Video Segment

DARPA Robotics Challenge 2013 - A Woodstock for Robots, A. Aden-Buie and J. Markoff, 2013.

Dynamics

- Rigid Body Dynamics
- Newton-Euler Formulation
- Articulated Multi-Body Dynamics
- Recursive Algorithm
- Lagrange Formulation
- Explicit Form

Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

Lagrangian → $L = K - U$ ← Kinetic Energy (K), Potential Energy (U)

Since $U = U(q)$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} + \frac{\partial U}{\partial q} = \tau$$

Inertial forces: $\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q}$

Gravity vector: $\frac{\partial U}{\partial q}$

Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right)-\frac{\partial K}{\partial q}=\tau-G; \quad G=\frac{\partial U}{\partial q}$$

Inertial forces

$$\Downarrow$$

$$M(q)\ddot{q}+V(q,\dot{q})=\tau-G(q)$$

Inertial forces

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = \tau - G \qquad K = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

$$\frac{\partial K}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\left[\frac{1}{2}\dot{q}^T M(q)\dot{q}\right] = M(q)\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) = \frac{d}{dt}(M\dot{q}) = M\ddot{q} + \dot{M}\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_n}\dot{q} \end{bmatrix} = M\ddot{q} + V(q,\dot{q})$$

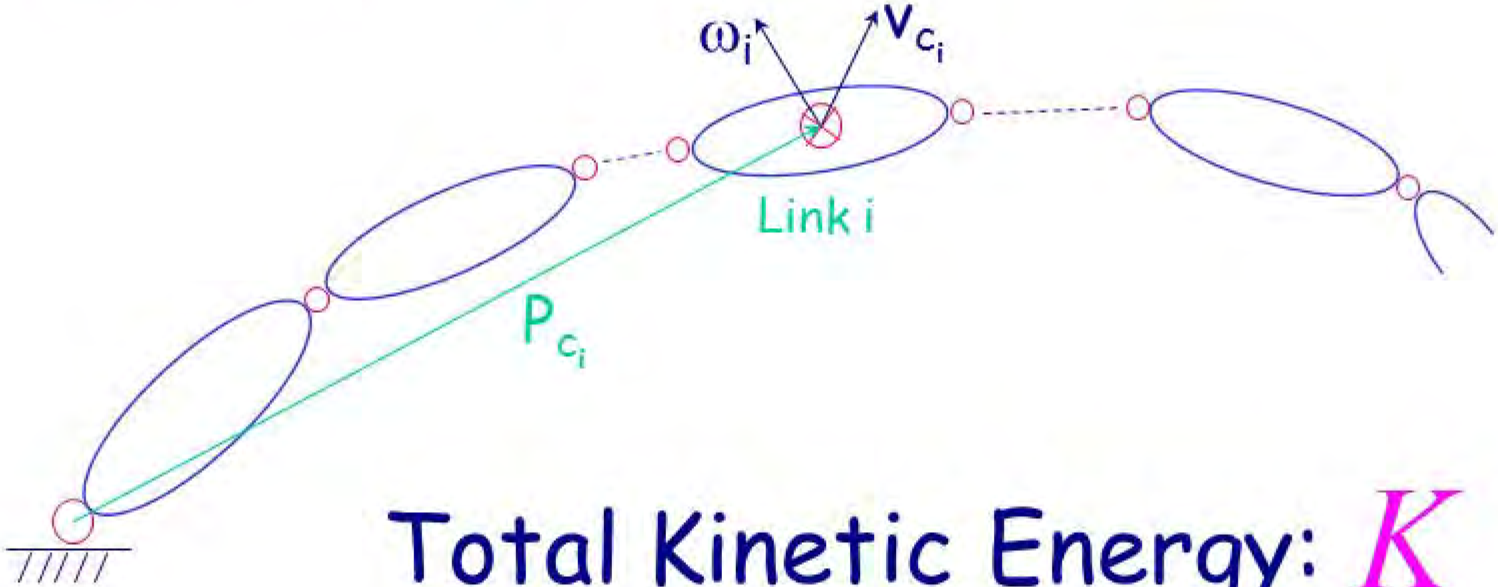
Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right)-\frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \dfrac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \dfrac{\partial M}{\partial q_n}\dot{q} \end{bmatrix} = M\ddot{q} + V(q,\dot{q})$$

$$\boxed{M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau}$$

$$M(q): K = \frac{1}{2}\dot{q}^T M \dot{q} \qquad M(q) \Rightarrow V(q,\dot{q})$$

Equations of Motion

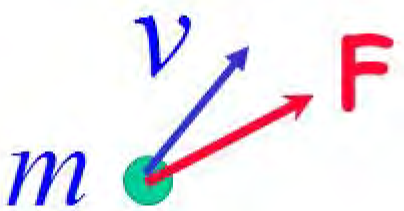


Total Kinetic Energy: K

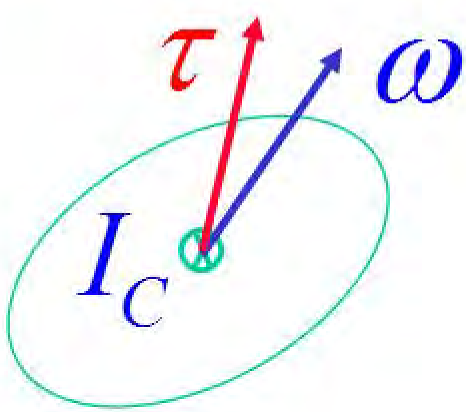
$$K = \sum K_{Link\,i} \equiv \frac{1}{2}\dot{q}^T M \dot{q}$$

Kinetic Energy

Work done by external forces to bring the system from rest to its current state.

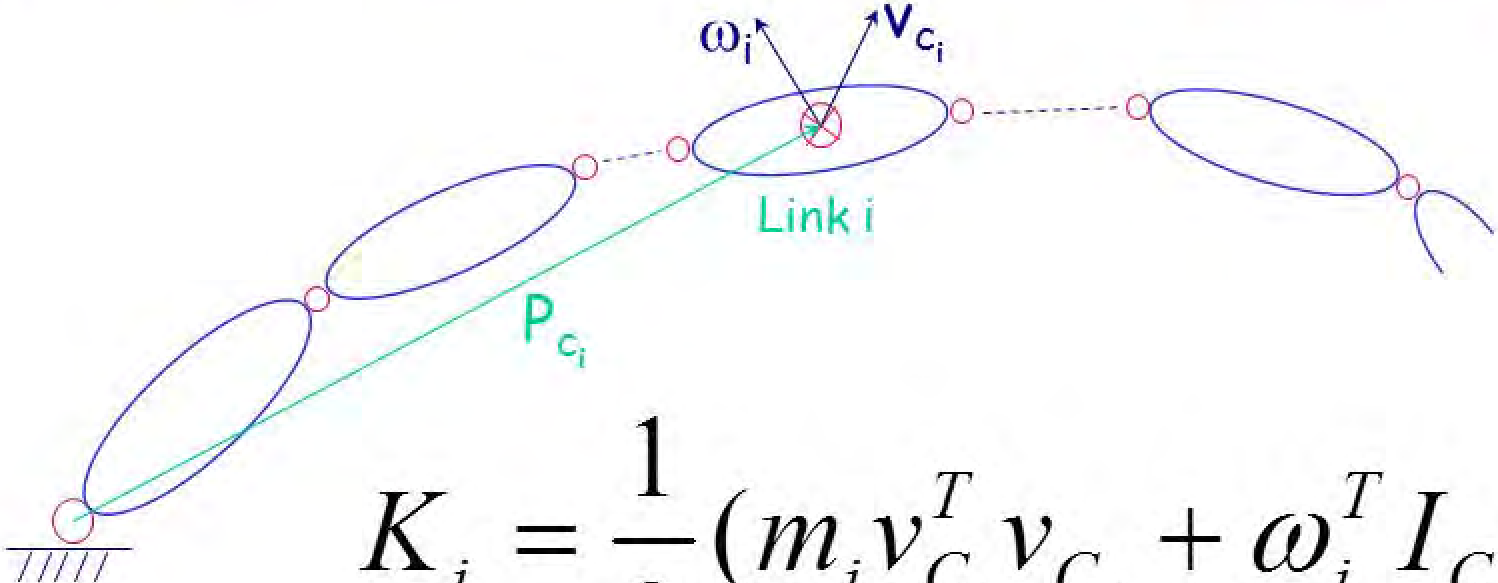


$$K = \frac{1}{2} m v^2$$



$$K = \frac{1}{2} \omega^T I_C \, \omega$$

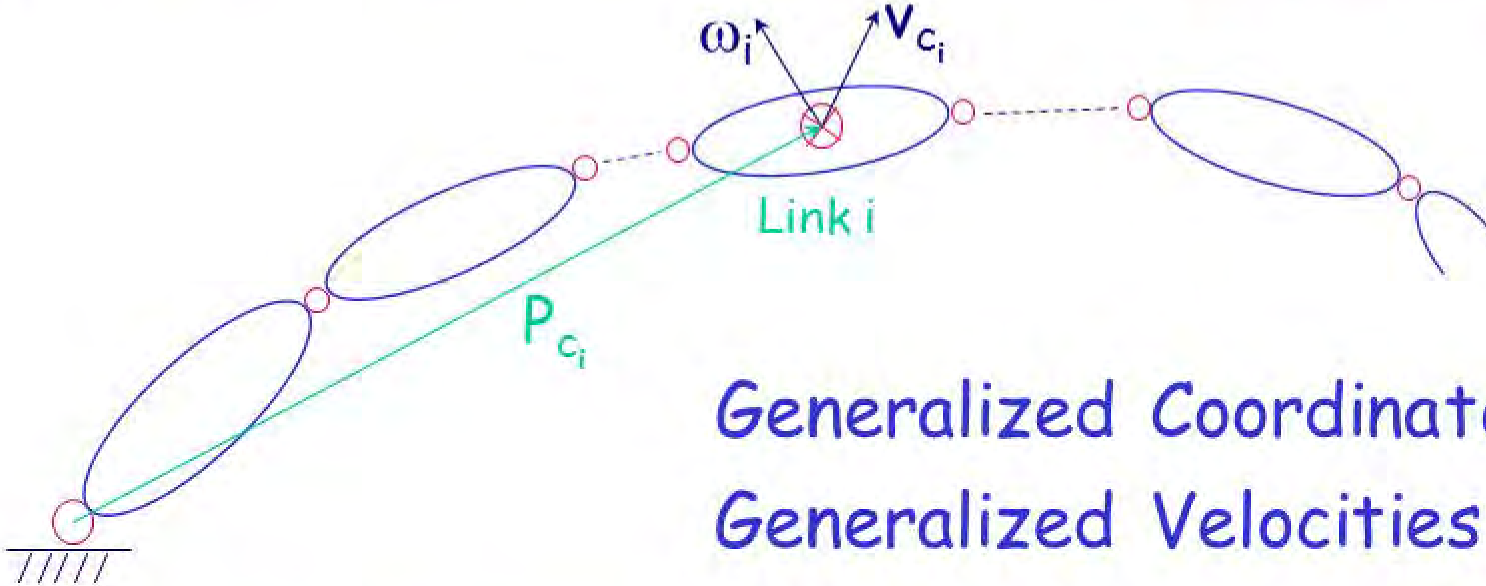
Equations of Motion — Explicit Form



$$K_i = \frac{1}{2}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{Ci} \omega_i)$$

Total Kinetic Energy ⇨ $K = \sum_{i=1}^{n} K_i$

Equations of Motion Explicit Form



Generalized Coordinates q

Generalized Velocities $\dot{q}$

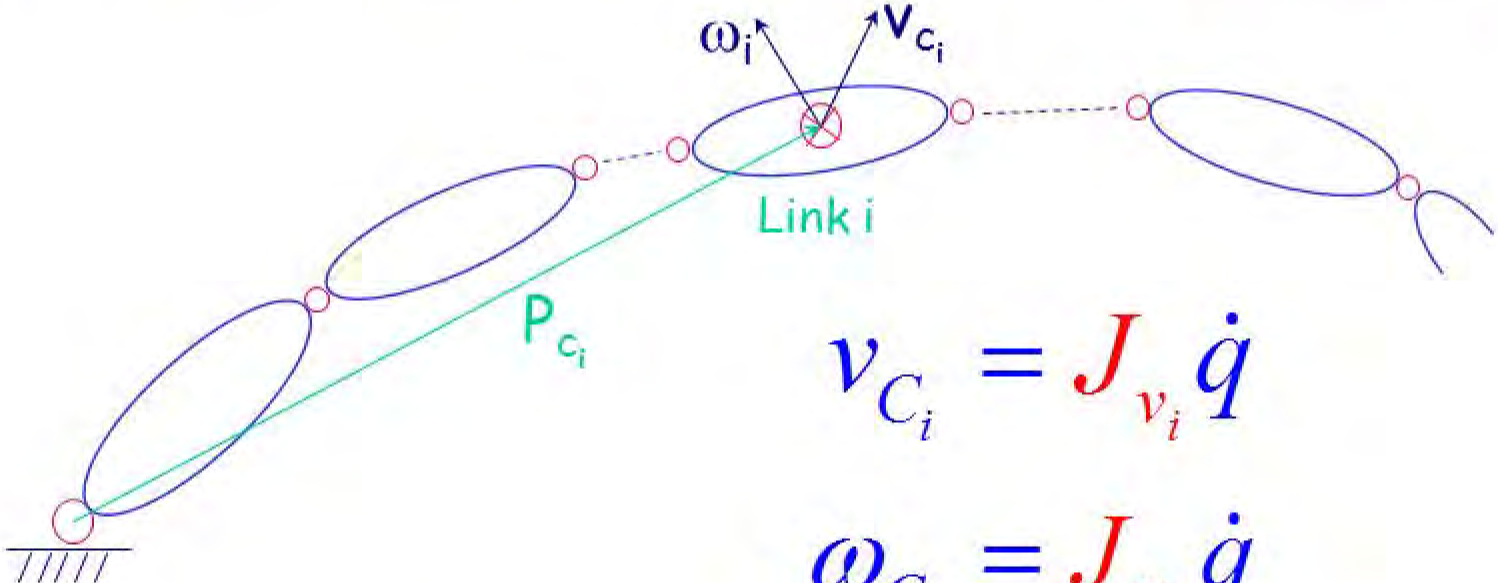
Kinetic Energy

Quadratic Form of Generalized Velocities

$$K = \frac{1}{2}\dot{q}^T M \dot{q}$$

$$\frac{1}{2}\dot{q}^T M \dot{q} \equiv \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{Ci}\omega_i)$$

Equations of Motion — Explicit Form



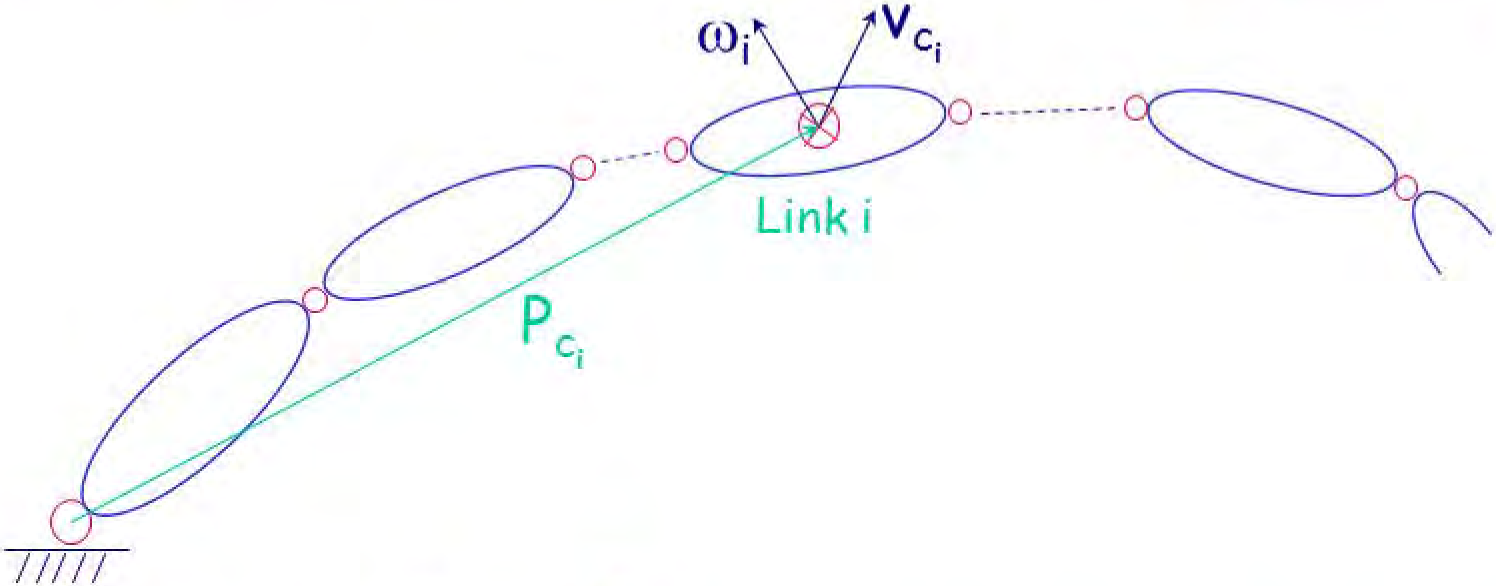
$$v_{C_i} = J_{v_i}\dot{q}$$

$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

$$\frac{1}{2}\dot{q}^T M\dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i}\omega_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n}(m_i\dot{q}^T J_{v_i}^T J_{v_i}\dot{q} + \dot{q}^T J_{\omega_i}^T I_{Ci} J_{\omega_i}\dot{q})$$

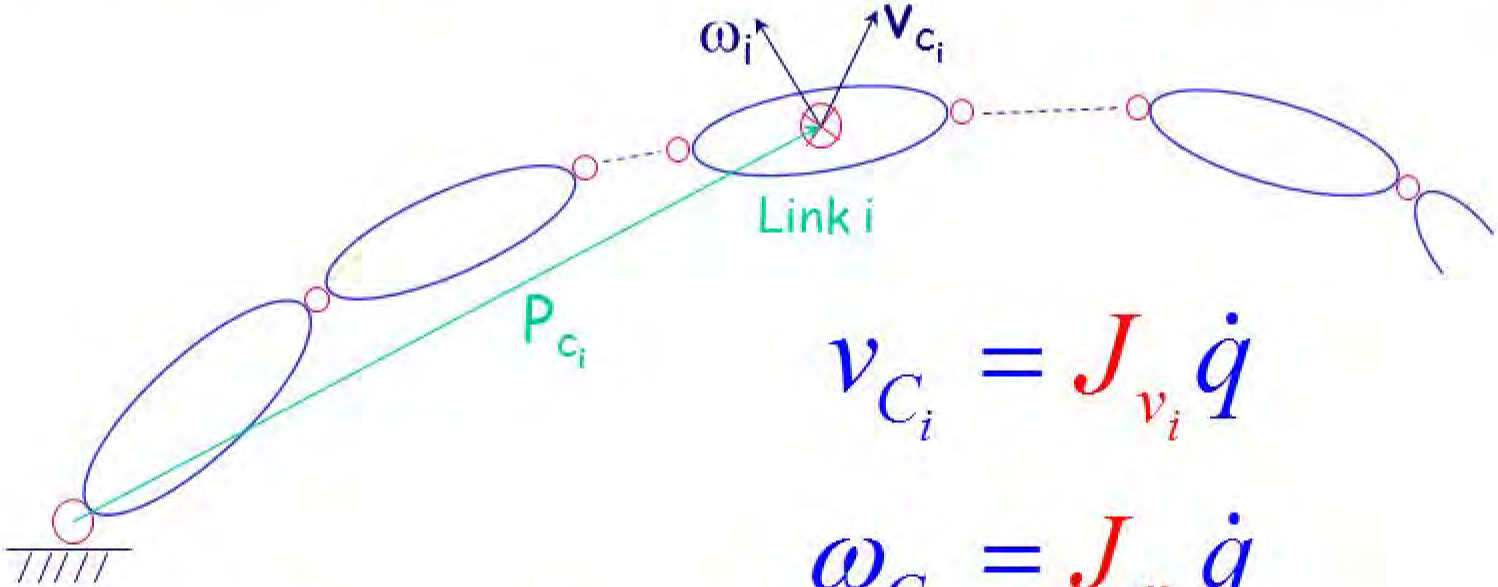
Equations of Motion Explicit Form



$$\frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{n} (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{Ci} J_{\omega_i}) \right] \dot{q}$$

$$M = \sum_{i=1}^{n} (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{Ci} J_{\omega_i})$$

Equations of Motion — Explicit Form

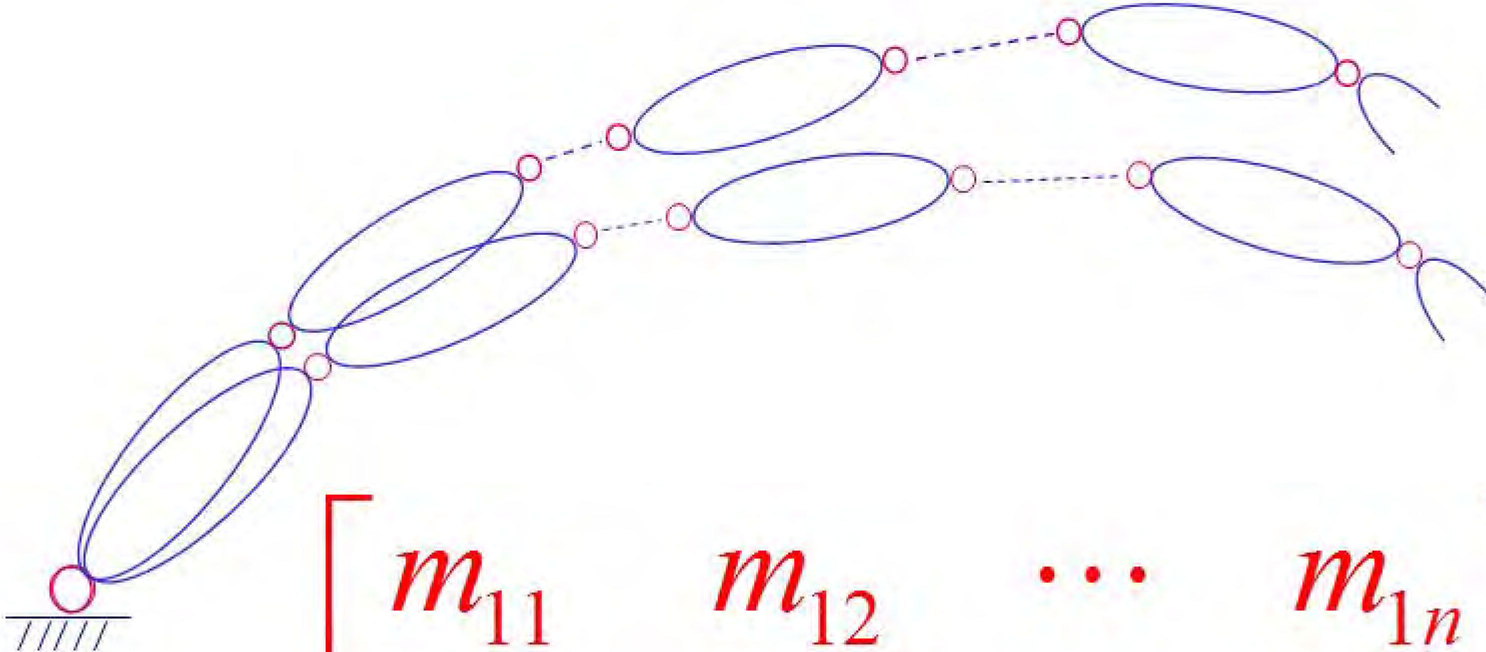


$$v_{C_i} = J_{v_i}\dot{q}$$

$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

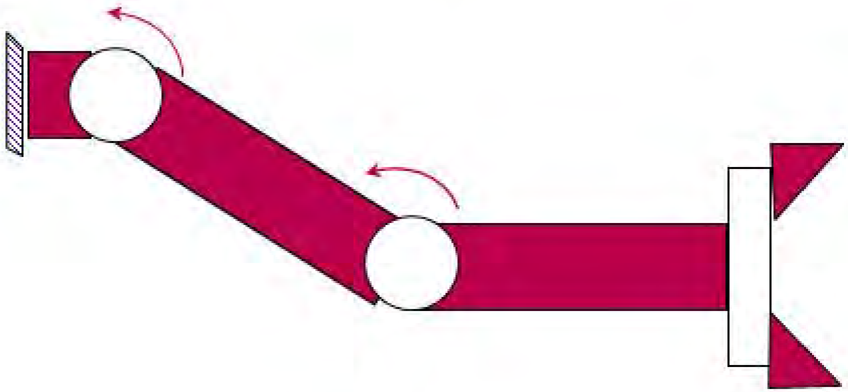
$$J_{v_i} = \left[\frac{\partial p_{C_i}}{\partial q_1} \quad \frac{\partial p_{C_i}}{\partial q_2} \quad \cdots \quad \frac{\partial p_{C_i}}{\partial q_i} \quad 0 \quad 0 \quad \cdots \quad 0\right]$$

$$J_{\omega_i} = \left[\bar{\varepsilon}_1 z_1 \quad \bar{\varepsilon}_2 z_2 \quad \cdots \quad \bar{\varepsilon}_i z_i \quad 0 \quad 0 \quad \cdots \quad 0\right]$$



$$M(q)_{(n\times n)} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

Vector $V(\mathbf{q}, \dot{\mathbf{q}})$ Centrifugal & Coriolis Forces



$$\begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_n}\dot{q} \end{bmatrix} = M\ddot{q} + V(q,\dot{q})$$

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau$$

$$M(q): K = \frac{1}{2}\dot{q}^T M \dot{q} \qquad M(q) \Rightarrow V(q,\dot{q})$$

Vector $V(\mathbf{q},\dot{\mathbf{q}})$

$\frac{\partial M}{\partial q_1}$ $\frac{\partial m_{11}}{\partial q_1}$

$$V = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T M_{q_1}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T M_{q_2}\dot{\mathbf{q}} \end{bmatrix} = \begin{pmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{12} & \dot{m}_{22} \end{pmatrix}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \begin{pmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{pmatrix}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \begin{pmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{pmatrix}\dot{\mathbf{q}} \end{bmatrix}$$

$$\dot{m}_{ij} = m_{ij1}\dot{q}_1 + m_{ij2}\dot{q}_2$$

$\frac{\partial m_{22}}{\partial q_2}$

$$V(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} \frac{1}{2}(m_{111} + m_{111} - m_{111}) & \frac{1}{2}(m_{122} + m_{122} - m_{221}) \\ \frac{1}{2}(m_{211} + m_{211} - m_{112}) & \frac{1}{2}(m_{222} + m_{222} - m_{222}) \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} m_{112} + m_{121} - m_{121} \\ m_{212} + m_{221} - m_{122} \end{bmatrix}\begin{bmatrix} \dot{q}_1\dot{q}_2 \end{bmatrix}$$

Christoffel Symbols

$$\frac{\partial m_{ij}}{\partial q_k}$$

$$b_{ijk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki})$$

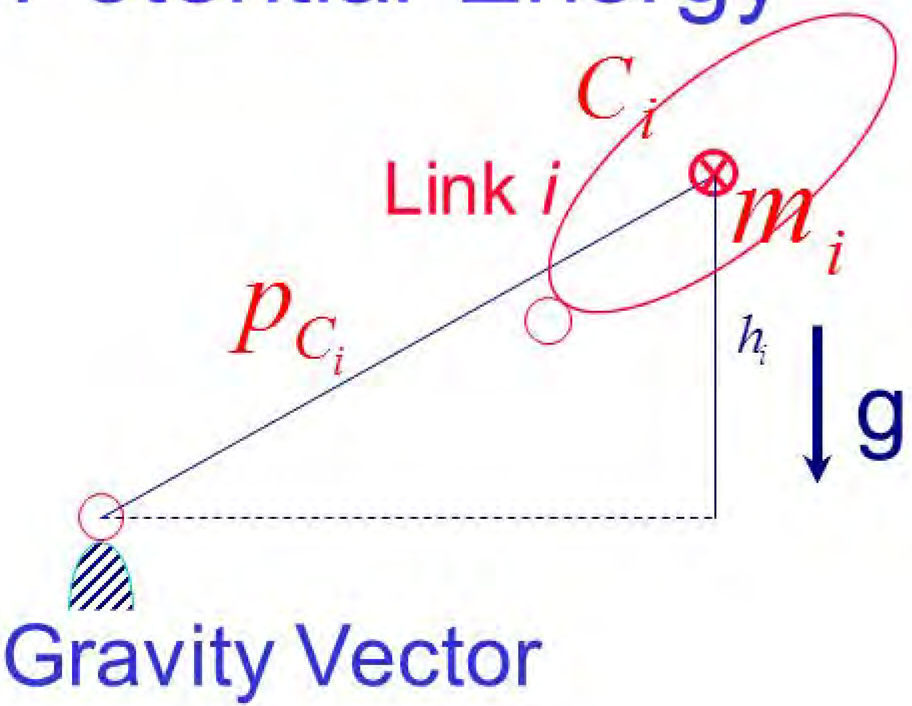
$$V = \begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

$C(\mathbf{q})$ $B(\mathbf{q})$

$$\underset{(n\times n)}{C(\mathbf{q})}\underset{(n\times 1)}{[\dot{\mathbf{q}}^2]} = \begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$$\underset{(n\times\frac{(n-1)n}{2})}{B(\mathbf{q})} \underset{(\frac{(n-1)n}{2}\times 1)}{[\dot{\mathbf{q}}\dot{\mathbf{q}}]} = \begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \vdots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix} \begin{bmatrix} \dot{q}_1\dot{q}_2 \\ \dot{q}_1\dot{q}_3 \\ \vdots \\ \dot{q}_{(n-1)}\dot{q}_n \end{bmatrix}$$

Potential Energy



$$U_i = m_i g_0 h_i + U_0$$

$$U_i = m_i(-g^T p_{C_i}) + U_0$$

$$U = \sum U_i$$

Lagrange Equations

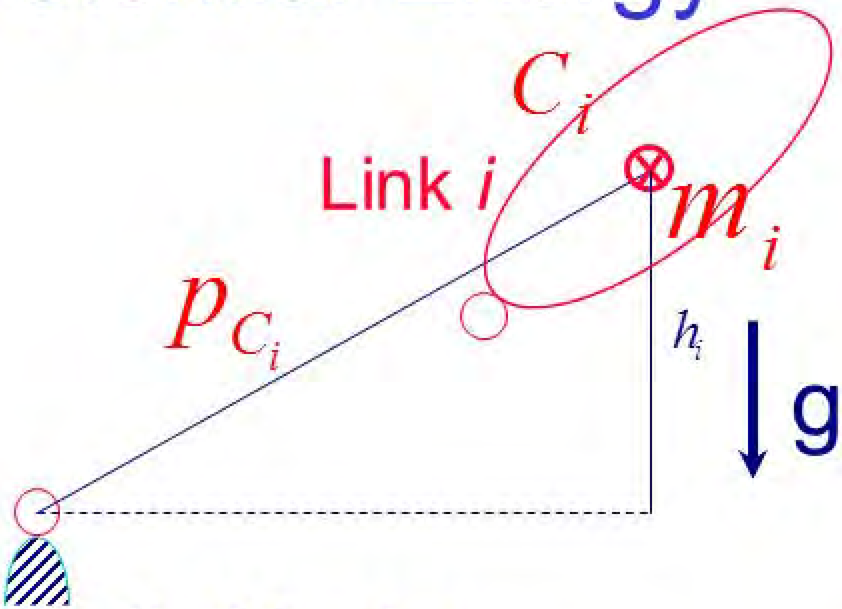
$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right)-\frac{\partial K}{\partial q}=\tau-G; \quad G=\frac{\partial U}{\partial q}$$

Inertial forces

⇓

$$M(q)\ddot{q}+V(q,\dot{q})=\tau-G(q)$$

Potential Energy



$$U_i = m_i g_0 h_i + U_0$$

$$U_i = m_i(-g^T p_{C_i}) + U_0$$

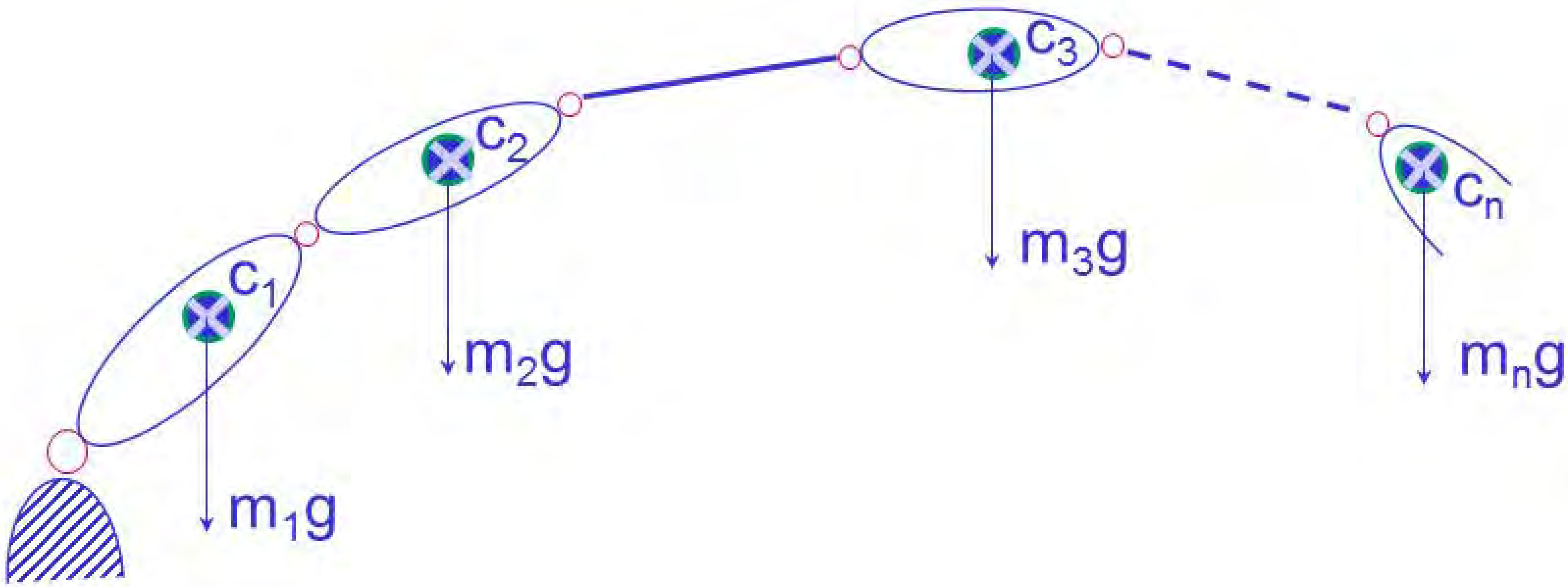
$$U = \sum U_i$$

Gravity Vector

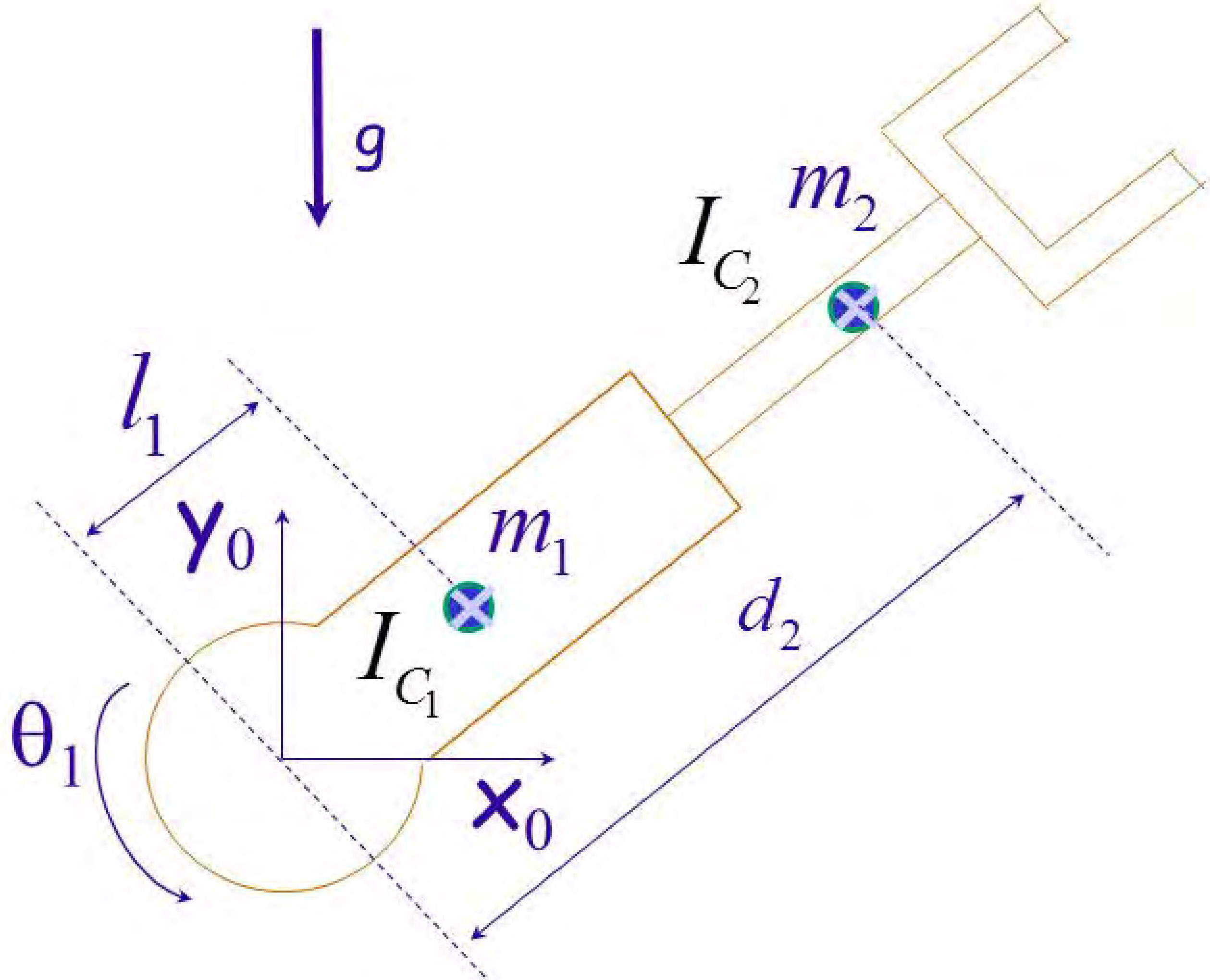
$$G_j = \frac{\partial U}{\partial q_j} = -\sum_{i=1}^{n} (m_i g^T \frac{\partial \mathbf{p}_{C_i}}{\partial q_j})$$

$$G = -\begin{pmatrix} J_{v_1}^T & J_{v_2}^T & \cdots & J_{v_n}^T \end{pmatrix} \begin{pmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{pmatrix}$$

Gravity Vector



$$G = -(J_{v_1}^T(m_1 g) + J_{v_2}^T(m_2 g) + \cdots + J_{v_n}^T(m_n g))$$



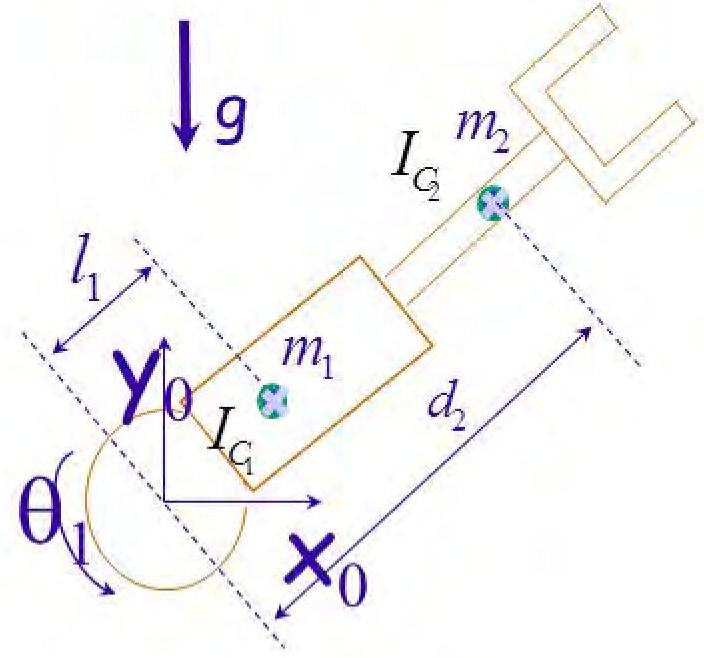
g
m2
I C2
l1
Y0
m1
I C1
d2
θ1
x0

Matrix M

$$M = m_1 J_{v_1}^T J_{v_1} + J_{\omega_1}^T I_{C_1} J_{\omega_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T I_{C_2} J_{\omega_2}$$

J_{v_1} and J_{v_2} : direct differentiation of the vectors:

$${}^0\mathbf{p}_{C_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}; \text{ and } {}^0\mathbf{p}_{C_2} = \begin{bmatrix} d_2 c_1 \\ d_2 s_1 \\ 0 \end{bmatrix}$$



In frame {0}, these matrices are:

$${}^0J_{v_1} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } {}^0J_{v_2} = \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix}$$

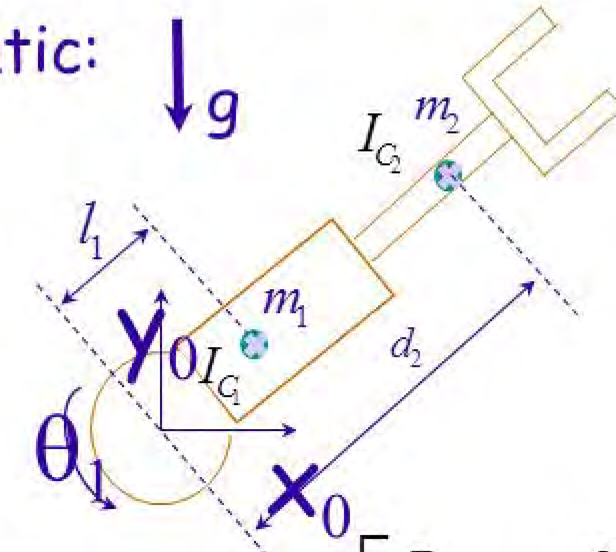
This yields

$$m_1({}^0J_{v_1}^T\, {}^0J_{v_1}) = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } m_2({}^0J_{v_2}^T\, {}^0J_{v_2}) = \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

The matrices J_{ω_1} and J_{ω_2} are given by

$$J_{\omega_1} = \begin{bmatrix} \overline{\epsilon}_1 \mathbf{z}_1 & 0 \end{bmatrix} = \text{ and } J_{\omega_2} = \begin{bmatrix} \overline{\epsilon}_1 \mathbf{z}_1 & \overline{\epsilon}_2 \mathbf{z}_2 \end{bmatrix}$$

Joint 1 is revolute and joint 2 is prismatic:



$$^1J_{\omega_1} = {}^1J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

And

$$({}^1J_{\omega_1}^T \, {}^1I_{C_1} \, {}^1J_{\omega_1}) = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } ({}^1J_{\omega_2}^T \, {}^1I_{C_2} \, {}^1J_{\omega_2}) = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}$$

Finally,

$$M = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$

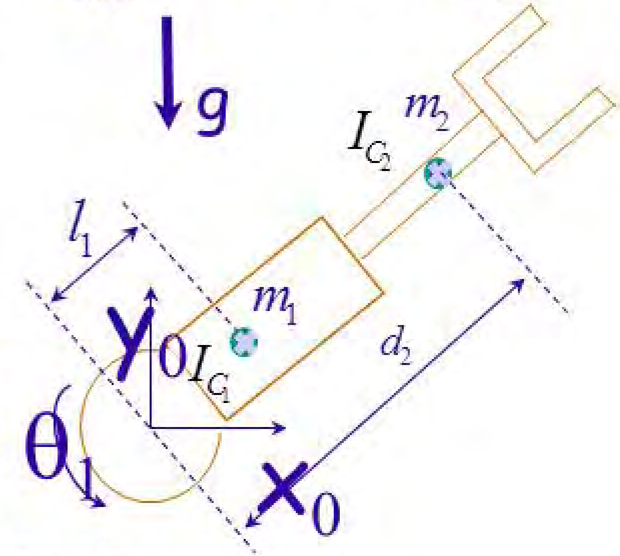
Centrifugal and Coriolis Vector *V*

$$b_{i,jk} = \frac{1}{2}\left(m_{ijk} + m_{ikj} - m_{jki}\right)$$

where $m_{ijk} = \dfrac{\partial m_{ij}}{\partial q_k}$; with $b_{iii} = 0$ and $b_{iji} = 0$ for $i > j$

For this manipulator, only m_{11} is configuration dependent - function of d_2. This implies that only m_{112} is non-zero,

$$m_{112} = 2m_2 d_2 .$$



Matrix *B*

$$B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}.$$

Matrix *C*

$$C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}.$$

Matrix *V*

$$V = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}\left[\dot{\theta}_1 \dot{d}_2\right] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$$

Vector V

$$V = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$$

The Gravity Vector G

$$\mathbf{G} = -\left[J_{v_1}^T m_1 \mathbf{g} + J_{v_2}^T m_2 \mathbf{g} \right].$$

In frame {0}, ${}^0\mathbf{g} = \begin{pmatrix} 0 & -g & 0 \end{pmatrix}^T$ and the gravity vector is

$${}^0\mathbf{G} = -\begin{bmatrix} -l_1 s_1 & l_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -d_2 s_1 & d_2 c_1 & 0 \\ c_1 & s_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

and

$${}^0\mathbf{G} = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix}$$

Equations of Motion

$$\begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.$$