l_3c_{123} & l_3c_{123} \\ 1 & 1 & 1 \end{pmatrix}$$



$${}^0J_E \quad \begin{bmatrix} l_1s_1 \quad l_2s_{12} \quad l_3s_{123} & l_2s_{12} \quad l_3s_{123} & l_3s_{123} \\ l_1c_1 \quad l_2c_{12} \quad l_3c_{123} & l_2c_{12} \quad l_3c_{123} & l_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Stanford Scheinman Arm





i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

$${}^{i-1}_{\ \ 1}T=\begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics: ${}^0_NT = {}^0_1T \ {}^1_2T \ \ldots \ {}^{N-1}_{\ \ N}T$

Stanford Scheinman Arm Jacobian

$${}^0J=\begin{pmatrix} \dfrac{\partial^0x_P}{\partial q_1} & \dfrac{\partial^0x_P}{\partial q_2} & \dfrac{\partial^0x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^0Z_1 & {}^0Z_2 & 0 & {}^0Z_4 & {}^0Z_5 & {}^0Z_6 \end{pmatrix}$$

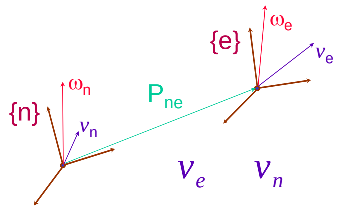
$$\begin{bmatrix} c_1d_2 & s_1s_2d_3 & c_1c_2d_3 & c_1s_2 & 0 & 0 & & & 0 \\ s_1d_2 & c_1s_2d_3 & s_1c_2d_3 & s_1s_2 & 0 & 0 & & & 0 \\ 0 & s_2d_3 & c_2 & 0 & 0 & & & 0 & \\ 0 & s_1 & 0 & c_1s_2 & c_1c_2s_4 & s_1c_4 & c_1c_2c_4s_5 & s_1s_4s_5 & c_1s_2c_5 \\ 0 & c_1 & 0 & s_1s_2 & s_1c_2s_4 & c_1c_4 & s_1c_2c_4s_5 & c_1s_4s_5 & s_1s_2c_5 \\ 1 & 0 & 0 & c_2 & s_2s_4 & & & s_2c_4s_5 & c_5c_2 \end{bmatrix}$$

Stanford Scheinman Arm Jacobian

$$\theta_5=k\pi$$

$$J=\begin{bmatrix} -c_1d_2-s_1s_2d_3 & c_1c_2d_3 & c_1s_2 & 0 & 0 & 0 \\ -s_1d_2+c_1s_2d_3 & s_1c_2d_3 & s_1s_2 & 0 & 0 & 0 \\ 0 & -s_2d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1s_2 & -c_1c_2s_4-s_1c_4 & c_1s_2 \\ 0 & c_1 & 0 & s_1s_2 & -s_1c_2s_4+c_1c_4 & s_1s_2 \\ 1 & 0 & 0 & c_2 & s_2s_4 & c_2 \end{bmatrix}$$

Jacobian at the End-Effector



$$v_e \quad v_n \quad n \quad P_{ne}$$

$$\begin{cases} v_e \quad v_n \quad P_{ne} \quad n \\ e \quad n \end{cases}$$

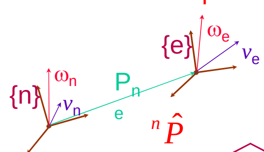
$$\begin{cases} v_e \quad v_n \quad P_{ne} \quad n \\ e \quad n \end{cases}$$

$$\begin{pmatrix} v_e \\ e \end{pmatrix} \quad \left(\begin{array}{c:c} \mathrm{I} & \hat{P}_{ne} \\ \hdashline \mathrm{O} & \mathrm{I} \end{array}\right) \begin{pmatrix} v_n \\ n \end{pmatrix}$$

$$J_e \dot{q} \quad \left(\begin{array}{c:c} \mathrm{I} & \hat{P}_{ne} \\ \hdashline \mathrm{O} & \mathrm{I} \end{array}\right) J_n \dot{q}$$

$$J_e \quad \left(\begin{array}{c:c} \mathrm{I} & \hat{P}_{ne} \\ \hdashline \mathrm{O} & \mathrm{I} \end{array}\right) J_n$$

Cross Product Operator (in diff. frames)



$${}^0J_e \quad \begin{pmatrix} I & {}^0\hat{P}_{ne} \\ 0 & I \end{pmatrix} {}^0J_n$$

$${}^n\hat{P}$$

$${}^0\hat{P} \neq {}^0_nR \; {}^n\hat{P}; \quad \widehat{{}^0P} \quad (\widehat{{}^0_nR.{}^nP}) \quad {}^0_nR.\widehat{{}^nP}$$

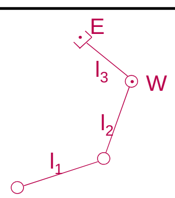
$${}^0P \quad {}^0 \quad {}^0_nR.({}^nP \quad {}^n \quad)$$

$${}^0\hat{P}.{}^0 \quad {}^0_nR.({}^n\hat{P}.{}^n \quad) \quad {}^0_nR.({}^n\hat{P}.{}^0_nR^T.{}^0 \quad)$$

$$\boxed{{}^0\hat{P} = {}^0_nR \; {}^n\hat{P} \; {}^0_nR^T}$$

$${}^iJ \quad \begin{pmatrix} {}^i_jR & 0 \\ 0 & {}^i_jR \end{pmatrix} {}^jJ$$

$${}^0J_e = \begin{pmatrix} {}^0_nR & -{}^0_nR\,{}^n\hat{P}_{ne}\,{}^0_nR^T \\ 0 & {}^0_nR \end{pmatrix} {}^nJ_n$$

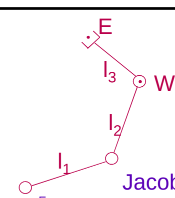


Wrist Point

$$x \quad l_1c_1 \quad l_2c_{12}$$
$$y \quad l_1s_1 \quad l_2s_{12}$$

End-Effector Point

$$x \quad l_1c_1 \quad l_2c_{12} \quad l_3c_{123}$$
$$y \quad l_1s_1 \quad l_2s_{12} \quad l_3s_{123}$$



Wrist Point

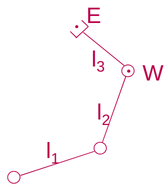
$$x \quad l_1c_1 \quad l_2c_{12}$$
$$y \quad l_1s_1 \quad l_2s_{12}$$

End-Effector Point

$$x \quad l_1c_1 \quad l_2c_{12} \quad l_3c_{123}$$
$$y \quad l_1s_1 \quad l_2s_{12} \quad l_3s_{123}$$

Jacobian (W)

$$J_W \begin{bmatrix} l_1s_1 & l_2s_{12} & l_2s_{12} & 0 \\ l_1c_1 & l_2c_{12} & l_2c_{12} & 0 \\ 0 & & 0 & 0 \\ 0 & & 0 & 0 \\ 0 & & 0 & 0 \\ 1 & & 1 & 1 \end{bmatrix}; \quad {}^0J_E \begin{pmatrix} I & {}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0J_W$$



Wrist Point

$x \quad l_1c_1 \quad l_2c_{12}$

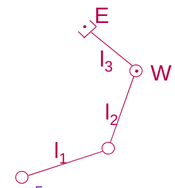
$y \quad l_1s_1 \quad l_2s_{12}$

End-Effector Point

$x \quad l_1c_1 \quad l_2c_{12} \quad l_3c_{123}$

$y \quad l_1s_1 \quad l_2s_{12} \quad l_3s_{123}$

$$J_w \begin{bmatrix} l_1s_1 \quad l_2s_{12} & l_2s_{12} & 0 \\ l_1c_1 \quad l_2c_{12} & l_2c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad {}^0J_E \begin{bmatrix} l_1s_1 \quad l_2s_{12} \quad l_3s_{123} & l_2s_{12} \quad l_3s_{123} & l_3s_{123} \\ l_1c_1 \quad l_2c_{12} \quad l_3c_{123} & l_2c_{12} \quad l_3c_{123} & l_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$



Wrist Point

$x \quad l_1c_1 \quad l_2c_{12}$

$y \quad l_1s_1 \quad l_2s_{12}$

End-Effector Point

$x \quad l_1c_1 \quad l_2c_{12} \quad l_3c_{123}$

$y \quad l_1s_1 \quad l_2s_{12} \quad l_3s_{123}$

$$J_w \begin{bmatrix} l_1s_1 \quad l_2s_{12} & l_2s_{12} & 0 \\ l_1c_1 \quad l_2c_{12} & l_2c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad {}^0J_E \begin{pmatrix} I & {}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0J_W$$

$${}^0P_{WE} \begin{bmatrix} l_3c_{123} \\ l_3s_{123} \\ 0 \end{bmatrix} \quad {}^0\hat{P}_{WE} \begin{pmatrix} 0 & 0 & l_3s_{123} \\ 0 & 0 & l_3c_{123} \\ l_3s_{123} & l_3c_{123} & 0 \end{pmatrix}$$

Resolved Motion Rate Control (Whitney 72)

$$x \quad J(\ )$$

Outside singularities

$$J^{\ 1}(\ )\ x$$

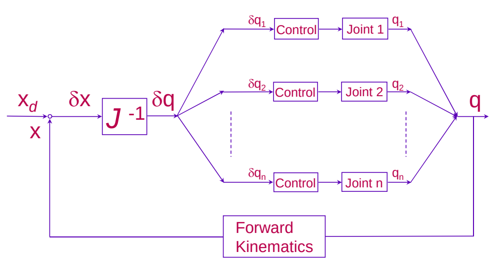
Arm at Configuration

$$x \quad f(\ )$$

$$x \quad x_d \quad x$$

$$J^{\ 1}\ x$$

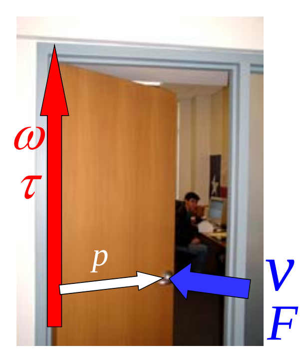
Resolved Motion Rate Control



Jacobian

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces ⇦

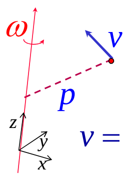
Angular/Linear – Velocities/Forces



$$v = \omega \times p$$

$$\tau = p \times F$$

Angular/Linear – Velocities/Forces

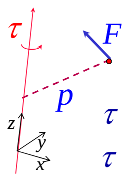


$$v = \omega \times p$$

$$v = -\hat{p}\,\omega$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -p_y \\ p_x \end{pmatrix} \dot{\theta}$$

$$v = J\,\dot{\theta}$$



$$\tau = p \times F$$

$$\tau = \hat{p}\;F$$

$$\tau = (-\hat{p})^T\;F$$

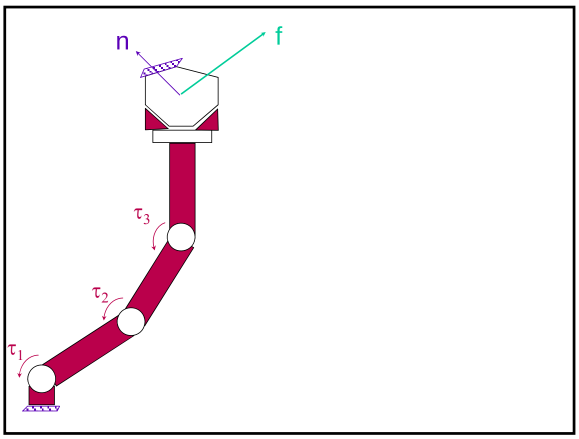
$$\tau = \begin{pmatrix} -p_y & p_x \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

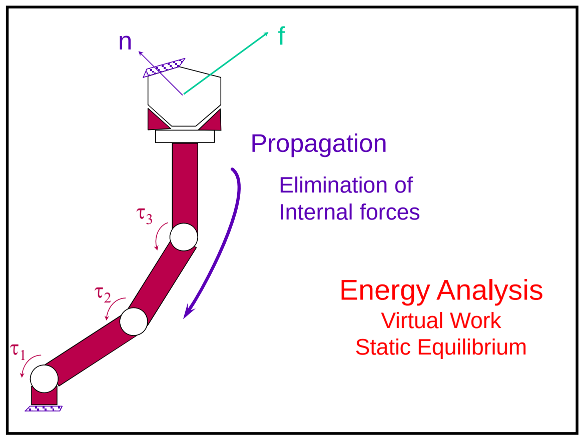
$$\tau = J^T F$$

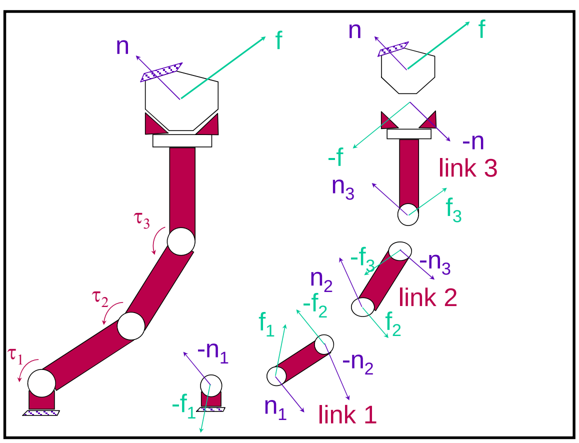
Velocity/Force Duality

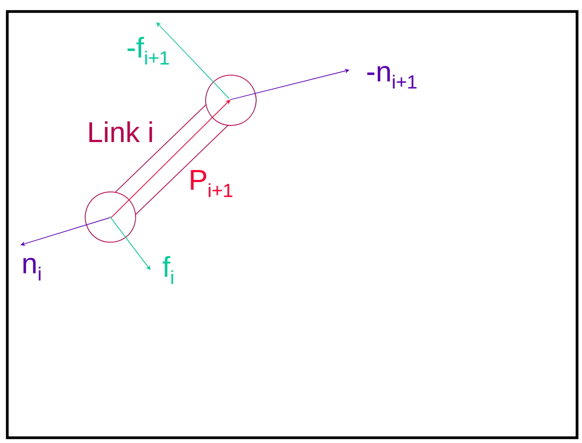
$$\dot{x} = J\;\dot{\theta}$$

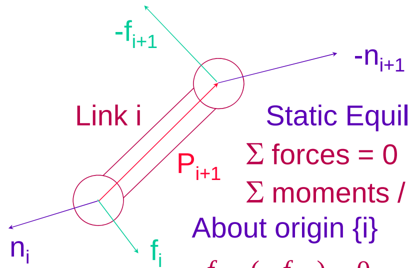
$$\tau = J^T F$$











Static Equilibrium

Σ forces = 0

Σ moments / a point = 0

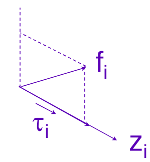
About origin {i}

$$f_i \quad (\quad f_{i \; 1}) \quad 0$$

$$n_i \quad (\quad n_{i \; 1}) \quad P_{i \; 1} \quad (\quad f_{i \; 1}) \quad 0$$

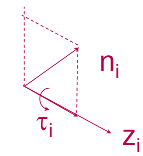
$$f_i \quad f_{i \; 1}$$

$$n_i \quad n_{i \; 1} \quad P_{i \; 1} \quad f_{i \; 1}$$



Prismatic Joint

$_i \quad f_i^{\;T} Z_i$



Revolute Joint

$_i \quad n_i^{\;T} Z_i$

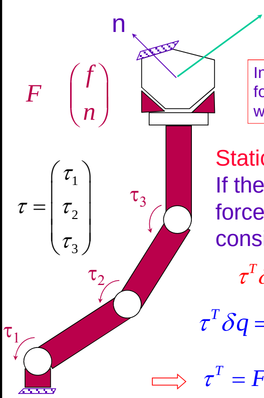
Algorithm

$${}^n f_n \quad {}^n f$$

$${}^n n_n \quad {}^n n \quad {}^n P_{n \; 1} \quad {}^n f$$

$${}^i f_i \quad {}_{i \; 1}{}^i R. {}^{i \; 1} f_{i \; 1}$$

$${}^i n_i \quad {}_{i \; 1}{}^i R. {}^{i \; 1} n_{i \; 1} \quad {}^i P_{i \; 1} \quad {}^i f_i$$



Virtual Work Principal

$F \quad \begin{pmatrix} f \\ n \end{pmatrix}$

$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$

Internal forces are workless

$$\delta w = \sum_i f_i \; \delta x_i$$

applied forces — virtual displacements

Static Equilibrium:
If the virtual work done by applied forces is zero in displacements consistent with constraints

$$\tau^T \delta q + (-F)^T \delta x = 0$$

$\tau^T \delta q = F^T \delta x$ using $\delta x = J \, \delta q$

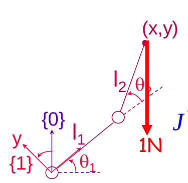
$\Longrightarrow \; \tau^T = F^T J \; \Longrightarrow \; \boxed{\tau = J^T F}$

Velocity/Force Duality

$$\boxed{\dot{x} = J \; \dot{\theta}}$$

$$\boxed{\tau = J^T F}$$

Example (Static Forces)



$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \end{pmatrix}$$

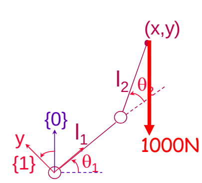
$$J^T = \begin{pmatrix} -(l_1 S1 + l_2 S12) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix}$$

$$\boxed{\tau = J^T F}$$

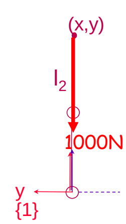
$l_1 = l_2 = 1; \quad \theta_1 = 0; \; \theta_2 = 60^\circ$

$$\tau = \begin{pmatrix} -(l_1 S1 + l_2 S12) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} l_1 C1 + l_2 C12 \\ l_2 C12 \end{bmatrix} = -\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Example (Static Forces)



$$\boxed{\tau = J^T F}$$



$$\tau = \begin{pmatrix} -(l_1 S1 + l_2 S12) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1K \end{bmatrix} = \begin{bmatrix} l_1 C1 + l_2 C12 \\ l_2 C12 \end{bmatrix} (-1K) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$l_1 = l_2 = 1; \quad \theta_1 = 90; \theta_2 = 0^\circ$