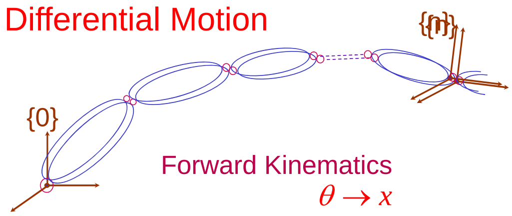
Movie Segment

Skillful manipulation based on high-speed sensory-motor fusion, T. Senoo et al., University of Tokyo, ICRA 2009 video proceedings

Instantaneous Kinematics

Differential Motion



Forward Kinematics

$\theta \rightarrow x$

Instantaneous Kinematics

$\theta + \delta\theta \rightarrow x + \delta x$

Relationship: $\boxed{\delta\theta \leftrightarrow \delta x}$

$\boxed{\dot{\theta} \leftrightarrow \dot{x}}$ ← Linear Velocity, Angular Velocity

Jacobian

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

Joint Coordinates

$$coordinate - i : \begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$$

Joint coordinate-i: $\boxed{q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i}$

with $\varepsilon_i = \begin{cases} 0 & \text{revolute} \\ 1 & \text{prismatic} \end{cases}$

and $\bar{\varepsilon}_i = 1 - \varepsilon_i$

Joint Coordinate Vector: $\boxed{q = (q_1 q_2 \ldots q_n)^T}$

Jacobians: Direct Differentiation

$$x = f(q); \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} f_1(q) \\ f_2(q) \\ \vdots \\ f_m(q) \end{pmatrix}$$

$$\delta x_1 = \frac{\partial f_1}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_1}{\partial q_n}\delta q_n$$

$$\vdots$$

$$\delta x_m = \frac{\partial f_m}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_m}{\partial q_n}\delta q_n$$

$$\delta x = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} . \delta q$$

$$\boxed{\delta x_{(m\times 1)} = J_{(m\times n)}(q)\, \delta q_{(n\times 1)}}$$

Jacobian

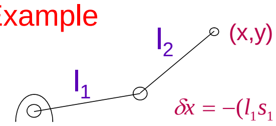
$$\delta x_{(m\times 1)} = J_{(m\times n)}(q)\, \delta q_{(n\times 1)}$$

$$\dot{x}_{(m\times 1)} = J_{(m\times n)}(q)\, \dot{q}_{(n\times 1)}$$

where

$$J_{ij}(q) = \frac{\partial}{\partial q_j} f_i(q)$$

Example



$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

$$\delta x = -(l_1 s_1 + l_2 s_{12})\delta\theta_1 - l_2 s_{12}\delta\theta_2$$

$$\delta y = (l_1 c_1 + l_2 c_{12})\delta\theta_1 + l_2 c_{12}\delta\theta_2$$

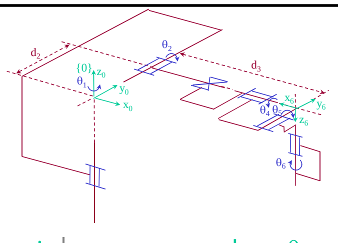
$$\delta X = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{bmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix}$$

$$\boxed{\delta x = J(\theta)\delta\theta}$$

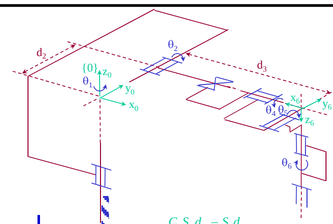
$$\dot{x} = J(\theta)\dot{\theta}$$

$$J \equiv \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \end{pmatrix} = \begin{bmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{bmatrix}$$

Stanford Scheinman Arm



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6



$$x = \begin{pmatrix} x_P \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{bmatrix} C_1 S_2 d_3 - S_1 d_2 \\ S_1 S_2 d_3 + C_1 d_2 \\ C_2 d_3 \\ C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \\ S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6) \\ S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6 \\ C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4S_5 + C_2C_5 \end{bmatrix}$$

Stanford Scheinman Arm

Position

$$x_p = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

$$\dot{x}_p = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} =$$ [redacted] $$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

$$\dot{x}_{p(3\times1)} = J_{x_p(3\times6)}(q)\,\dot{q}_{(6\times1)}$$

Linear Velocity **V**

Orientation: Direction Cosines

$$x_R = \begin{bmatrix} r_1(q) \\ r_2(q) \\ r_3(q) \end{bmatrix}$$

$$\dot{x}_R = J_{X_R}(q)\dot{q}$$

$$\dot{x}_R = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix}_{(9x1)} = \begin{pmatrix} \frac{\partial r_1}{\partial q_1} & \cdots & \frac{\partial r_1}{\partial q_6} \\ \frac{\partial r_2}{\partial q_1} & \cdots & \frac{\partial r_2}{\partial q_6} \\ \frac{\partial r_3}{\partial q_1} & \cdots & \frac{\partial r_3}{\partial q_6} \end{pmatrix}_{(9x6)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_6 \end{pmatrix}_{(6x1)}$$

$$x_R = \begin{vmatrix} C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \\ S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6) \\ S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6 \\ C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4S_5 + C_2C_5 \end{vmatrix}$$

$$\dot{x}_R = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix}_{(9x1)} = \begin{pmatrix} \frac{\partial r_1}{\partial q_1} & \cdots & \frac{\partial r_1}{\partial q_6} \\ \frac{\partial r_2}{\partial q_1} & \cdots & \frac{\partial r_2}{\partial q_6} \\ \frac{\partial r_3}{\partial q_1} & \cdots & \frac{\partial r_3}{\partial q_6} \end{pmatrix}_{(9x6)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_6 \end{pmatrix}_{(6x1)}$$

Max rank: ______

Representations

$$x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$$

- Cartesian
- Spherical
- Cylindrical
- ….

- Euler Angles
- Direction Cosines
- Euler Parameters
- ….

Jacobian for a representation X

$$\dot{x}_P = J_{x_P}(q)\,\dot{q}$$
$$\dot{x}_R = J_{x_R}(q)\,\dot{q}$$

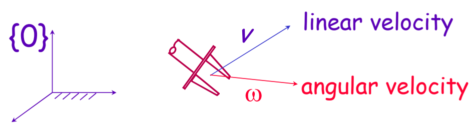
$$\begin{pmatrix} \dot{x}_P \\ \dot{x}_R \end{pmatrix} = \begin{pmatrix} J_{x_P}(q) \\ J_{x_R}(q) \end{pmatrix} \dot{q}$$

Cartesian & Direction Cosines

$$\dot{x}_{(12x1)} = J_x(q)_{(12x6)}\,\dot{q}_{(6x1)}$$

The Jacobian is dependent on the ___________

Basic Jacobian



$$\begin{pmatrix} v \\ \omega \end{pmatrix}_{(6x1)} = J_0(q)_{(6xn)}\,\dot{q}_{(nx1)}$$

$$\dot{x}_P = E_P(x_P)v$$
$$\dot{x}_R = E_R(x_R)\omega$$

Examples

$*\quad \dot{x}_P = \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} v$

$$E_P(x_P) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Examples

$*\quad \dot{x}_R = \begin{pmatrix}\dot{\alpha}\\ \dot{\beta}\\ \gamma\end{pmatrix} = \begin{pmatrix}-\frac{s\alpha.c\beta}{s\beta} & \frac{c\alpha.c\beta}{s\beta} & 1\\ c\alpha & s\alpha & 0\\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0\end{pmatrix}\omega$

$$E_R(x_R) = \begin{pmatrix}-\frac{s\alpha.c\beta}{s\beta} & \frac{c\alpha.c\beta}{s\beta} & 1\\ c\alpha & s\alpha & 0\\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0\end{pmatrix}$$

Jacobian for X

Given a representation $x = \begin{bmatrix}x_P\\ x_R\end{bmatrix}$

$$\dot{x} = J_x(q)\,\dot{q}$$

$$J_x(q) = E(x)\;J_0(q)$$

Basic Jacobian $\begin{pmatrix}v\\ \omega\end{pmatrix} = J_0(q)\;\dot{q}$

Jacobian and Basic Jacobian

$$\begin{cases} v = J_v\,\dot{q}\\ \omega = J_\omega\,\dot{q}\end{cases}$$

$\dot{x}_P = E_P.v \Rightarrow \dot{x}_P = (E_P.J_v)\dot{q}$

$\dot{x}_R = E_R.\omega \Rightarrow \dot{x}_R = (E_R.J_\omega)\dot{q}$

$$\begin{cases} J_{x_P} = E_P.J_v\\ J_{x_R} = E_R.J_\omega\end{cases}$$

$$J_x = \begin{pmatrix}J_{X_P}\\ J_{X_R}\end{pmatrix} = \begin{pmatrix}E_P & 0\\ 0 & E_R\end{pmatrix}\begin{pmatrix}J_v\\ J_\omega\end{pmatrix}$$

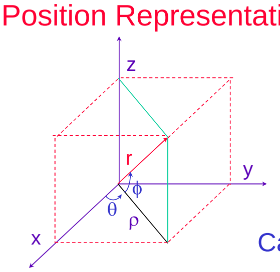
$$J_x(q) = E(x)\quad J_0(q)$$

$$\begin{pmatrix}v\\ \omega\end{pmatrix} = J_0(q)\;\dot{q}$$

With Cartesian Coordinates

$E_P = I_3\;;\; J_{x_P} = J_v\;;$ and $E = \begin{pmatrix}I & 0\\ 0 & E_R\end{pmatrix}$

Position Representations



Cartesian: (x, y, z)

Cylindrical: (ρ, θ, z)

Spherical: (r, θ, φ)

Position Representations

Cartesian Coordinates (x, y, z)

$$E_P(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \;\; y \;\; z)^T = (\rho\cos\theta \;\; \rho\sin\theta \;\; z)^T$

$$E_P(X) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

Using

$$(x \;\; y \;\; z)^T = (\rho\cos\theta\sin\phi \;\; \rho\sin\theta\sin\phi \;\; \rho\cos\theta)^T$$

$$E_P(X) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & -\sin\phi/\rho \end{pmatrix}$$

Euler Angles

$$x_R = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}; E_R(x_R) = \begin{pmatrix} -\frac{s\alpha . c\beta}{s\beta} & \frac{c\alpha . c\beta}{s\beta} & 1 \\ c\alpha & s\alpha & 0 \\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0 \end{pmatrix}$$

Singularity of the representation for $\beta = k\pi$

Jacobian for X

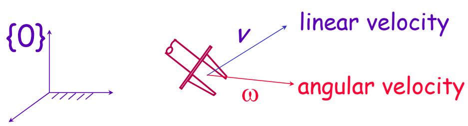
Given a representation $x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$

$$\dot{x} = J_x(q)\,\dot{q}$$

$$J_x(q) = E(x)\; J_0(q)$$

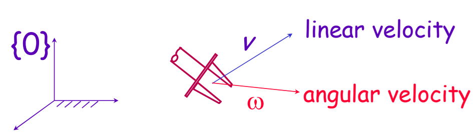
Basic Jacobian $\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q)\;\dot{q}$

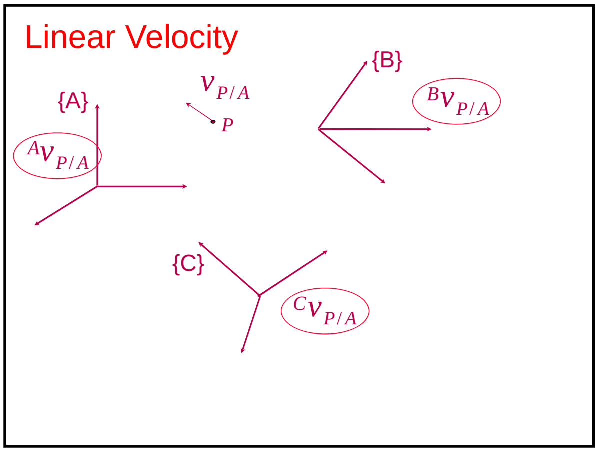
Jacobian

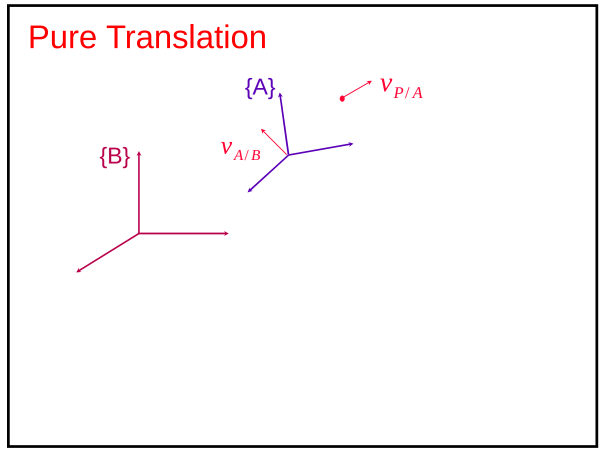


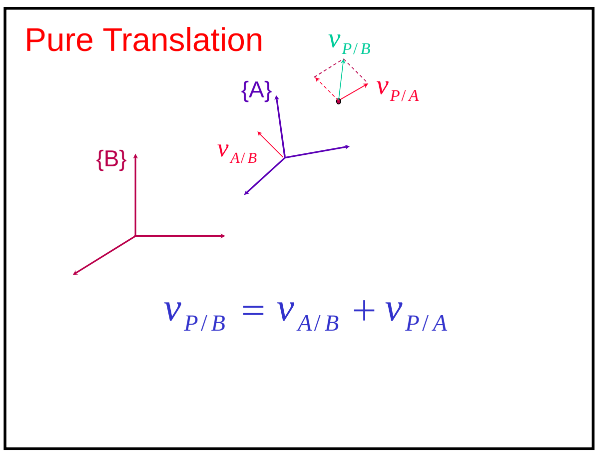
$$\begin{pmatrix} v \\ \omega \end{pmatrix}_{(6x1)} = J\,(q)_{(6xn)}\,\dot{q}_{(nx1)}$$

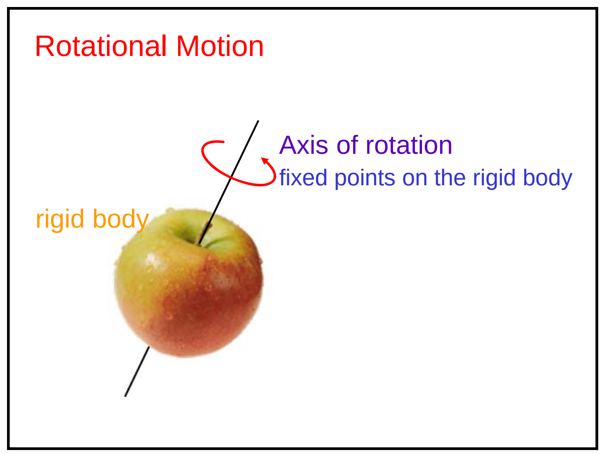
Linear & Angular Velocities

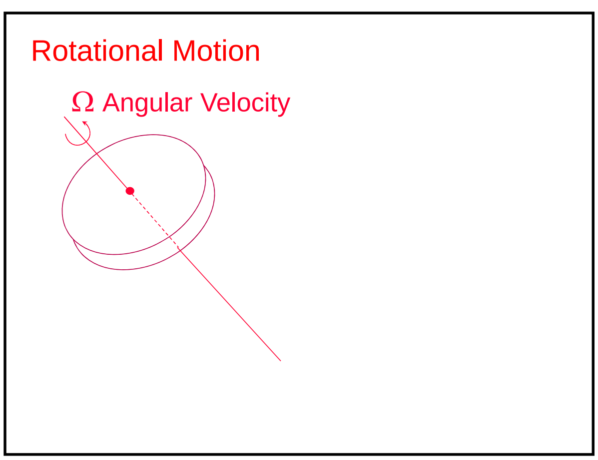


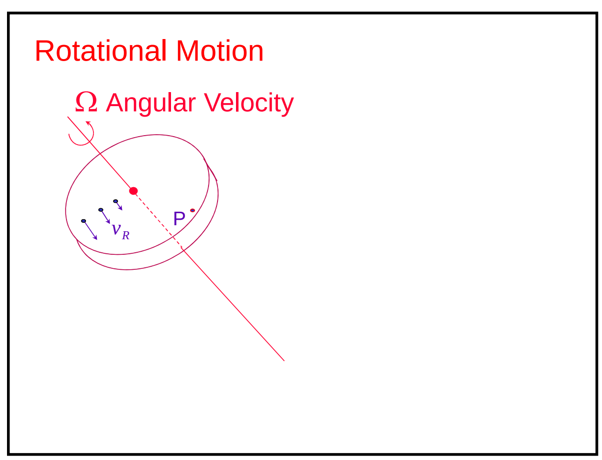

Linear Velocity
{A}
{B}
{C}
$v_{P/A}$
P
${}^{A}v_{P/A}$
${}^{B}v_{P/A}$
${}^{C}v_{P/A}$


Pure Translation
{A}
{B}
$v_{P/A}$
$v_{A/B}$


Pure Translation
$v_{P/B}$
{A}
$v_{P/A}$
{B}
$v_{A/B}$
$v_{P/B} = v_{A/B} + v_{P/A}$

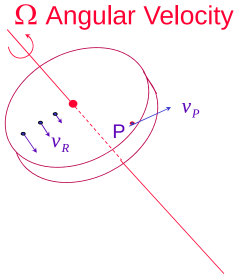

Rotational Motion
Axis of rotation
fixed points on the rigid body
rigid body


Rotational Motion
Ω Angular Velocity


Rotational Motion
Ω Angular Velocity
P
v_R

Rotational Motion

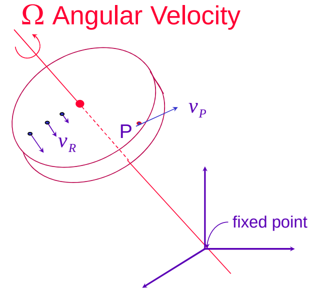
Ω Angular Velocity



$v_P = ?$

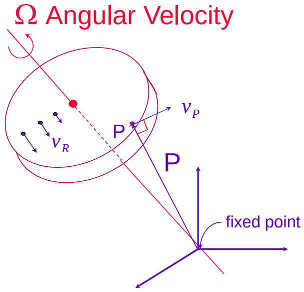
Rotational Motion

Ω Angular Velocity



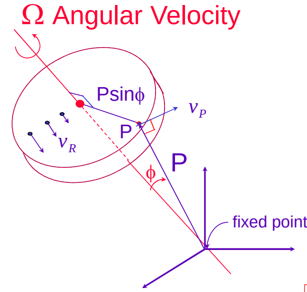
Rotational Motion

Ω Angular Velocity



Rotational Motion

Ω Angular Velocity



v_P is proportional to:
- $\|\Omega\|$
- $\|P\sin\phi\|$

and
- $v_P \perp \Omega$
- $v_P \perp P$

$v_P = \Omega \times P$

Cross Product Operator

$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$c = a \times b \Rightarrow c = \hat{a}b$$

vectors $\Rightarrow$ matrices

$a \times \Rightarrow \hat{a}$: a skew-symmetric matrix

$$c = \hat{a}b = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$c = \hat{a}b$$

Cross Product Operator

$$v_P = \Omega \times P \Rightarrow v_P = \hat{\Omega}P$$

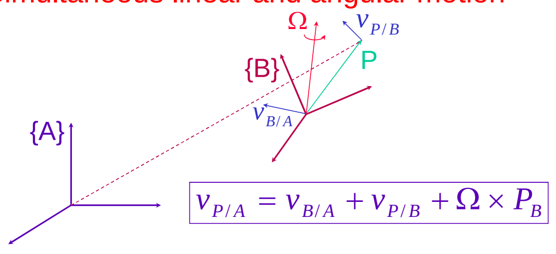
$\Omega\times \Rightarrow \hat{\Omega}$: a skew-symmetric matrix

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}; P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$v_P = \hat{\Omega}P = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \cdot \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$v_P = \hat{\Omega}P$$

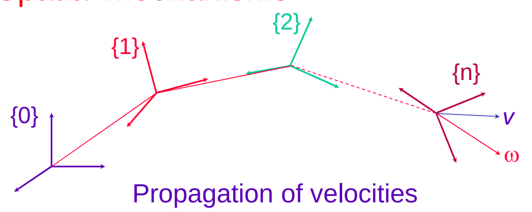
Simultaneous linear and angular motion



$$v_{P/A} = v_{B/A} + v_{P/B} + \Omega \times P_B$$

$$ {}^{A}v_{P/A} = {}^{A}v_{B/A} + {}^{A}_{B}R.{}^{B}v_{P/B} + {}^{A}\Omega_{B} \times {}^{A}_{B}R.{}^{B}P_{B} $$

Spatial Mechanisms

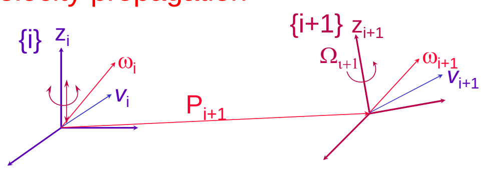


Propagation of velocities

$\dot{x}$ — v : linear velocity; ω : angular velocity

$$\dot{x} = J(\theta).\dot{\theta}$$

Velocity propagation



Linear: $v_{i+1} = v_i + \omega_i \times P_{i+1} + \dot{d}_{i+1}.Z_{i+1}$

Angular: $\omega_{i+1} = \omega_i + \Omega_{i+1}$

$\Omega_{i+1} = \dot{\theta}_{i+1}.Z_{i+1}$

Velocity propagation

Joint 1

v_1 and ω_1 in frame {1}

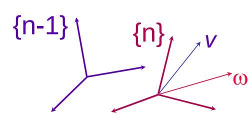
Joint i+1

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R.{}^{i}\omega_i + \dot{\theta}_{i+1}.{}^{i+1}Z_{i+1} $$

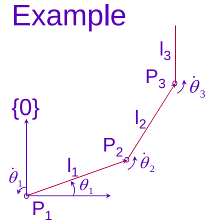
$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R.({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1}.{}^{i+1}Z_{i+1} $$

$\Rightarrow$ ${}^{n}\omega_n$ and ${}^{n}v_n$

$$ \begin{pmatrix} {}^{0}v_n \\ {}^{0}\omega_n \end{pmatrix} = \begin{pmatrix} {}^{0}_{n}R & 0 \\ 0 & {}^{0}_{n}R \end{pmatrix} . \begin{pmatrix} {}^{n}v_n \\ {}^{n}\omega_n \end{pmatrix} $$



Example



$$v_{i+1} = v_i + \omega_i \times P_{i+1}$$

- $v_{P_1} = 0$ ${}^{0}\omega_1 = \dot{\theta}_1.{}^{0}Z_1$
- $v_{P_2} = v_{P_1} + \omega_1 \times P_2$
- $v_{P_3} = v_{P_2} + \omega_2 \times P_3$

$$ {}^{0}v_{P_2} = 0 + \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1.c_1 \\ l_1.s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} .\dot{\theta}_1 $$

$$ {}^{0}v_{P_3} = {}^{0}v_{P_2} + {}^{0}\omega_2 \times {}^{0}P_3 $$

$$ {}^{0}v_{P_3} = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} .\dot{\theta}_1 + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .(\dot{\theta}_1 + \dot{\theta}_2).{}^{0}P_3 $$

$$ {}^{0}P_3 \leftarrow \begin{bmatrix} l_2.c_{12} \\ l_2.s_{12} \\ 0 \end{bmatrix} $$

$$ = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} .\dot{\theta}_1 + \begin{bmatrix} -l_2.s_{12} \\ l_2.c_{12} \\ 0 \end{bmatrix} .(\dot{\theta}_1 + \dot{\theta}_2) $$

$$ {}^{0}\omega_3 = (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3).{}^{0}Z_0 $$

$$ {}^{0}v_{P_3} = \underbrace{\begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{J_v} . \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} $$

$$ {}^{0}\omega_3 = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{J_\omega} . \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} $$

$$ \begin{pmatrix} v \\ \omega \end{pmatrix} = J . \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} $$