Video Segment

From Compliant Balancing to Dynamic Walking, ATR/NICT, Japan, ICRA 2010

Dynamics

- Rigid Body Dynamics
- Newton-Euler Formulation
- Articulated Multi-Body Dynamics
- Recursive Algorithm
- Lagrange Formulation
- Explicit Form

MA23

COEFFICIENTS DYNAMIQUES DU MA 23

$$A_{11} = I_{z1} + M_2 SS2\, d_2^2 + SS2\, I_{y2} + CC2\, I_{z2} + M_3 (SS2\, \ell_2^2 + 2S2S(2+3)\, \ell_2 d_3 + SS(2+3) d_3^2)$$
$$+SS(2+3)\, I_{y3} + CC(2+3)\, I_{z3} + M_4 (SS2\, \ell_2^2 + 2S2S(2+3)\, \ell_2 h_1 + 2S2C(2+3)\, C4\, \ell_2\, d_{y4}$$
$$+SS(2+3)\, h_1^2 + 2SC(2+3)C4 d_{y4} h_1 + CC(2+3)\, d_{y4}^2 + SS(2+3)SS4 d_{y4}^2) + SS(2+3)SS4\, I_{x4}$$
$$+SS(2+3)\, CC4\, I_{y4} - 2SC(2+3)C4\, I_{yz4} + CC(2+3)\, I_{z4} + M_5 (SS2\, \ell_2^2 + 2S2S(2+3)\, \ell_2 h_2$$
$$+ 2S2C(2+3)C4\, \ell_2\, \ell_{y4} + 2S2S(2+3)\, C5\, \ell_2 d_5 + 2S2C(2+3)C4S5\, \ell_2 d_5 + SS(2+3)\, h_2^2 +$$
$$2SC(2+3)C4\, \ell_{y4} h_2 + 2SS(2+3)C5 h_2 d_5 + 2SC(2+3)\, C4S5 h_2 d_5 + CC(2+3)\, \ell_{y4}^2 +$$
$$2SC(2+3)C4C5\, \ell_{y4} d_5 + 2CC(2+3)S5\, \ell_{y4}\, d_5 + SS(2+3)CC4CC5\, d_5^2 + 2SC(2+3)C4SC5\, d_5^2 +$$
$$CC(2+3)\, SS5 d_5^2 + SS(2+3)SS4\, \ell_{y4}^2 + 2SS(2+3)SS4S5\, \ell_{y4} d_5 + SS(2+3)SS4 d_5^2) + SS(2+3)SS4 I_{x5}$$
$$+SS(2+3)CC4CC5 I_{y5} + 2SC(2+3)C4SC5 I_{y5} + CC(2+3)SS5 I_{y5} + SS(2+3)CC4SS5 I_{z5} - 2SC(2+3)C4SC5 I$$
$$+CC(2+3)CC5 I_{z5} + M_6 (SS2\, \ell_9^2 + 2S2S(2+3)\, \ell_9 d_6 + SS(2+3) d_6^2) + SS(2+3) I_{y6} + CC(2+3) I_{z6}$$

$$A_{22} = A_{33} + M_2 d_2^2 + I_{x2} + M_3\, (2C3\, \ell_2 d_3 + \ell_2^2\,) + M_4 (2C3\, \ell_2 h_1 + \ell_2^2 - 2S3C4\, \ell_2 d_{y4})$$
$$+M_5 (\, \ell_2^2 + 2C3\, \ell_2 h_2 + 2C3C5\, \ell_2 d_5 - 2S3C4\, \ell_2\, \ell_{y4} - 2S3C4S5\, \ell_2 d_5) + M_6 (2C3\, \ell_9 d_6 + \ell_9^2\,)$$

$$A_{33} = M_3 d_3^2 + I_{x3} + M_4 (h_1^2 + CC4\, d_{y4}^2\,) + CC4\, I_{x4} + SS4\, I_{y4} + M_5 (h_2^2 + 2C5 h_2 d_5 + SS4CC5 d_5^2$$
$$+CC4\, \ell_{y4}^2 + 2CC4S5\, \ell_{y4} d_5 + CC4\, d_5^2) + CC4 I_{x5} + SS4CC5 I_{y5} + SS4SS5 I_{z5} + M_6 d_6^2 + I_{x6}$$

$$A_{44} = M_4 d_{y4}^2 + I_{z4} + M_5\, (\, \ell_{y4}^2 + 2S5\, \ell_{y4} d_5 + SS5 d_5^2\,) + SS5 I_{y5} + CC5 I_{z5}$$

$$A_{55} = M_5 d_5^2 + I_{x5}$$

$$A_{66} = I_{z5}$$

$$A_{12} = A_{13} - M_4 C2S4\, \ell_2 d_{y4} + M_5 (\, -C2S4\, \ell_2\, \ell_{y4} - C2S4S5\, \ell_2\, d_5)$$

$$A_{13} = M_4\, (-C(2+3)S4\, d_{y4}\, h_1 + S(2+3)SC4 d_{y4}^2) - S(2+3)SC4 I_{x4} - S(2+3)SC4 I_{y4} + C(2+3)S4 I_{yz4}$$
$$-M_5 (-C(2+3)S4\, \ell_{y4} h_2 - C(2+3)S4C5\, \ell_{y4} d_5 - S(2+3)SC4CC5 d_5^2 - C(2+3)S4S5 h_2 d_5$$
$$-C(2+3)S4SC5 d_5^2 + S(2+3)SC4\, \ell_{y4}^2 + 2S(2+3)SC4S5\, \ell_{y4} d_5 + S(2+3)SC4 d_5^2\,) + S(2+3)SC4 I_{x5}$$
$$-S(2+3)\, SC4CC5\, I_{y5} - C(2+3)S4SC5 I_{y5} - S(2+3)SC4SS5 I_{z5} + C(2+3)S4SC5 I_{z5}$$

Joint Space Dynamics

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \Gamma$$

$q:$ Generalized Joint Coordinates

$M(q):$ Mass Matrix - Kinetic Energy Matrix

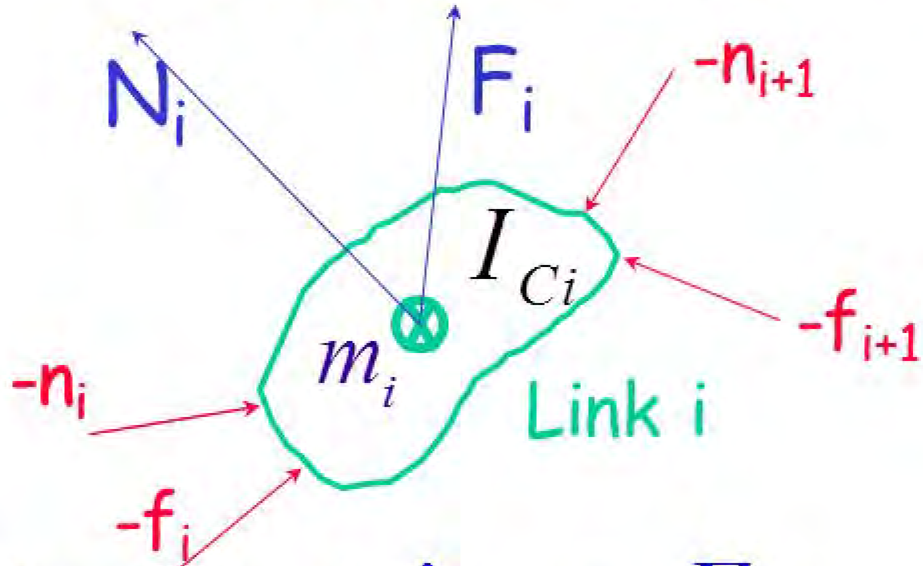
$V(q,\dot{q}):$ Centrifugal and Coriolis forces

$G(q):$ Gravity forces

$\Gamma:$ Generalized forces

Formulations

Newton-Euler



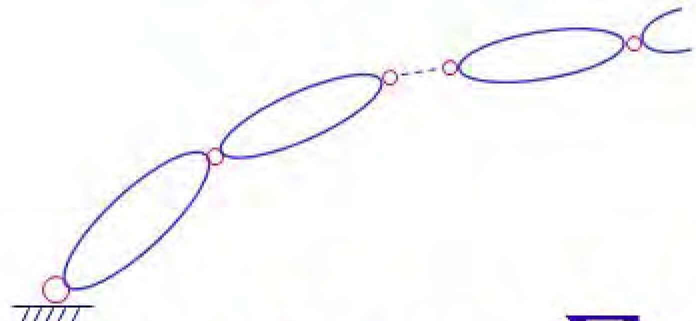
Newton: $m\dot{\mathbf{v}}_C = F$

Euler: $N_i = I_{C_i}\dot{\omega}_i + \omega_i \times I_{C_i}\omega_i$

Eliminate Internal Forces

$$\tau_i = \begin{cases} n_i^T . Z_i & \text{revolute} \\ f_i^T . Z_i & \text{prismatic} \end{cases}$$

Lagrange



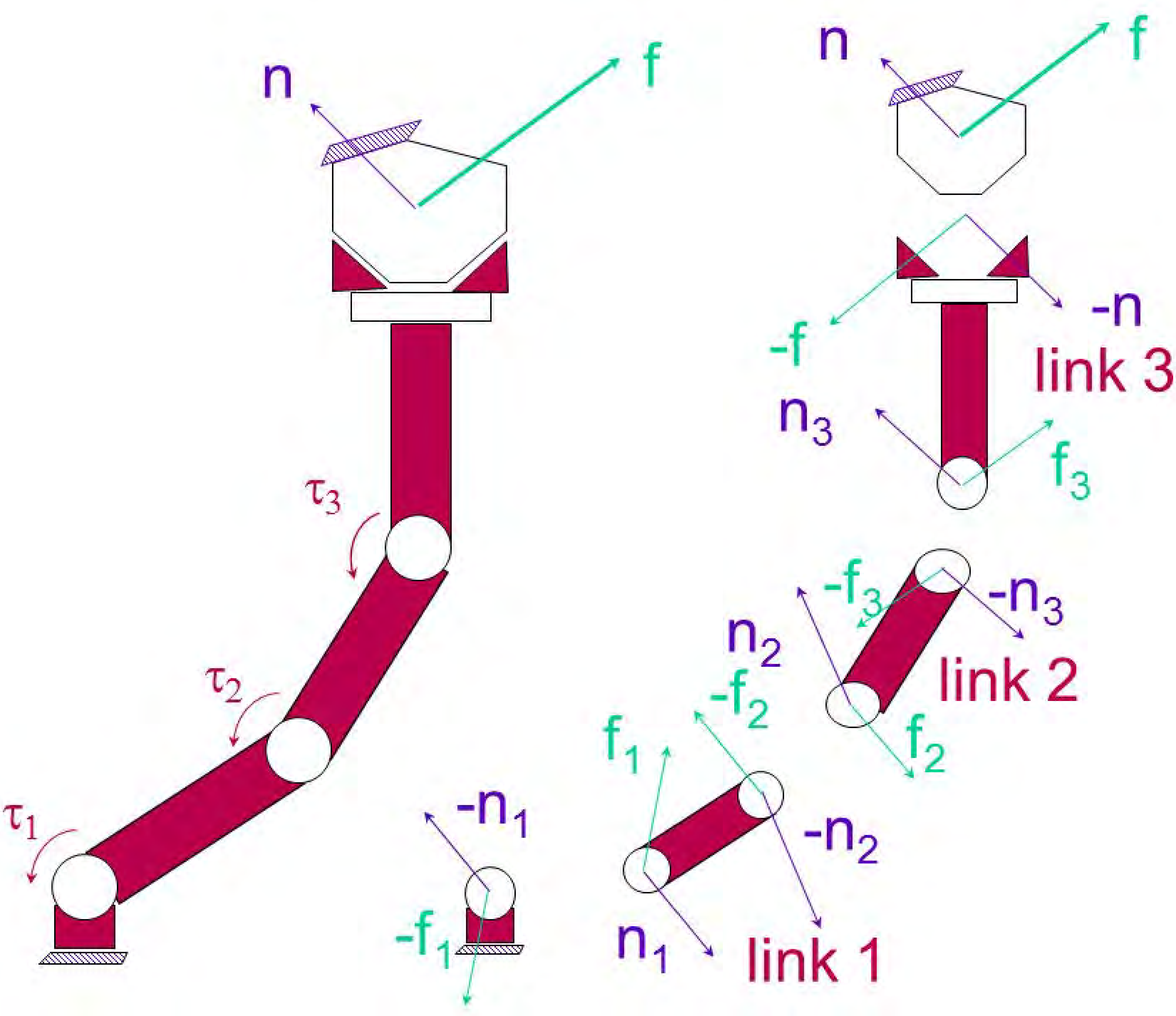
Kinetic Energy: $\sum_i K_i$

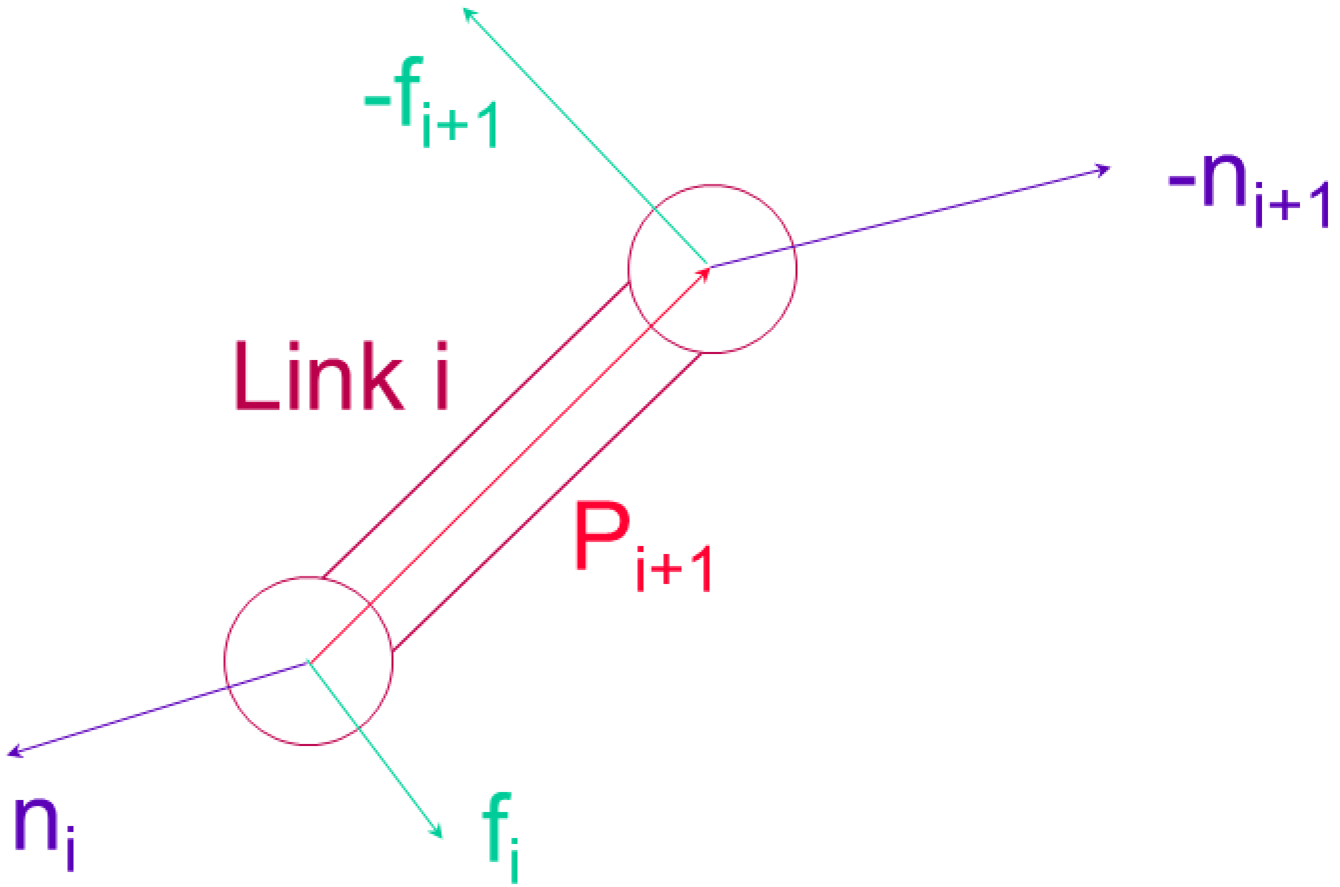
Potential Energy

Generalized Coordinates

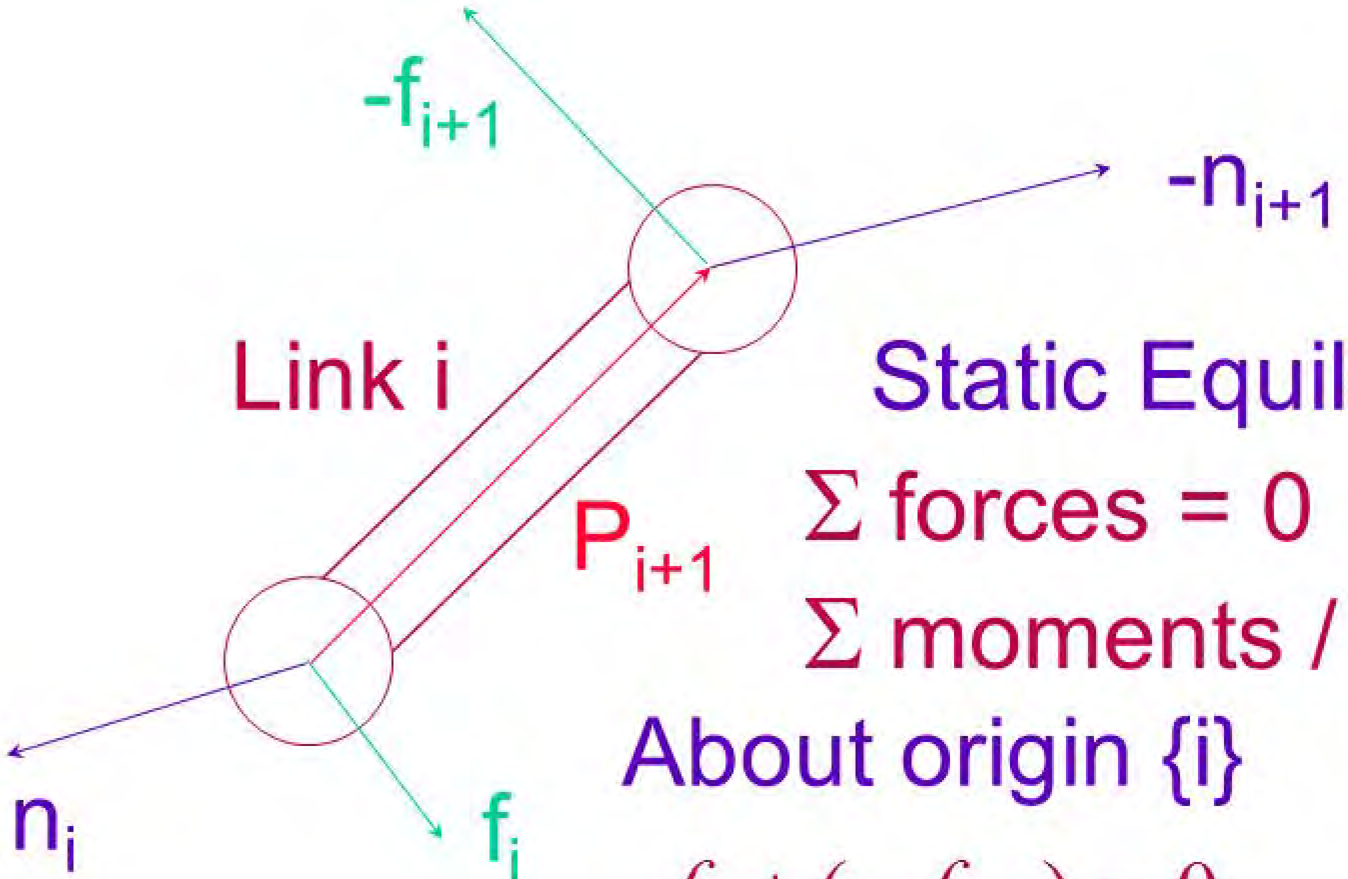
$$K = \frac{1}{2}\dot{q}^T M \dot{q}$$

$$M\ddot{q} + V + G = \tau$$


n
f
τ3
τ2
τ1
n
f
-f
-n
link 3
n3
f3
-f3
-n3
n2
link 2
-f2
f2
f1
-n1
-n2
-f1
n1
link 1



$-f_{i+1}$
$-n_{i+1}$
Link i
P_{i+1}
n_i
f_i



Static Equilibrium

Σ forces = 0

Σ moments / a point = 0

About origin {i}

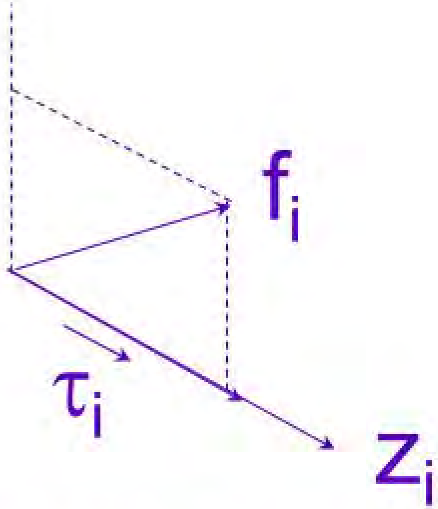
$$f_i + (-f_{i+1}) = 0$$

$$n_i + (-n_{i+1}) + P_{i+1} \times (-f_{i+1}) = 0$$

$$\Downarrow$$

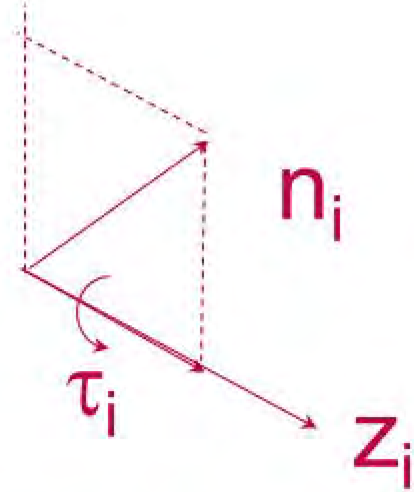
$$f_i = f_{i+1}$$

$$n_i = n_{i+1} + P_{i+1} \times f_{i+1}$$



Prismatic Joint

$$\tau_i = f_i^T Z_i$$



Revolute Joint

$$\tau_i = n_i^T Z_i$$

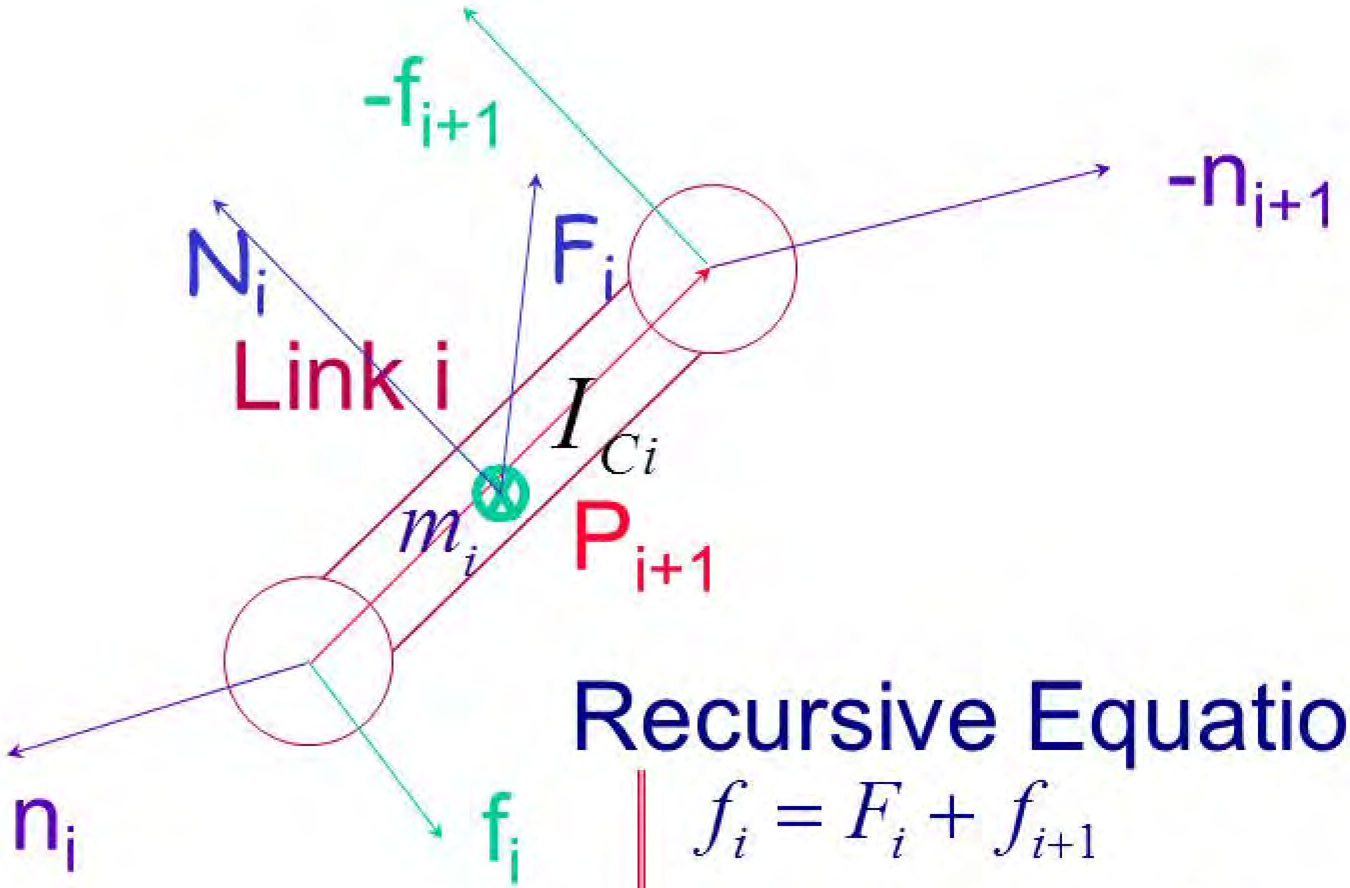
Algorithm

$${}^n f_n = {}^n f$$

$${}^n n_n = {}^n n + {}^n P_{n+1} \times {}^n f$$

$${}^i f_i = {}_{i+1}^{i}R . {}^{i+1} f_{i+1}$$

$${}^i n_i = {}_{i+1}^{i}R . {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i$$



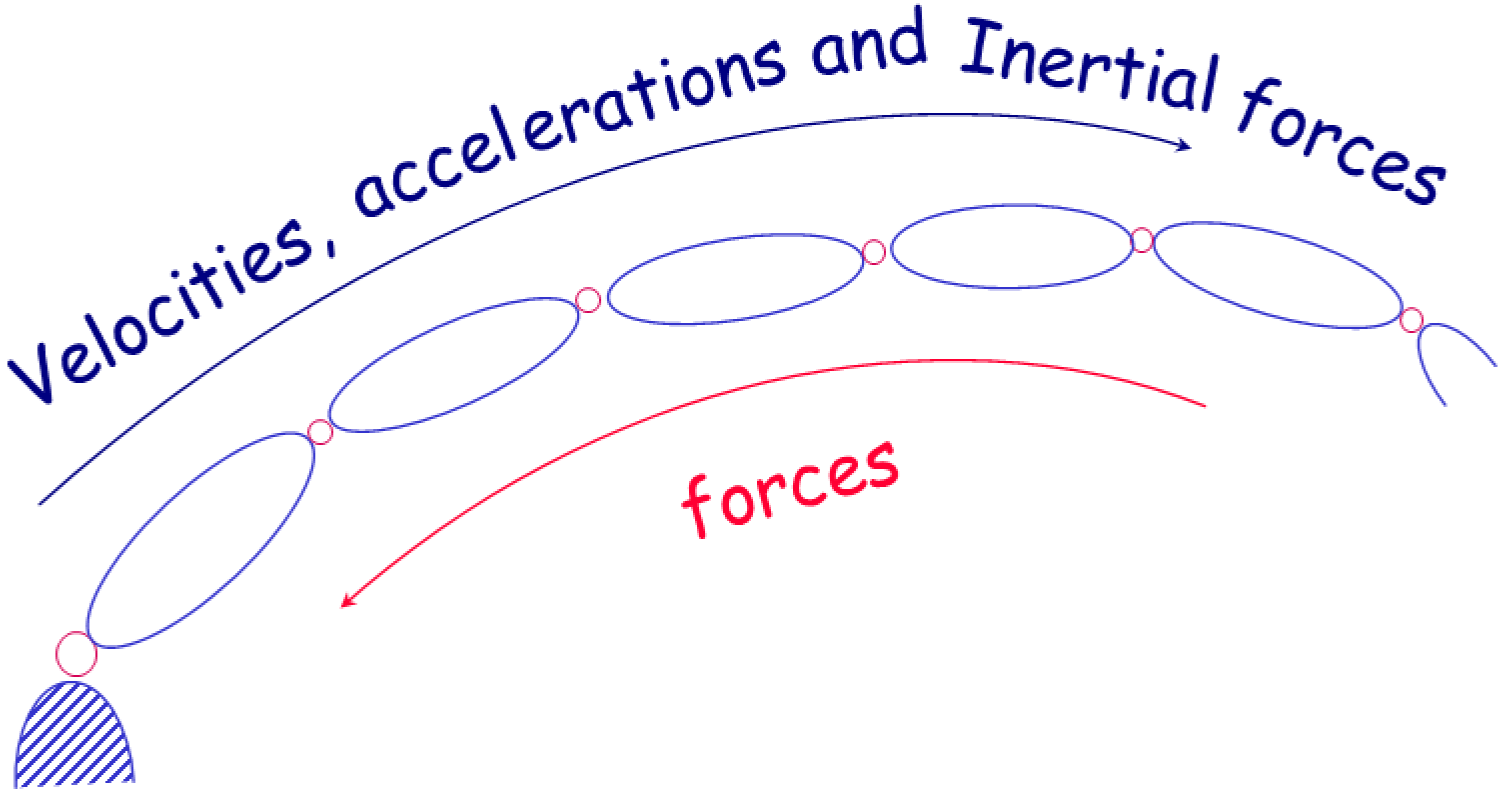
Recursive Equations

$$f_i = F_i + f_{i+1}$$

$$n_i = N_i + n_{i+1} + \mathbf{p}_{C_i} \times F_i + \mathbf{p}_{i+1} \times f_{i+1}$$

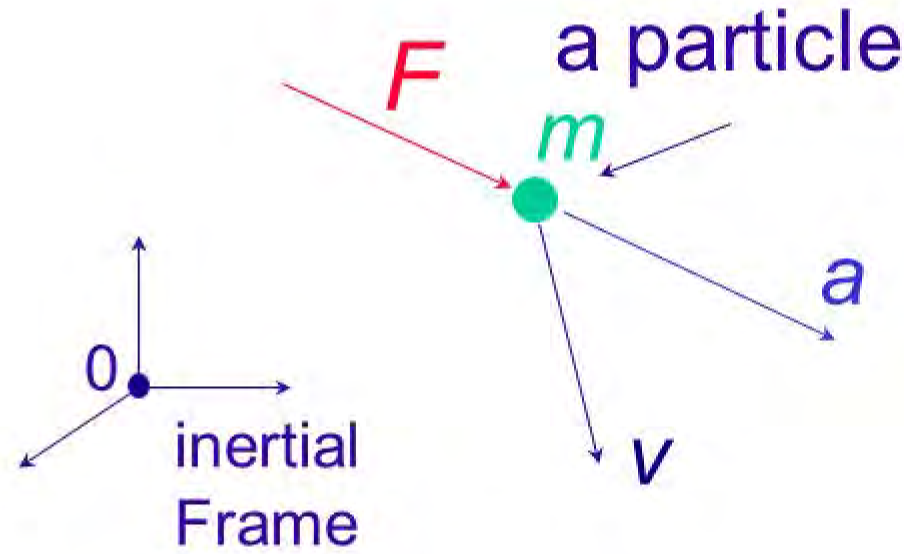
$$\tau_i = \begin{cases} n_i . Z_i & \text{revolute} \\ f_i . Z_i & \text{prismatic} \end{cases}$$

Newton-Euler Algorithm



Newton's Law

$$\underline{F} = m\underline{a}$$



$$\frac{d}{dt}(mv) = F$$

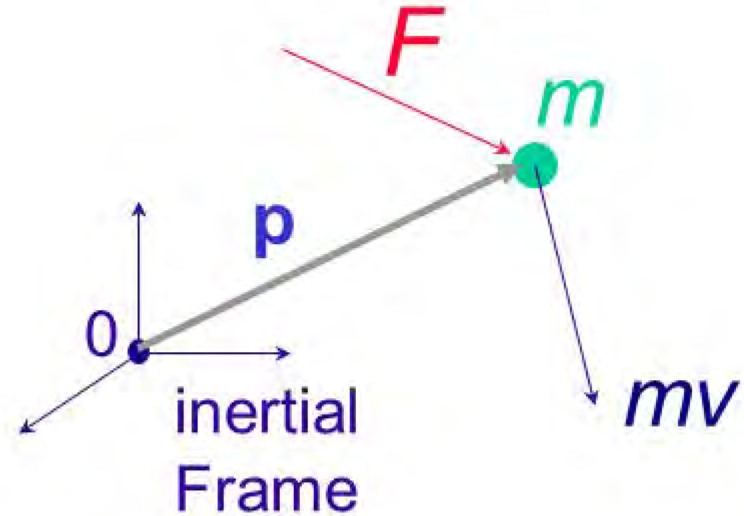
rate of change of the linear momentum is equal to the applied force

Linear Momentum

$$\varphi = mv$$

Angular Momentum

$$m\dot{\mathbf{v}} = F$$



take the moment /0

$$\mathbf{p} \times m\dot{\mathbf{v}} = \mathbf{p} \times F$$

N

$$\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = \mathbf{p} \times m\dot{\mathbf{v}} + \underbrace{\mathbf{v} \times m\mathbf{v}}_{=0} = \mathbf{p} \times m\dot{\mathbf{v}}$$

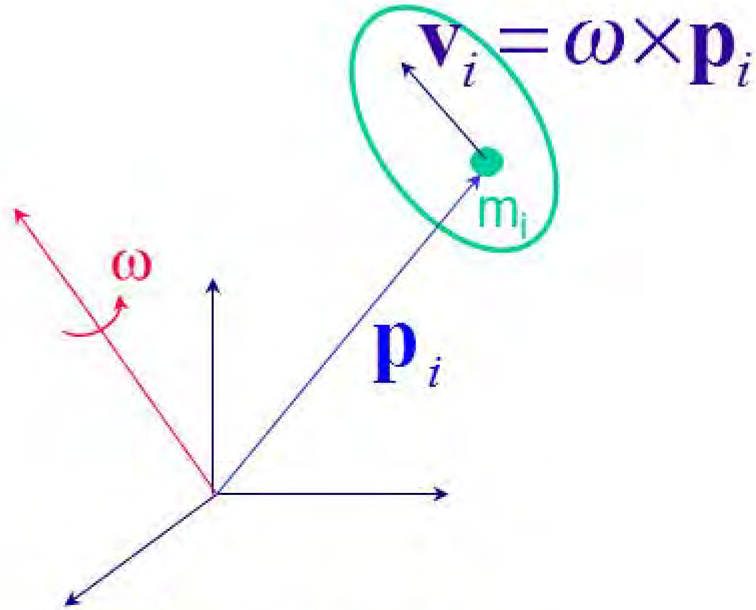
$$\boxed{\frac{d}{dt}(\mathbf{p} \times m\,\mathbf{v}) = N}$$

applied moment

angular momentum $\phi = p \times mv$

Rigid Body

Rotational Motion



$$\text{Angular Momentum} = \sum_i \mathbf{p}_i \times m_i \mathbf{v}_i$$

$$\phi = \sum_i m_i \mathbf{p}_i \times (\omega \times \mathbf{p}_i)$$

$$m_i \rightarrow \rho dv \qquad (\rho : density)$$

$$\phi = \int_V p \times (\omega \times p) \rho dv$$

$$\phi = \int_V p \times (\omega \times p)\rho dv$$

$$\mathbf{p} \times (\omega \times \mathbf{p}) = \hat{\mathbf{p}}(-\hat{\mathbf{p}})\omega$$

$$\phi = [\int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv]\omega$$

Inertia Tensor

$$\phi = I\omega$$

$$I = \int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv$$

Linear Momentum

$$\varphi = mv$$

Angular Momentum

$$\phi = I\omega$$

Newton Equation

$$\frac{d}{dt}(mv) = F$$

$$\dot{\varphi} = F$$

$$ma = F$$

Euler Equation

$$\frac{d}{dt}(I\omega) = N$$

$$\dot{\phi} = N$$

$$I\dot{\omega} + \omega \times I\omega = N$$

Inertia Tensor

$$I = \int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv \qquad (-\hat{\mathbf{p}}\hat{\mathbf{p}}) = (\mathbf{p}^T\mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T$$

$$I = \int_V [(\mathbf{p}^T\mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T]\rho dv$$

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad ;\mathbf{p}^T\mathbf{p} = x^2 + y^2 + z^2$$

$$(\mathbf{p}^T\mathbf{p})I_3 = (x^2 + y^2 + z^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{p}\mathbf{p}^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix}(x \quad y \quad z) = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}$$

$$(-\hat{\mathbf{p}}\hat{\mathbf{p}}) = \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & z^2 + x^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}$$

Inertia Tensor

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Moments of Inertia

$$I_{xx} = \iiint (y^2 + z^2)\rho dxdydz$$

$$I_{yy} = \iiint (z^2 + x^2)\rho dxdydz$$

$$I_{zz} = \iiint (x^2 + y^2)\rho dxdydz$$

Products of Inertia

$$I_{xy} = \iiint xy\rho dxdydz$$

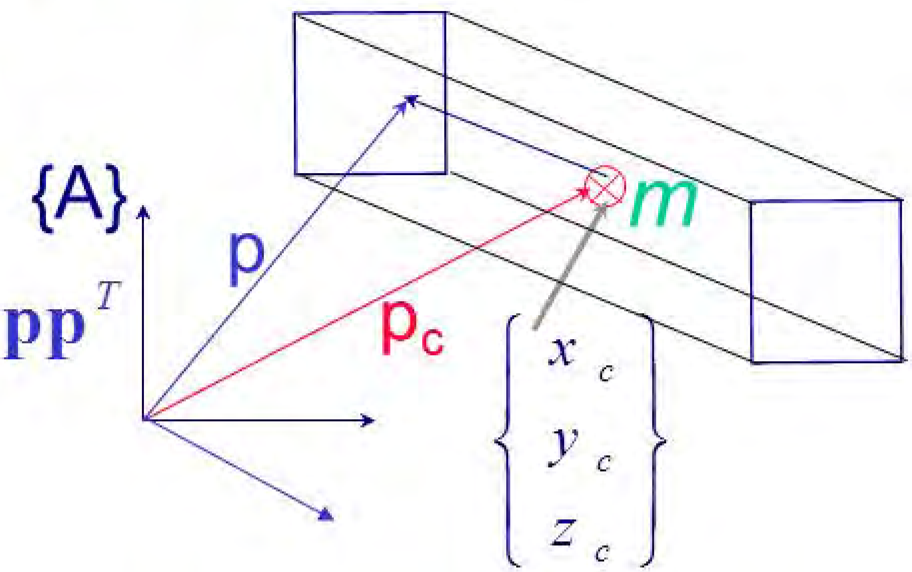
$$I_{xz} = \iiint xz\rho dxdydz$$

$$I_{yz} = \iiint yz\rho dxdydz$$

Parallel Axis theorem

$$I = \int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv$$

$$(-\hat{\mathbf{p}}\hat{\mathbf{p}}) = (\mathbf{p}^T\mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T$$

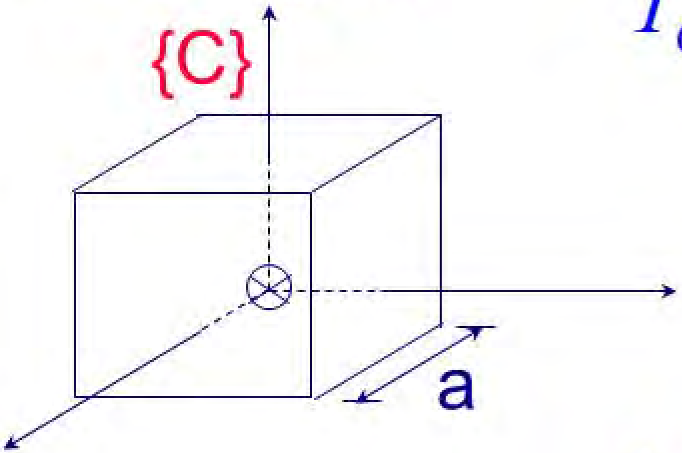


$$I_A = I_C + m\left[(\mathbf{p}_C{}^T\mathbf{p}_C)I_3 - \mathbf{p}_C\mathbf{p}_C{}^T\right]$$

$$I_{A\,zz} = I_{C\,zz} + m(x_C^2 + y_C^2)$$

$$I_{A\,xy} = I_{Cxy} + m x_C y_C$$

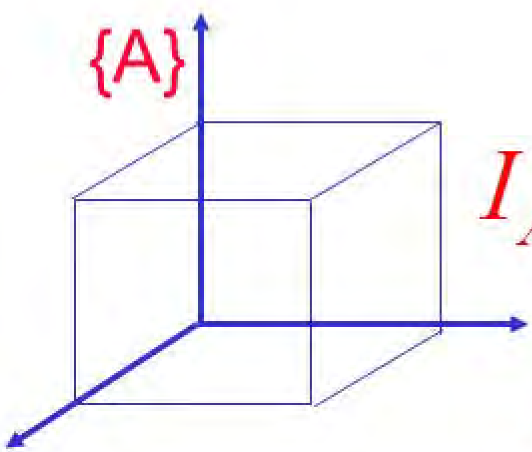
Example



$$I_{Czz} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \iint \rho(x^2 + y^2)\,dx\,dy\,dz$$

$$I_{Czz} = \frac{1}{6}\rho a^5; \qquad m = \rho a^3$$

$$I_{Cxx} = I_{Cyy} = I_{Czz} = \frac{ma^2}{6}$$

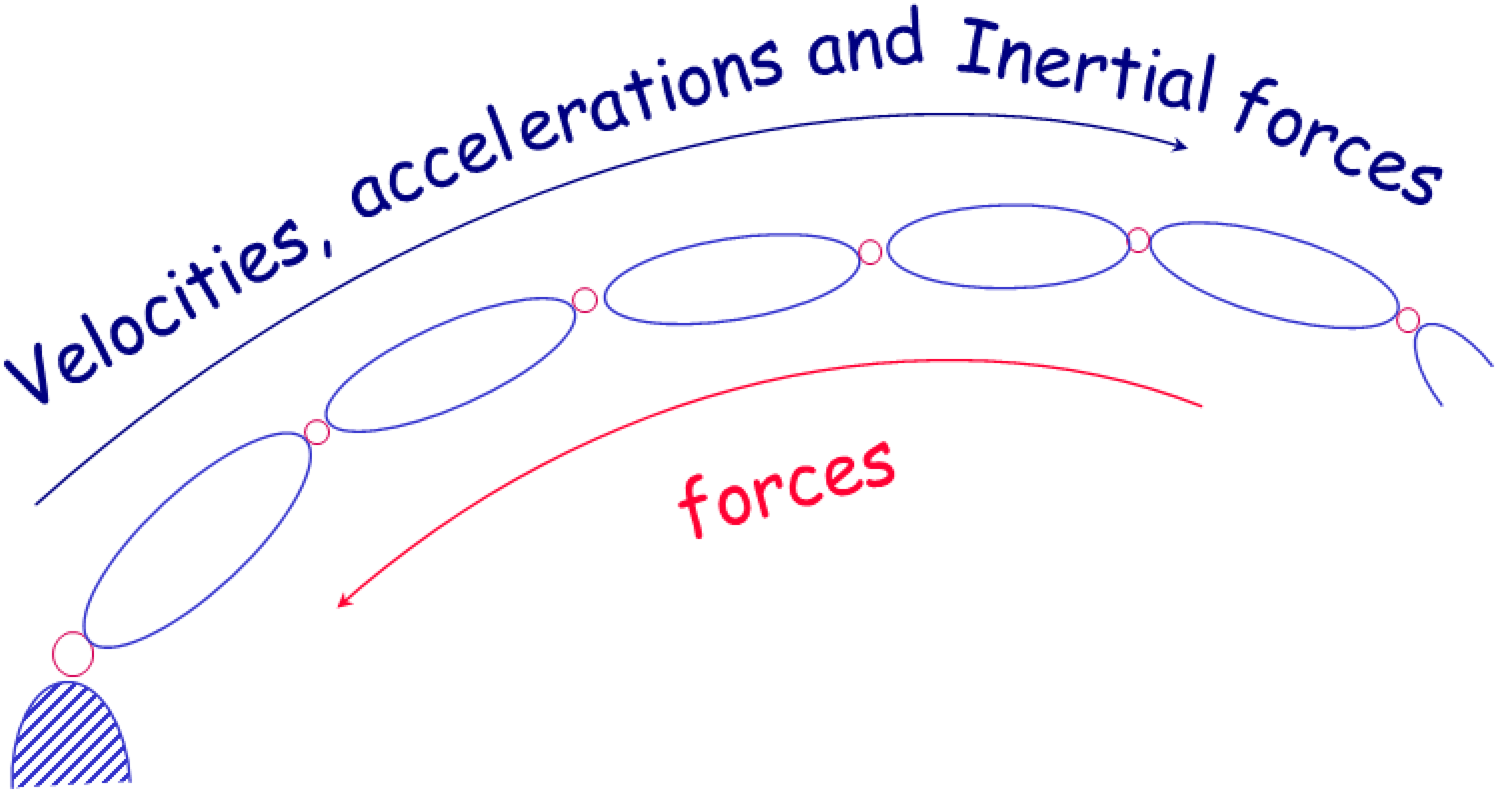


$${}^{A}x_c = {}^{A}y_c = {}^{A}z_c = \frac{a}{2}$$

$$I_{Axx} = I_{Ayy} = I_{Azz} = I_{Czz} + \frac{ma^2}{2} = \frac{2}{3}ma^2$$

$$I_{Axy} = I_{Axz} = I_{Ayz} = \frac{ma^2}{4}$$

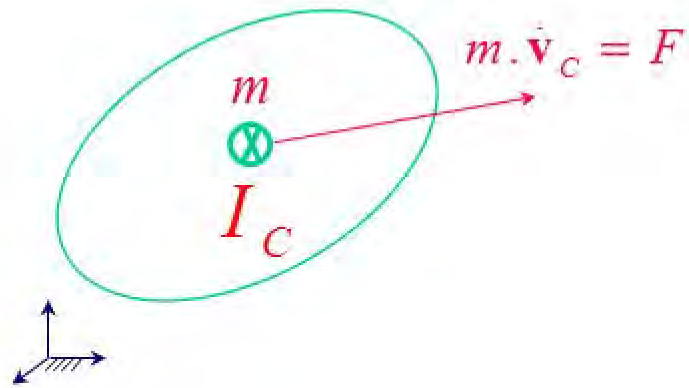
Newton-Euler Algorithm



Newton-Euler Equations

Translational Motion

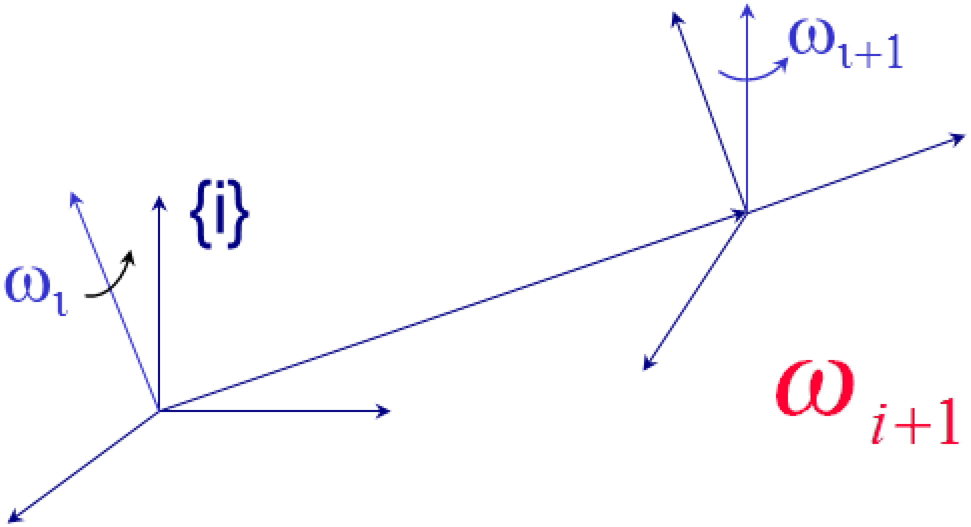
$$m\dot{\mathbf{v}}_C = F$$



Rotational Motion

$$I_C\dot{\omega} + \omega \times I_C\omega = N$$

Angular Acceleration

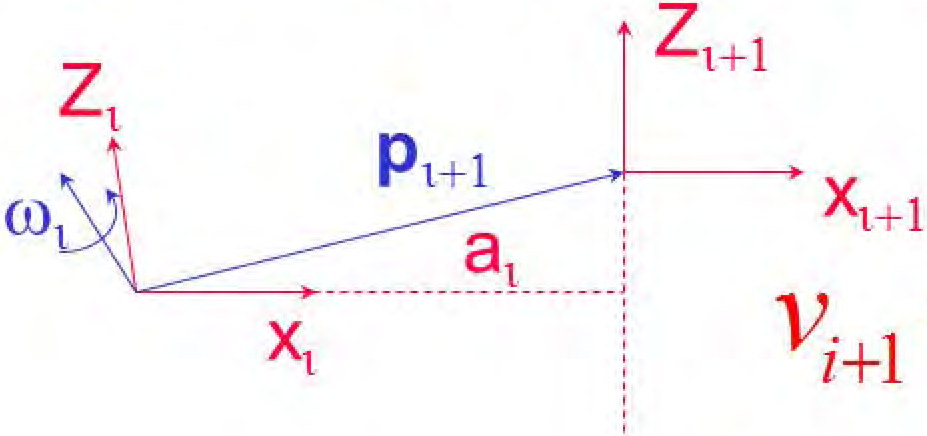


$$\omega_{i+1} = \omega_i + \Omega_{i+1}$$

$$\Omega_{i+1} = \dot{\theta}_{i+1} Z_{i+1}$$

$$\dot{\omega}_{i+1} = \dot{\omega}_i + \dot{\theta}_{i+1}\left(\omega_i \times Z_{i+1}\right) + \ddot{\theta}_{i+1} Z_{i+1}$$

Linear Acceleration



$$v_{i+1} = v_i + \omega_i \times p_{i+1} + V_{i+1}$$

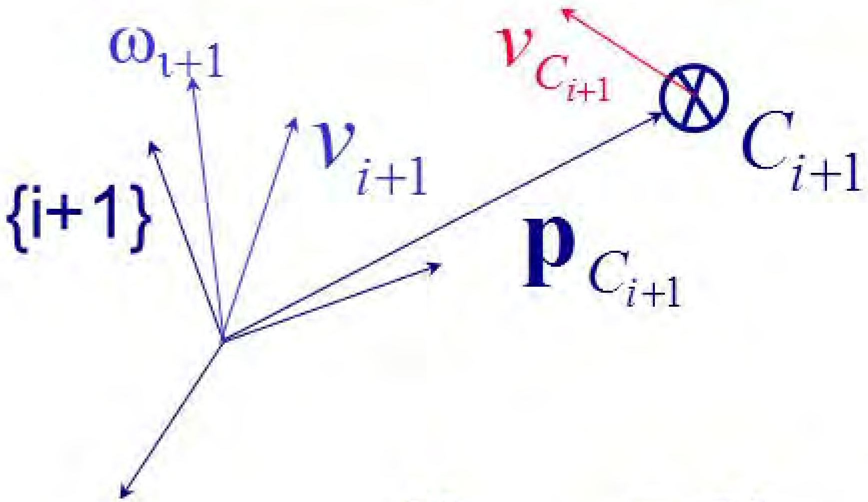
$$V_{i+1} = \dot{d}_{i+1} Z_{i+1}$$

$$P_{i+1} = a_i x_i + d_{i+1} Z_{i+1}$$

$$\dot{\mathbf{v}}_{i+1} = \dot{\mathbf{v}}_i + \dot{\omega}_i \times \mathbf{p}_{i+1} + \omega_i \times \dot{\mathbf{p}}_{i+1} + \dot{V}_{i+1}$$

$$\dot{\mathbf{v}}_{i+1} = \dot{\mathbf{v}}_i + \dot{\omega}_i \times \mathbf{p}_{i+1} + \omega_i \times (\omega_i \times \mathbf{p}_{i+1}) + 2\dot{d}_{i+1}\omega_i \times Z_{i+1} + \ddot{d}_{i+1} Z_{i+1}$$

Velocity and Acceleration at center of mass

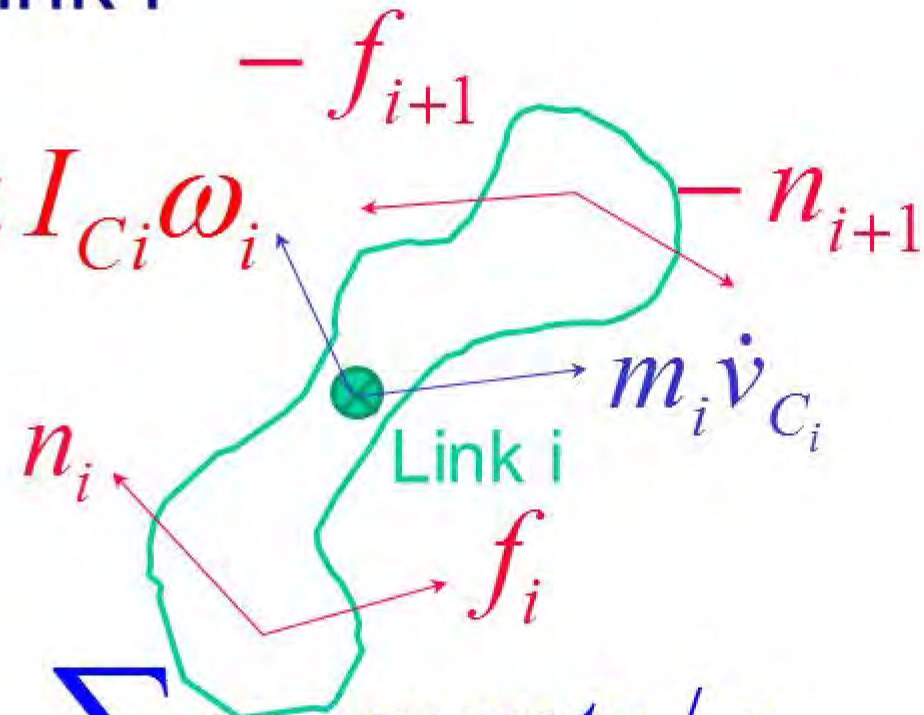


$$\mathbf{v}_{C_{i+1}} = \mathbf{v}_{i+1} + \omega_{i+1} \times \mathbf{p}_{C_{i+1}}$$

$$\dot{\mathbf{v}}_{C_{i+1}} = \dot{\mathbf{v}}_{i+1} + \dot{\omega}_{i+1} \times \mathbf{p}_{C_{i+1}} + \omega_{i+1} \times (\omega_{i+1} \times \mathbf{p}_{C_{i+1}})$$

Dynamic forces on Link i

$$I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$$



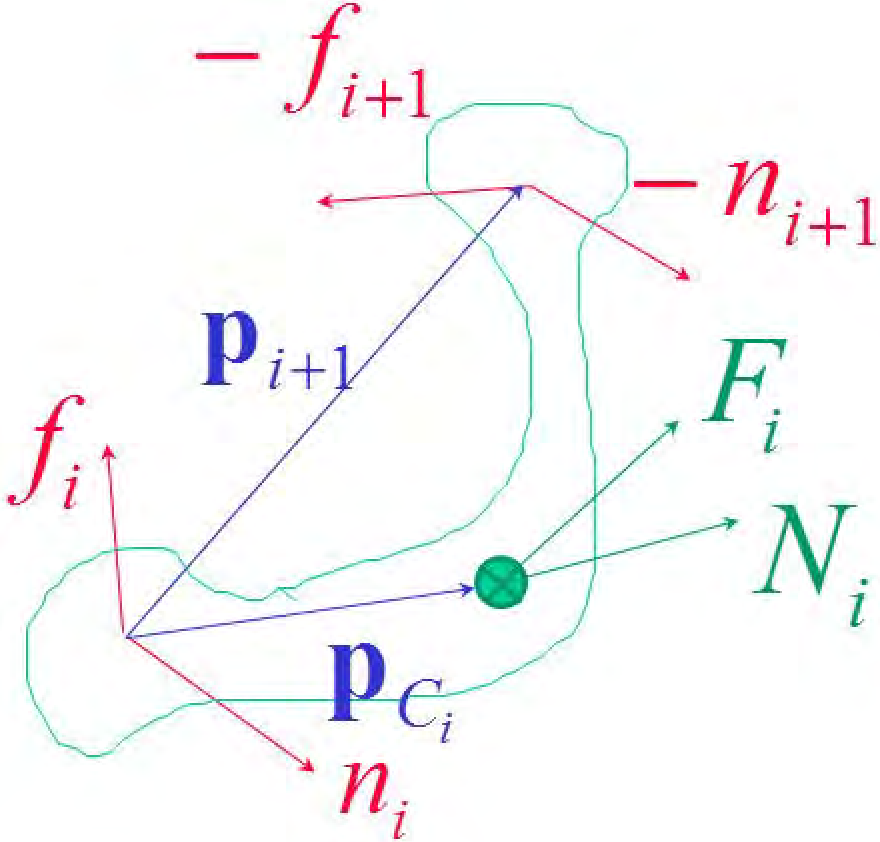
$$m_i \dot{\mathbf{v}}_{C_i} = \sum forces$$

$$I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i = \sum moments / c_i$$

Inertial forces/moments

$$F_i = m_i \dot{\mathbf{v}}_{C_i}$$

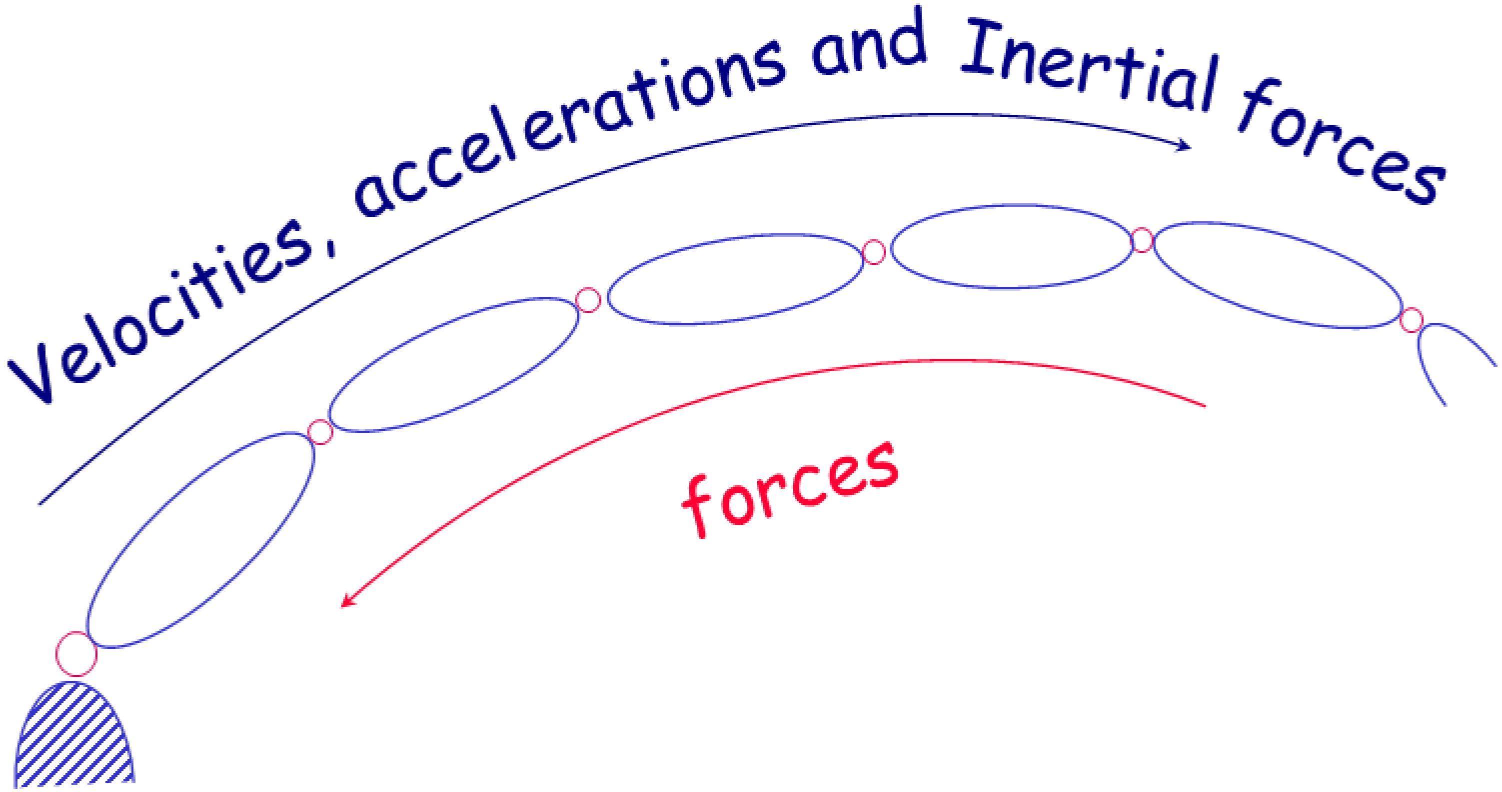
$$N_i = I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$$



$$F_i = f_i - f_{i+1}$$

$$N_i = n_i - n_{i+1} + (-\mathbf{p}_{C_i}) \times f_i + (\mathbf{p}_{i+1} - \mathbf{p}_{C_i}) \times (-f_{i+1})$$

Newton-Euler Algorithm



Recursive Equations

$$f_i = F_i + f_{i+1}$$

$$n_i = N_i + n_{i+1} + \mathbf{p}_{C_i} \times F_i + \mathbf{p}_{i+1} \times f_{i+1}$$

$$\tau_i = \begin{cases} n_i . Z_i & \text{revolute} \\ f_i . Z_i & \text{prismatic} \end{cases}$$

with
$$F_i = m_i \dot{\mathbf{v}}_{C_i}$$
$$N_i = I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$$

where $\omega_{i+1} = \omega_i + \Omega_{i+1} = \omega_i + \dot{\theta}_{i+1} Z_{i+1}$

$$\dot{\omega}_{i+1} = \dot{\omega}_i + \omega_i \times Z_{i+1}\dot{\theta}_{i+1} + \ddot{\theta}_{i+1} Z_{i+1}$$

$$\dot{\mathbf{v}}_{i+1} = \dot{\mathbf{v}}_i + \dot{\omega}_i \times \mathbf{p}_{i+1} + \omega_i \times (\omega_i \times \mathbf{p}_{i+1}) + 2\dot{d}_{i+1}\omega_i \times Z_{i+1} + \ddot{d}_{i+1} Z_{i+1}$$

$$\dot{\mathbf{v}}_{C_{i+1}} = \dot{\mathbf{v}}_{i+1} + \dot{\omega}_{i+1} \times \mathbf{p}_{C_{i+1}} + \omega_{i+1} \times (\omega_{i+1} \times \mathbf{p}_{C_{i+1}})$$

Outward iterations: i : 0 $\longrightarrow$ 5

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}Z_{i+1}$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\, {}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\, {}^{i}\omega_i \times {}^{i+1}Z_{i+1}\dot{\theta}_{i+1} + \ddot{\theta}_{i+1}\, {}^{i+1}Z_{i+1}$$

$$^{i+1}\dot{\mathbf{v}}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}\mathbf{p}_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}\mathbf{p}_{i+1}) + {}^{i}\dot{\mathbf{v}}_i)$$

$$^{i+1}\dot{\mathbf{v}}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}\mathbf{p}_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}\mathbf{p}_{C_{i+1}}) + {}^{i+1}\dot{\mathbf{v}}_{i+1}$$

$$^{i+1}F_{i+1} = m_{i+1}\, {}^{i+1}\dot{\mathbf{v}}_{C_{i+1}}$$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\omega_{i+1}$$

Inward iterations: i : 6 $\longrightarrow$ 1

$$^{i}f_i = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}\mathbf{p}_{C_i} \times {}^{i}F_i + {}^{i}\mathbf{p}_{i+1} \times {}^{i}_{i+1}R\, {}^{i+1}f_{i+1}$$

$$\tau_i = {}^{i}n_i^T\, {}^{i}Z_i$$

Gravity: set $^{0}\dot{\mathbf{v}}_0 = 1G$