introduction to Robotics

Lecture Notes, CS223A, Winter 2013-2014

O. Khatib and K. Kolarov

Movie Segment

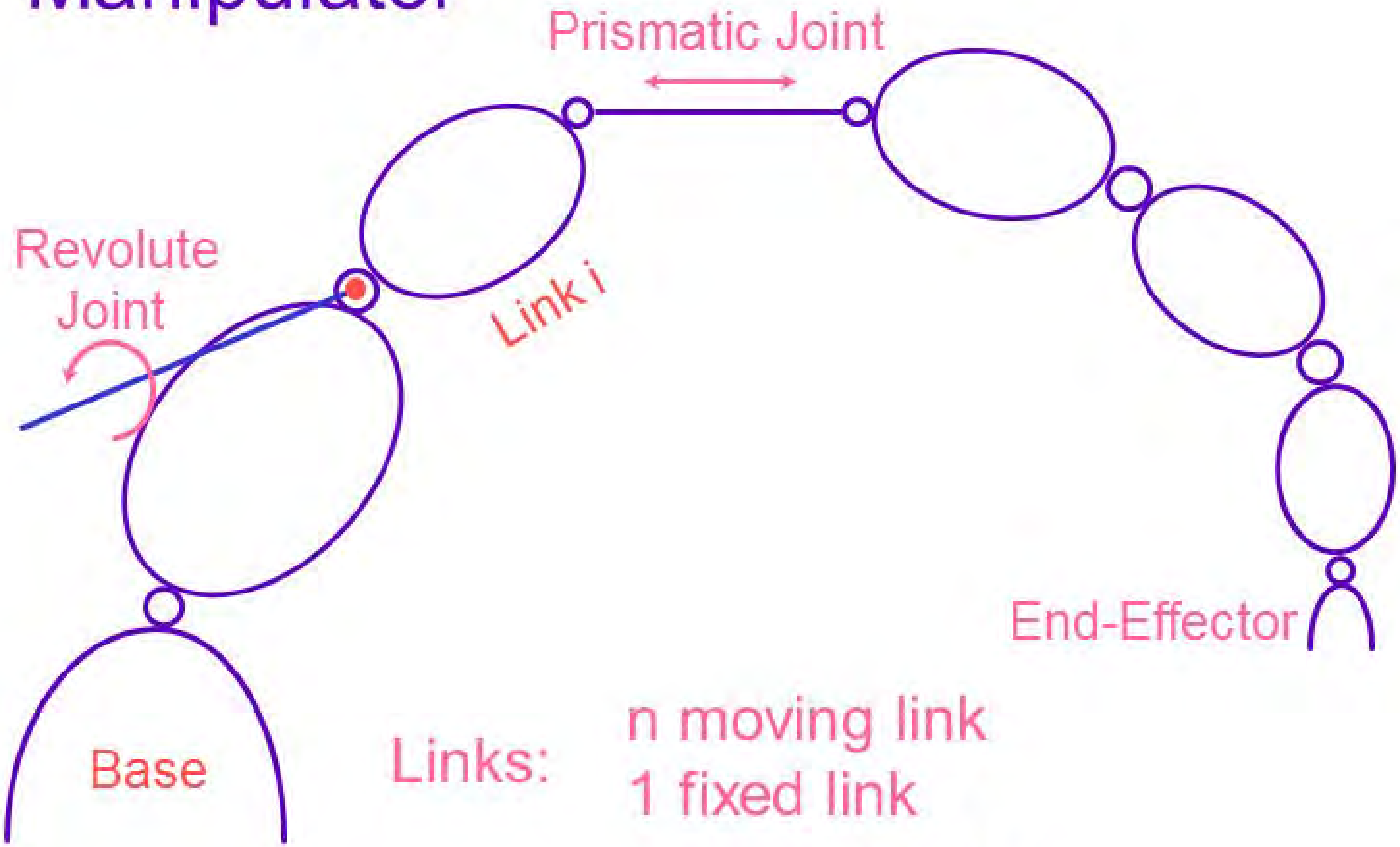
Pet-Proto Robot Navigates Obstacles, Boston Dynamics, 2012

Kinematics

Spatial Descriptions

- Task Description
- Transformations
- Representations

Manipulator

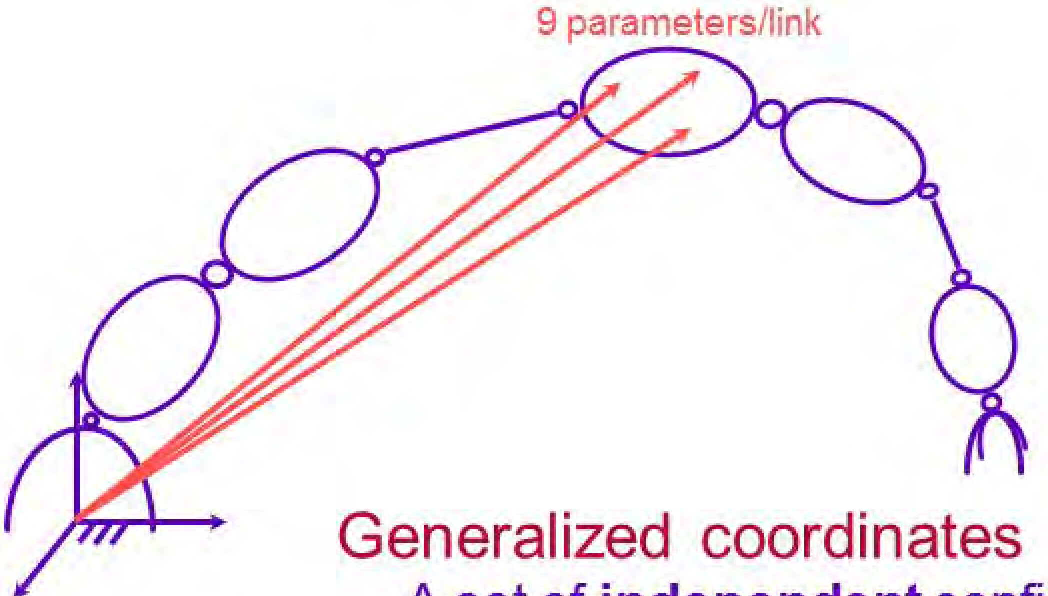


Links: n moving link
1 fixed link

Joints: Revolute (1 DOF)
Prismatic (1 DOF)

Configuration Parameters

A set of position parameters that describes the full configuration of the system.



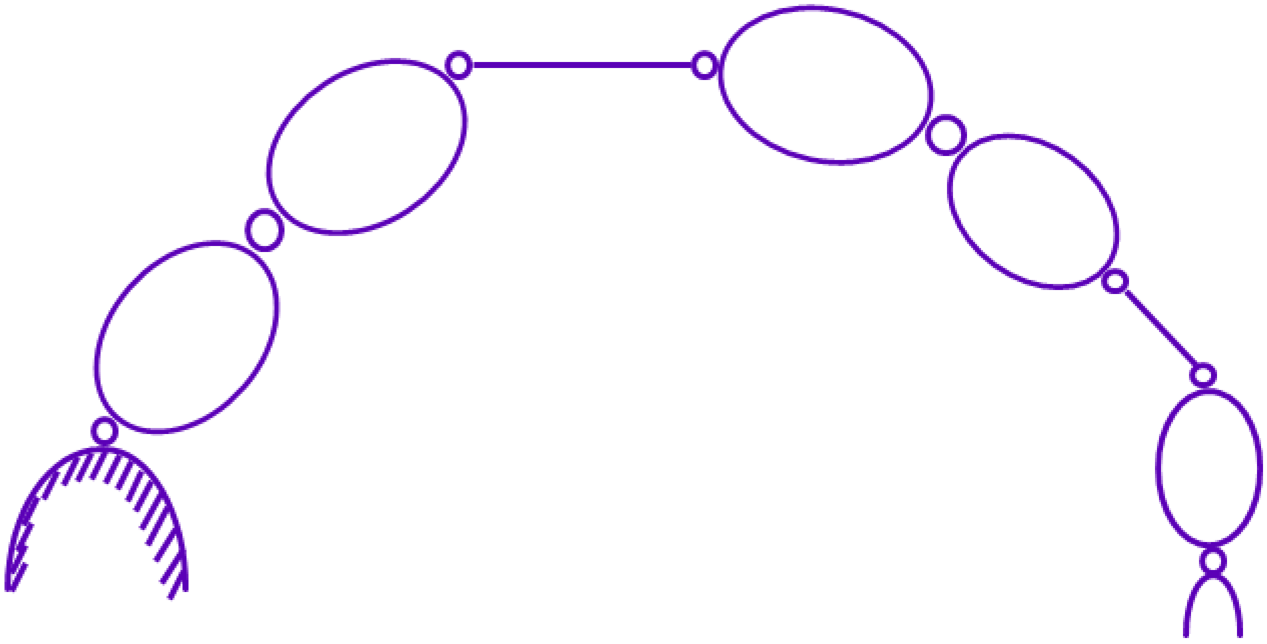
Generalized coordinates

A set of **independent** configuration parameters

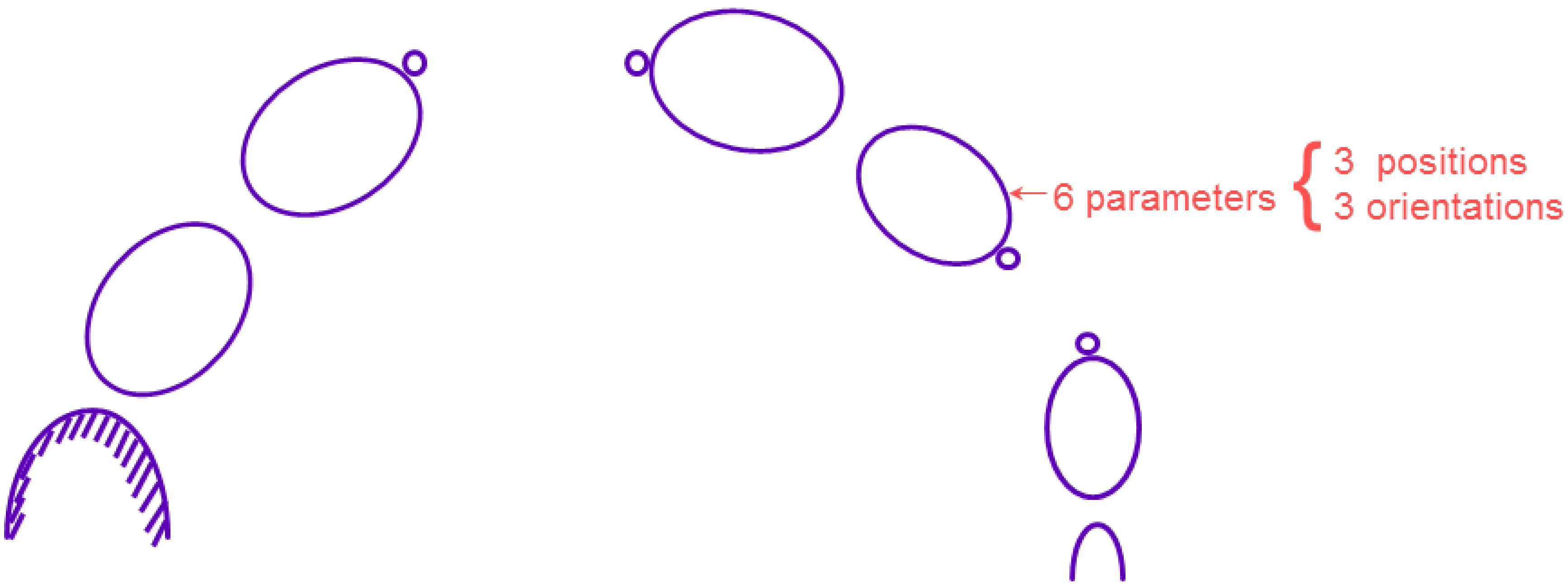
Degrees of Freedom

Number of generalized coordinates

Generalized Coordinates



Generalized Coordinates



n moving links: 6n parameters

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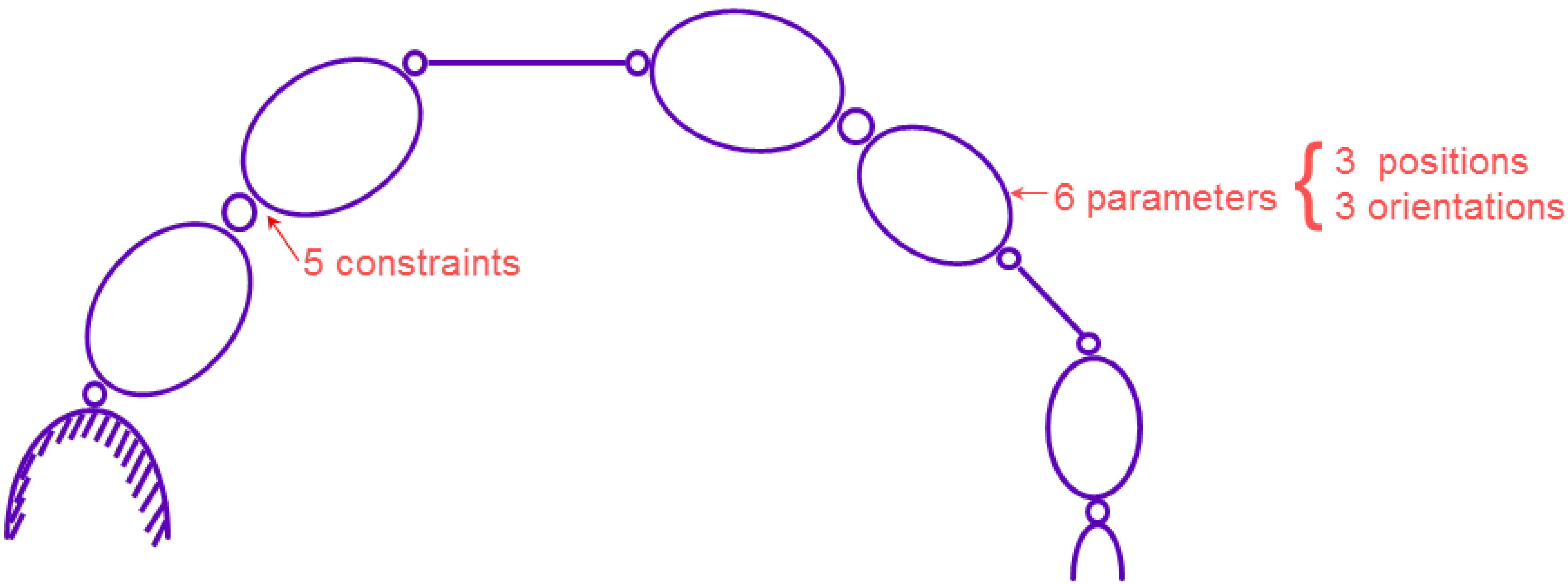
Generalized Coordinates

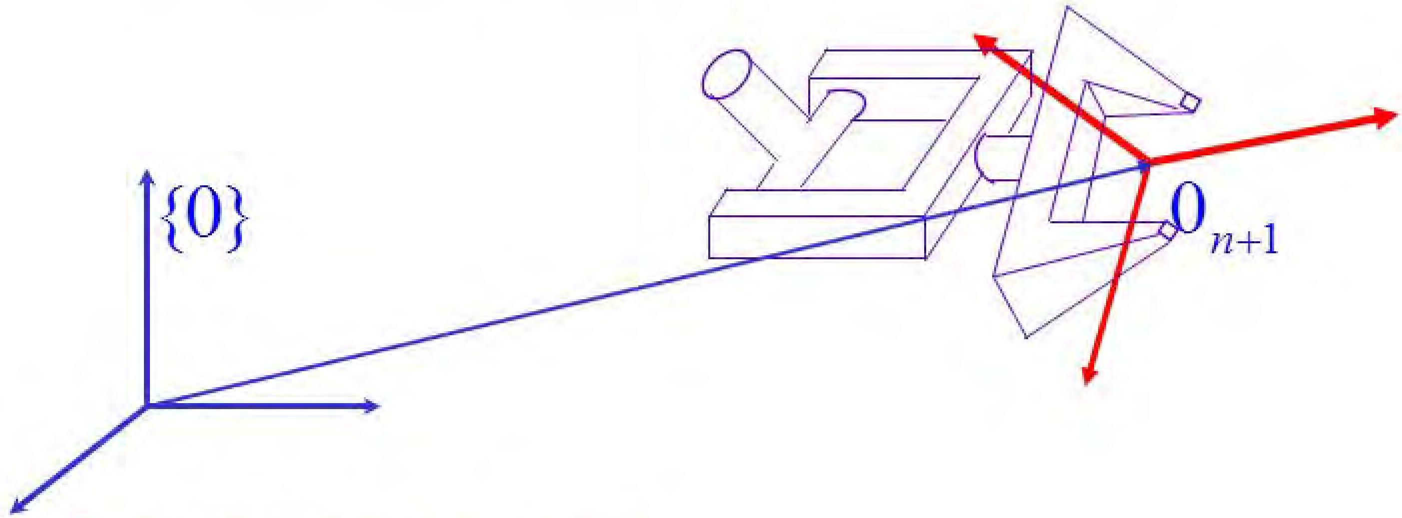


n moving links: 6n parameters

n 1 d.o.f. joints: 5n constraints

d.o.f. (system): 6n - 5n = n

End-Efector Configuration Parameters



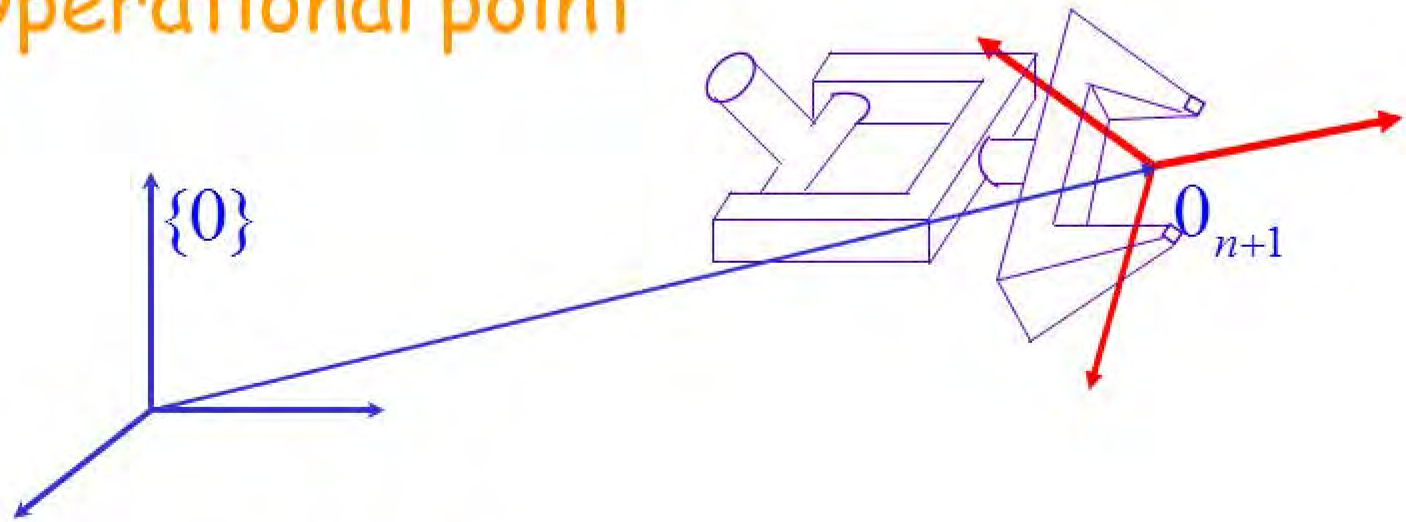
A set of m parameters:

$$(x_1, x_2, x_3, \ldots, x_m)$$

that completely specifies the end-effector position and orientation with respect to $\{0\}$

Operational Coordinates

O_{n+1} : Operational point

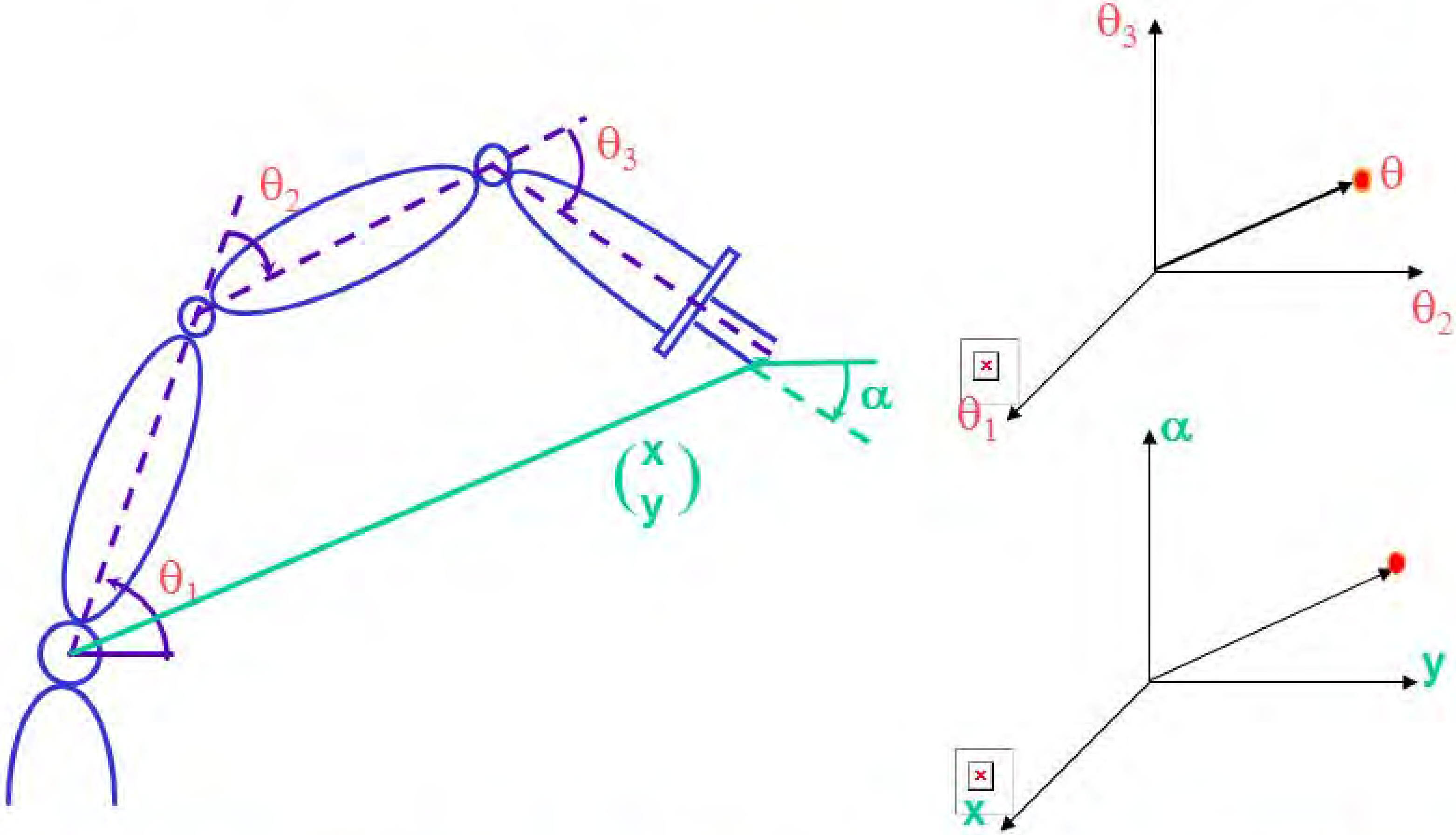


A set $x_1, x_2, \ldots, x_{m_0}$

of m_0 independent configuration parameters

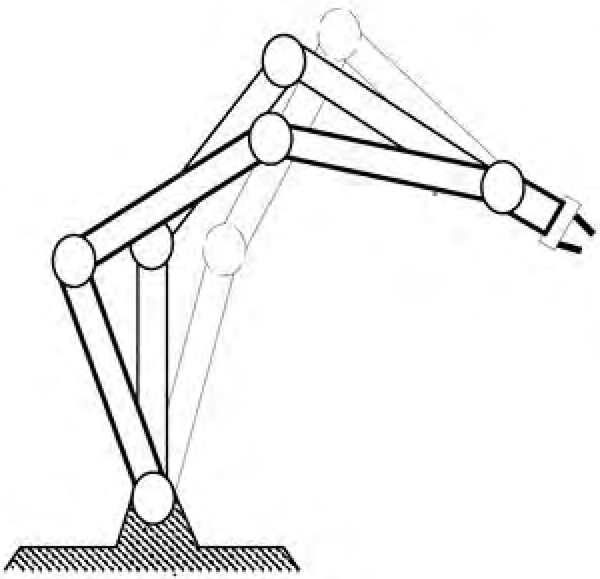
m_0 : number of degrees of freedom of the end-effector.

Joint Coordinates → Joint Space



Operational Coordinates → Operational Space

Redundancy

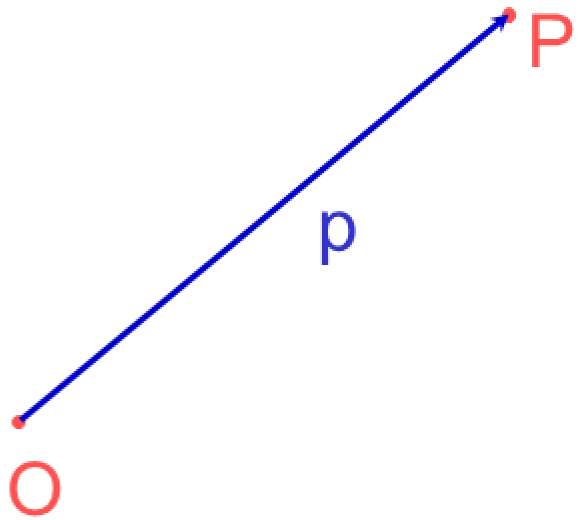


A robot is said to be redundant if

$$n > m_0$$

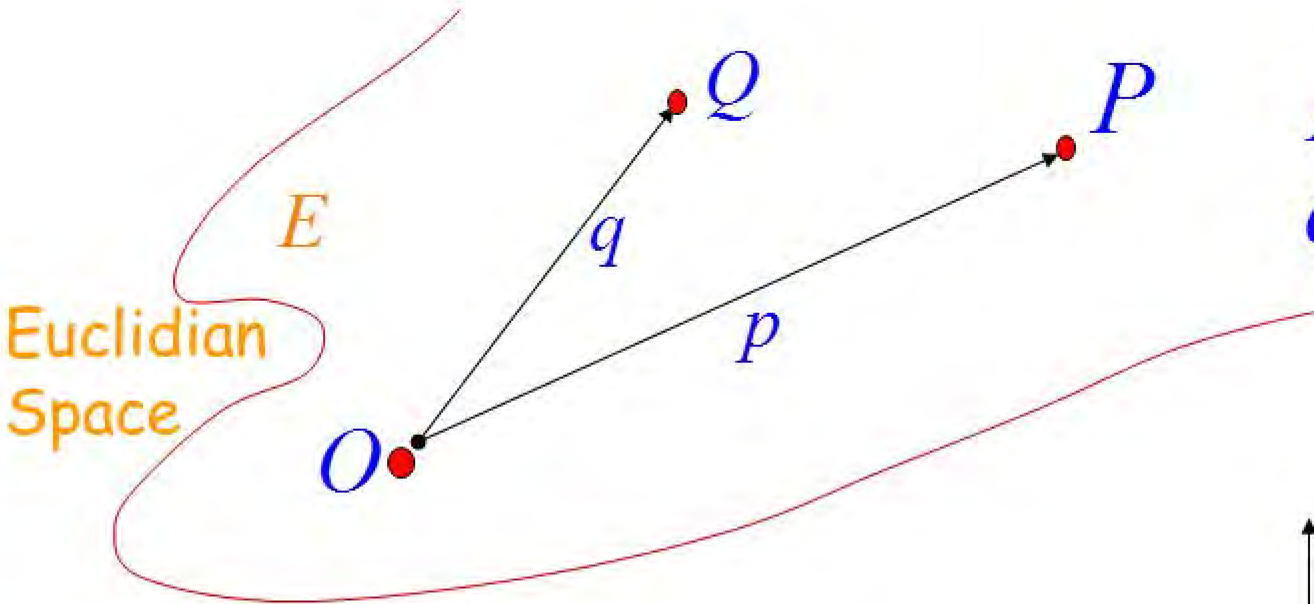
Degrees of redundancy: $n - m_0$

Position of a Point

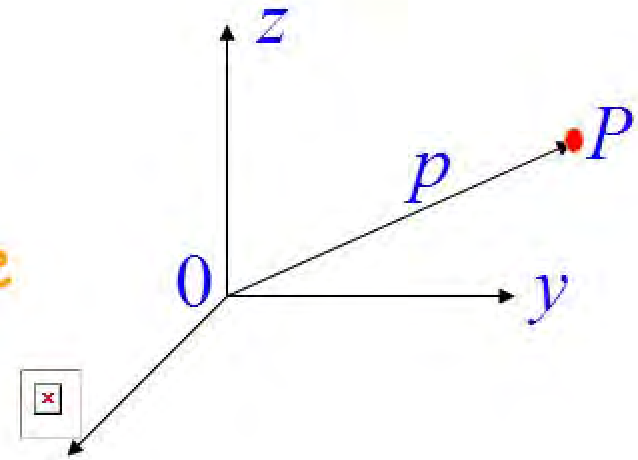


With respect to a fixed origin O, the position of a point P is described by the vector OP or simply by p.

Rigid Body Configuration

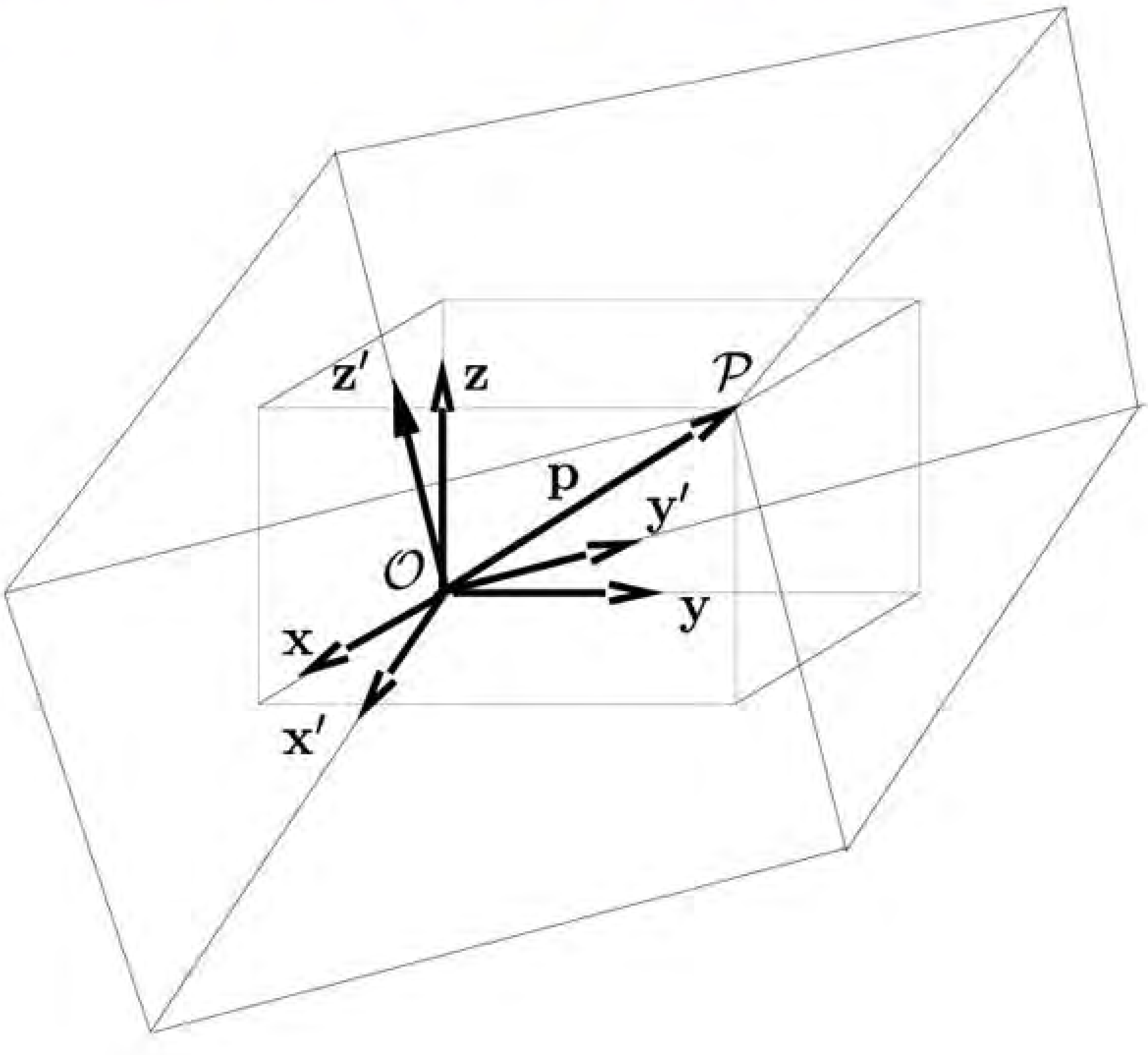


$$p = \overrightarrow{OP}$$
$$q = \overrightarrow{OQ}$$

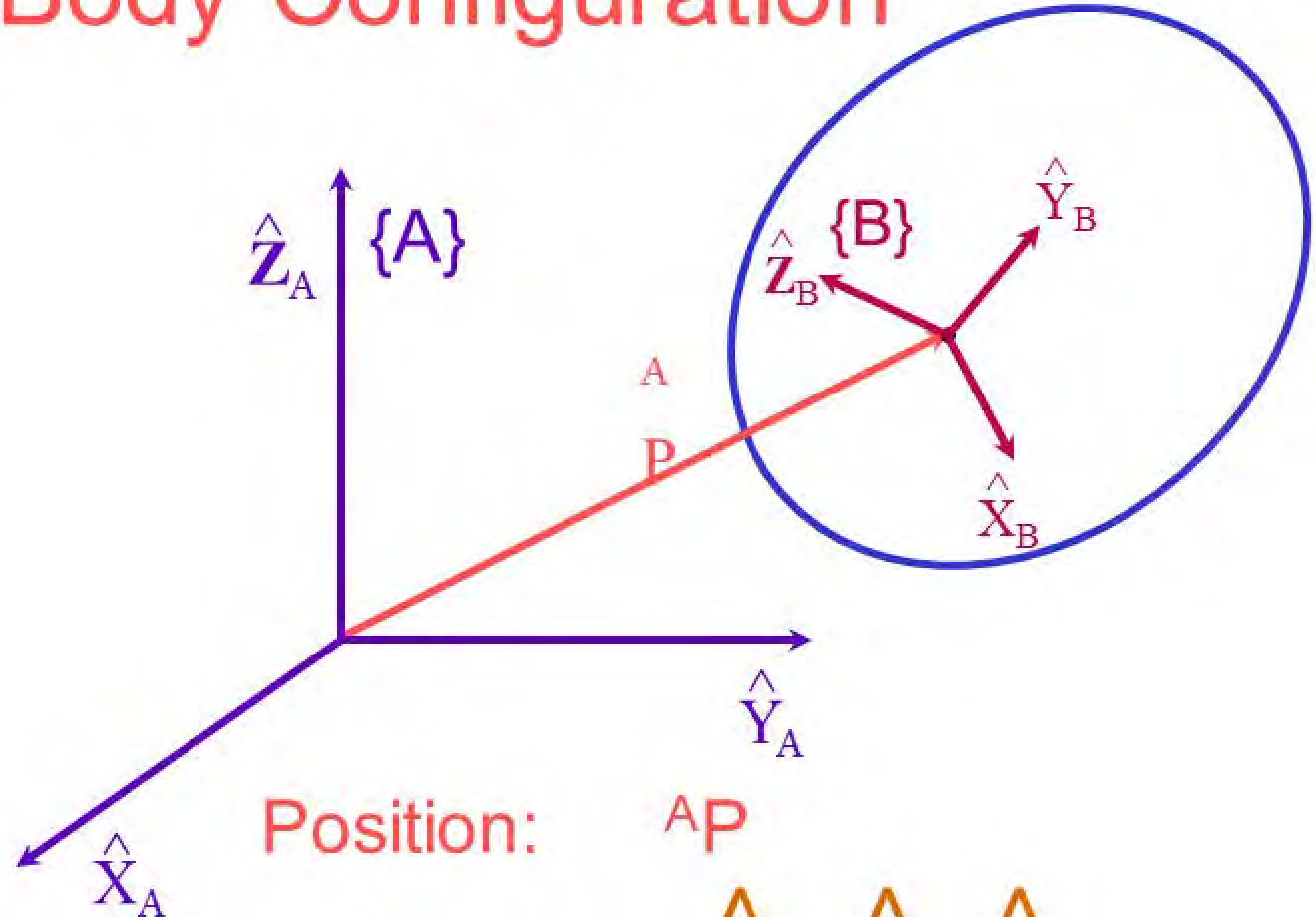


Cartesian Frame

Coordinate Frames



Rigid Body Configuration



Position: ^{A}P

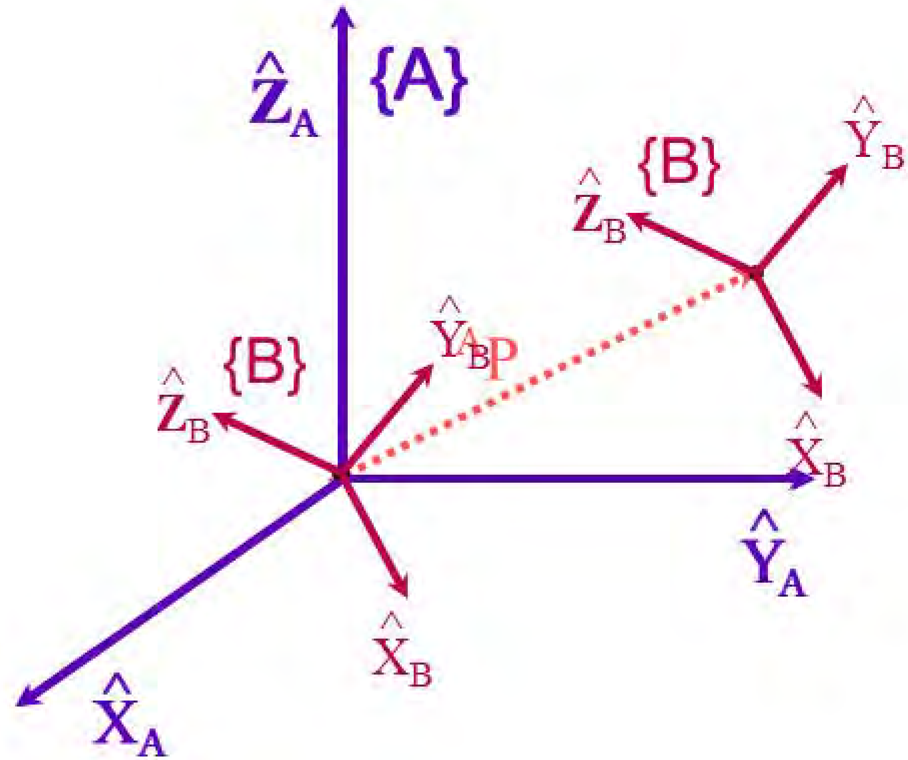
Orientation: $\{{}^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B\}$

describes rotations of {B} with respect to {A}

⇒ **Rotation Matrix**

Rotation Matrix

$$ {}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$



$$ {}^{A}\hat{X}_B = {}^{A}_{B}R \, {}^{B}\hat{X}_B $$

$$ {}^{A}\hat{X}_B = {}^{A}_{B}R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad {}^{A}\hat{Y}_B = {}^{A}_{B}R \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad {}^{A}\hat{Z}_B = {}^{A}_{B}R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow $$

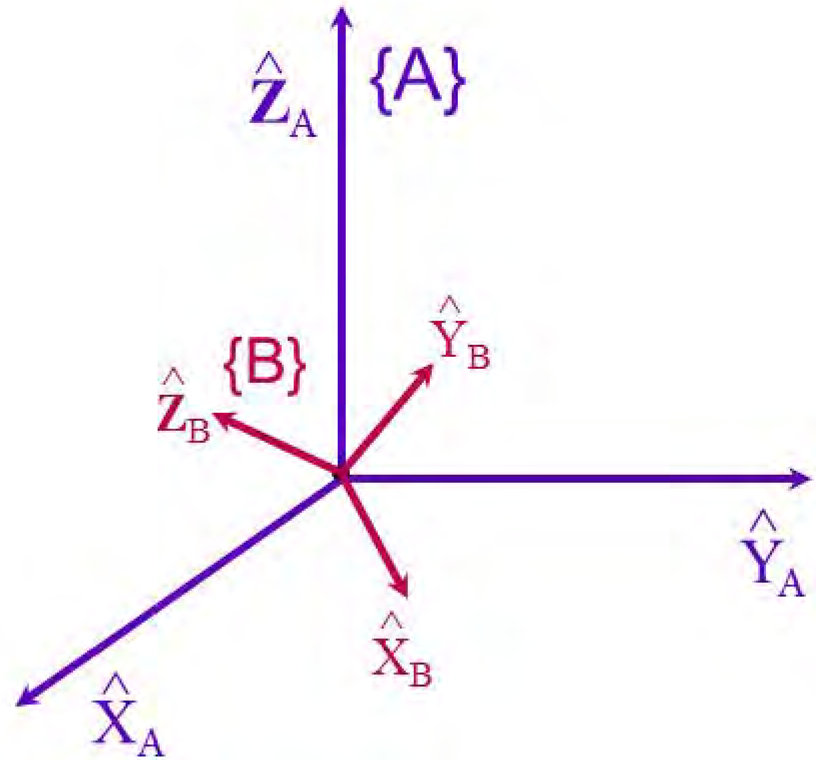
$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} $$

Movie Segment

Pet-Proto Robot Navigates Obstacles, Boston Dynamics, 2012

Rotation Matrix

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} $$



Dot Product

$$ {}^{A}\hat{X}_{B} = \begin{bmatrix} \hat{X}_{B} . \hat{X}_{A} \\ \hat{X}_{B} . \hat{Y}_{A} \\ \hat{X}_{B} . \hat{Z}_{A} \end{bmatrix} $$

$$ {}^{A}_{B}R = \begin{bmatrix} \hat{X}_{B} . \hat{X}_{A} & \hat{Y}_{B} . \hat{X}_{A} & \hat{Z}_{B} . \hat{X}_{A} \\ \hat{X}_{B} . \hat{Y}_{A} & \hat{Y}_{B} . \hat{Y}_{A} & \hat{Z}_{B} . \hat{Y}_{A} \\ \hat{X}_{B} . \hat{Z}_{A} & \hat{Y}_{B} . \hat{Z}_{A} & \hat{Z}_{B} . \hat{Z}_{A} \end{bmatrix} $$

$$ {}^{B}X_{A}^{T} $$

Rotation Matrix

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}_{A}^{T} \\ {}^{B}\hat{Y}_{A}^{T} \\ {}^{B}\hat{Z}_{A}^{T} \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}_{A} & {}^{B}\hat{Y}_{A} & {}^{B}\hat{Z}_{A} \end{bmatrix}^{T} = {}^{B}_{A}R^{T} $$

$$ {}^{A}_{B}R = {}^{B}_{A}R^{T} $$

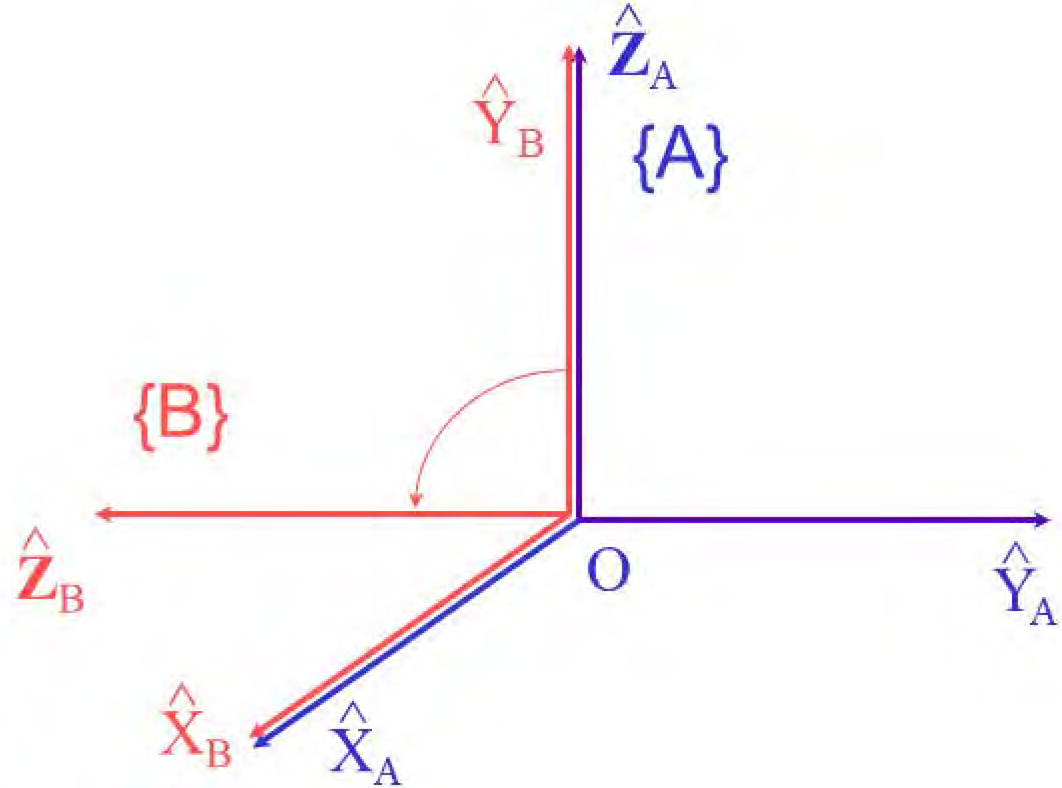
Inverse of Rotation Matrices

$$ {}^{A}_{B}R^{-1} = {}^{B}_{A}R = {}^{A}_{B}R^{T} $$

$$ \boxed{{}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T}} $$

Orthonormal Matrix

Example

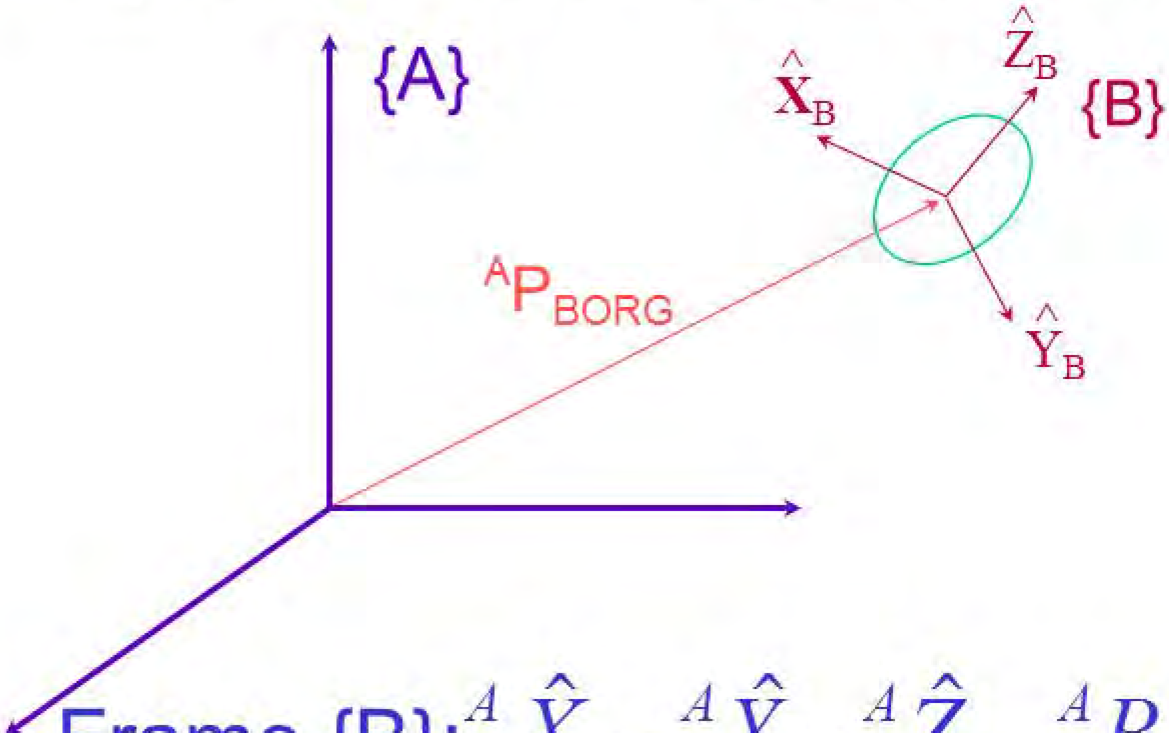


$$
{}^{A}_{B}R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{matrix} \leftarrow {}^{B}\hat{X}_A^T \\ \leftarrow {}^{B}\hat{Y}_A^T \\ \leftarrow {}^{B}\hat{Z}_A^T \end{matrix}
$$

$$
\begin{matrix} \uparrow & \uparrow & \uparrow \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{matrix}
$$

Description of a Frame

with respect to another reference frame



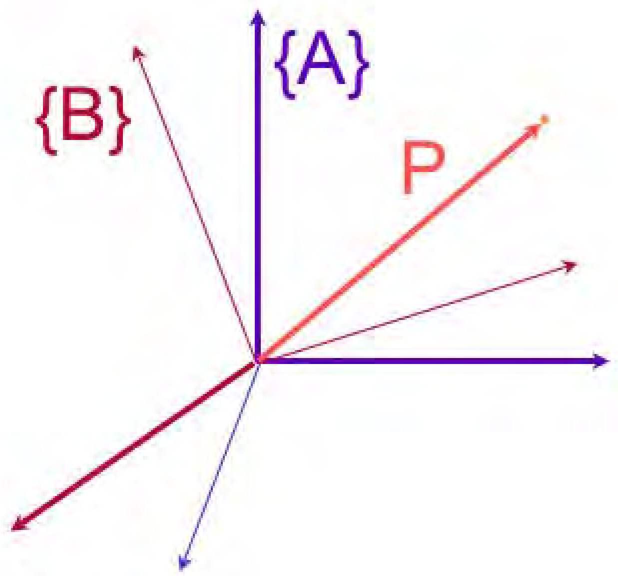
Frame {B}: $^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B, {}^{A}P_{Borg}$

$$\{B\} = \left\{ {}^{A}_{B}R \quad {}^{A}P_{Borg} \right\}$$

Mapping

changing descriptions from frame to frame

Rotations



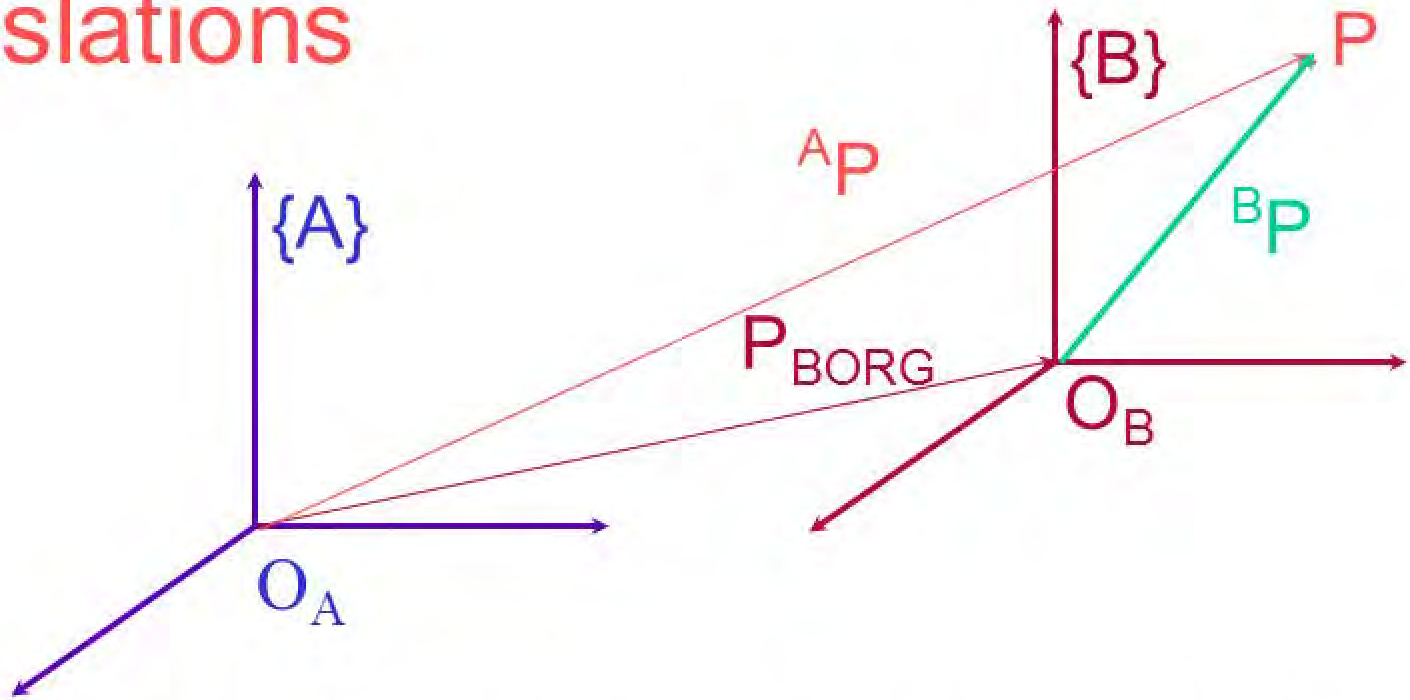
If P is given in {B}: ^{B}P

$$^{A}P = \begin{pmatrix} {}^{B}\hat{X}_{A} \cdot {}^{B}P \\ {}^{B}\hat{Y}_{A} \cdot {}^{B}P \\ {}^{B}\hat{Z}_{A} \cdot {}^{B}P \end{pmatrix} = \begin{pmatrix} {}^{B}\hat{X}_{A}^{T} \\ {}^{B}\hat{Y}_{A}^{T} \\ {}^{B}\hat{Z}_{A}^{T} \end{pmatrix} \cdot {}^{B}P$$

$$\Downarrow$$

$$^{A}P = {}^{A}_{B}R \; {}^{B}P$$

Translations



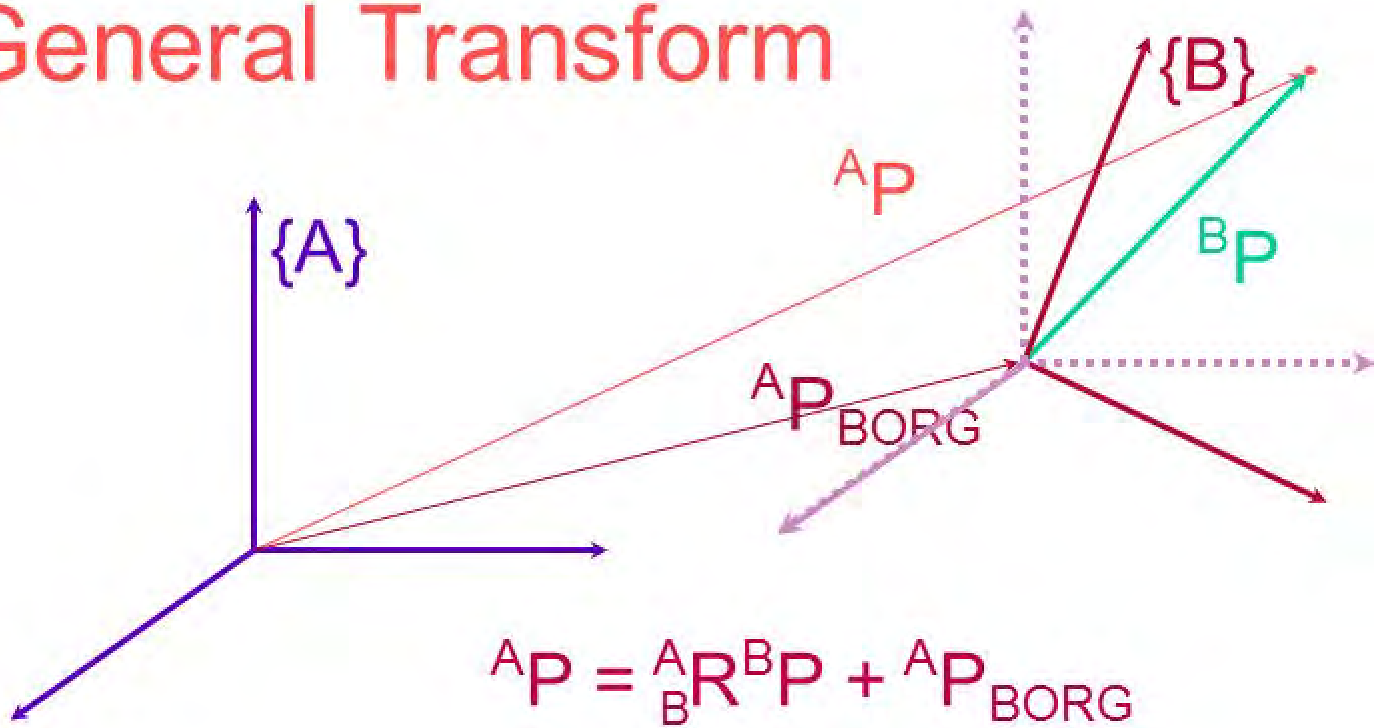
changing the position description of a point P

$\overrightarrow{O_BP} \Rightarrow \overrightarrow{O_AP}$ **(Two different vectors)**

P_{BORG} : $P_{O_B} \Rightarrow P_{O_A}$

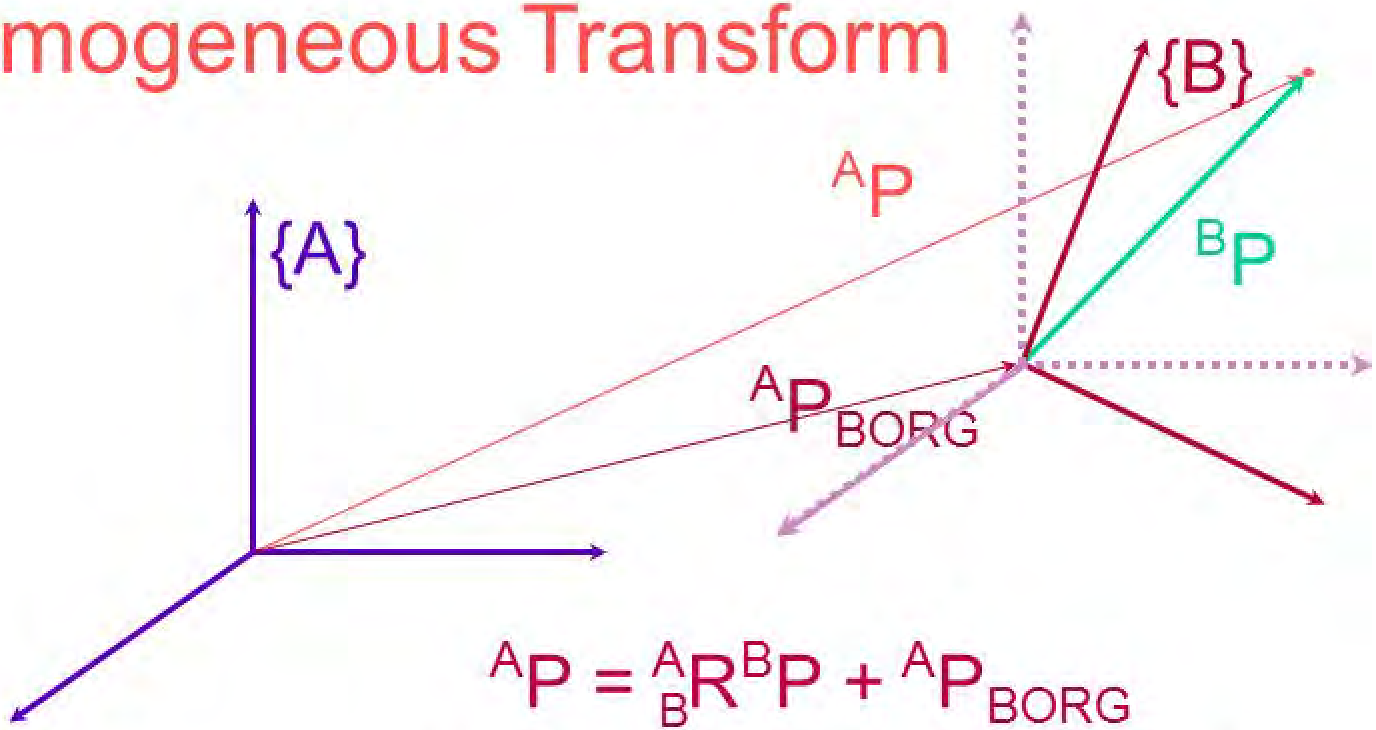
$$P_{O_A} = P_{O_B} + P_{BORG}$$

General Transform



$$^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$

Homogeneous Transform

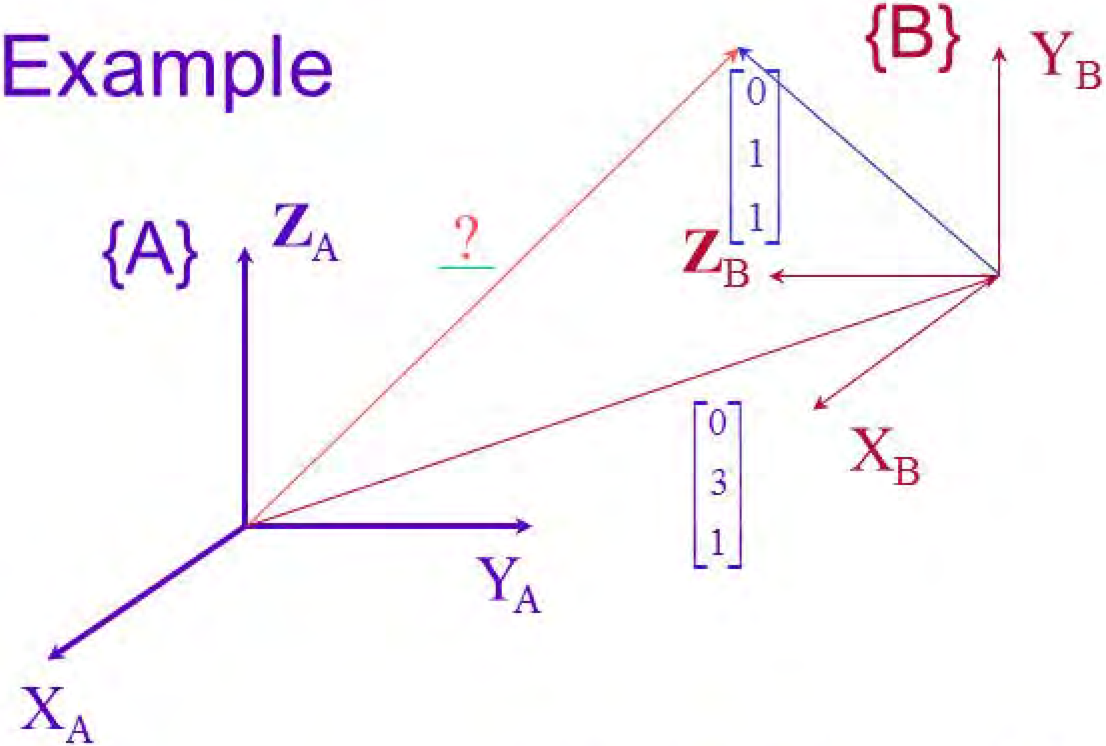


$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} $$

$$ \begin{bmatrix} {}^{A}P \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{Borg} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ \hline 1 \end{bmatrix} $$

$$ \underset{(4x1)}{{}^{A}P} = \underset{(4x4)}{{}^{A}_{B}T} \; \underset{(4x1)}{{}^{B}P} $$

Example



Homogeneous Transform

$$ {}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{B}P = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} $$

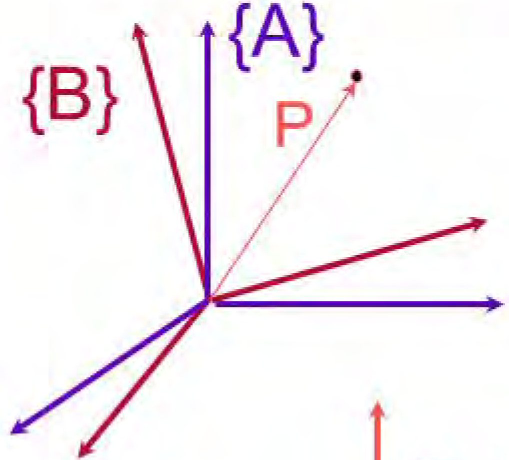
$$ {}^{A}P = {}^{A}_{B}T \cdot {}^{B}P \Rightarrow {}^{A}P = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} $$

Operators

Mapping: changing descriptions from frame to frame

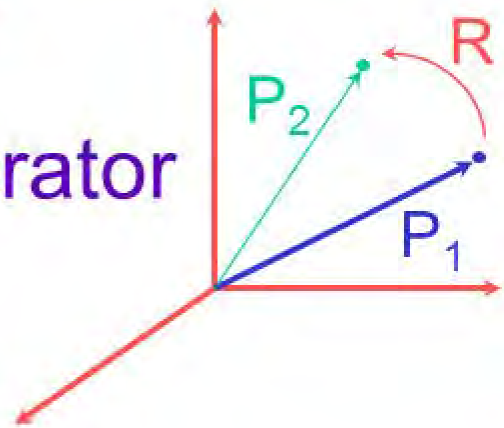
Operators: moving points (within the same frame)

Mapping



$$^{A}P = {}^{A}_{B}R \, {}^{B}P$$

Rotational Operator



$$R: \; P_1 \longrightarrow P_2$$

$$\boxed{P_2 = R \; P_1}$$

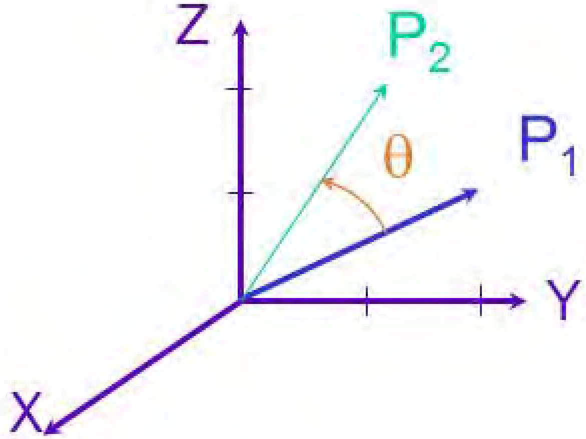
Rotational Operators

$$R_K(\theta): \quad P_1 \longrightarrow P_2$$

$$P_2 = R_K(\theta)\, P_1$$

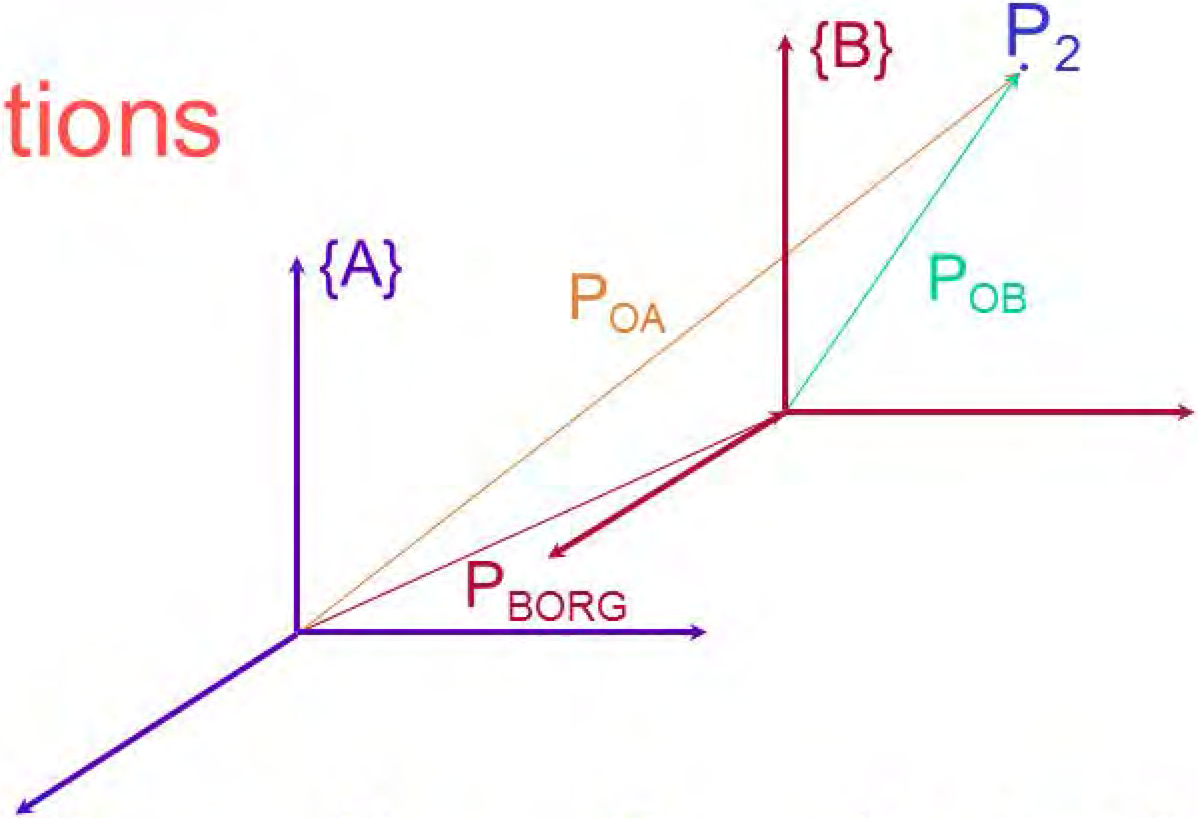
Example

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



$$P_2 = R_X(\theta) P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Translations

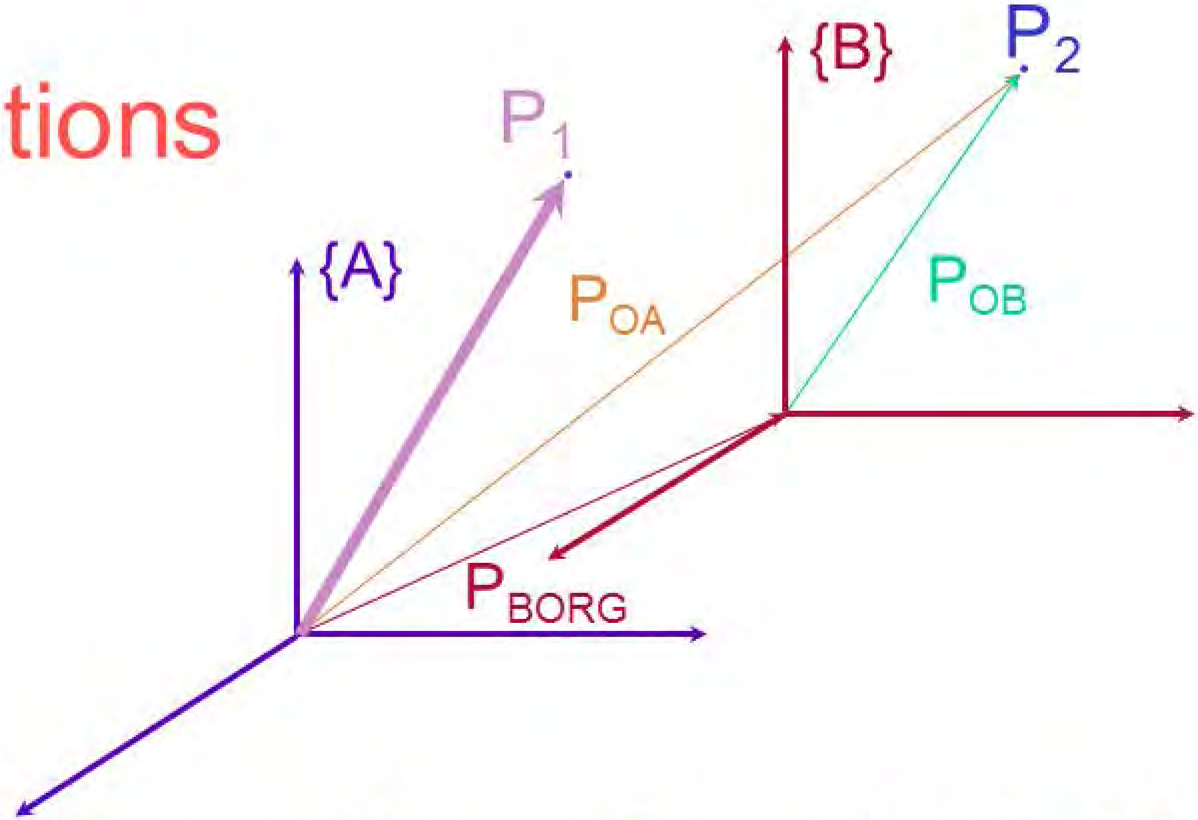


Mapping: $P_{BORG}: \quad P_{OB} \longrightarrow P_{OA}$ (same point)

2 diff. vectors

$$P_{OA} = P_{OB} + P_{BORG}$$

Translations



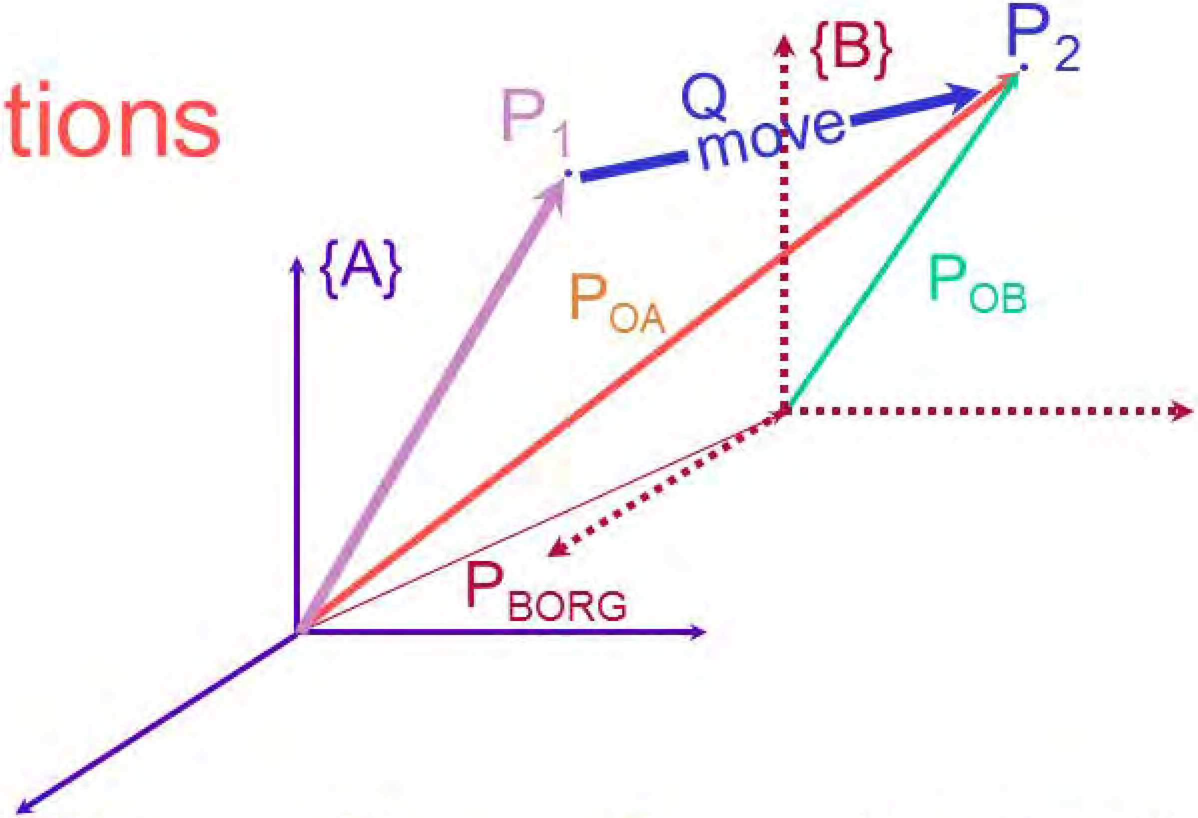
Mapping: P_{BORG} : $P_{OB} \longrightarrow P_{OA}$ (same point)

2 diff. vectors

$$P_{OA} = P_{OB} + P_{BORG}$$

Translational Operator:

Translations



Mapping: P_{BORG} : $P_{OB} \longrightarrow P_{OA}$ (same point)

2 diff. vectors

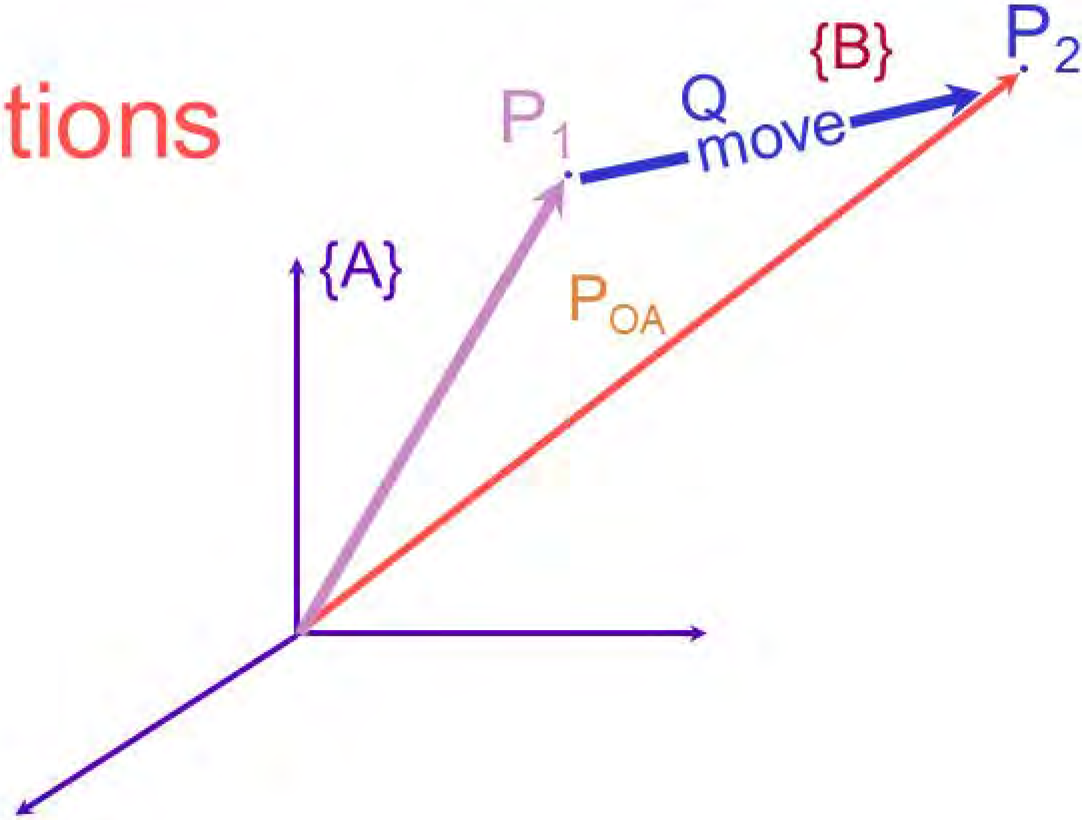
$$P_{OA} = P_{OB} + P_{BORG}$$

Translational Operator:

Q : $P_1 \longrightarrow P_2$ (2 points, 2 diff vectors)

$$P_2 = P_1 + Q$$

Translations

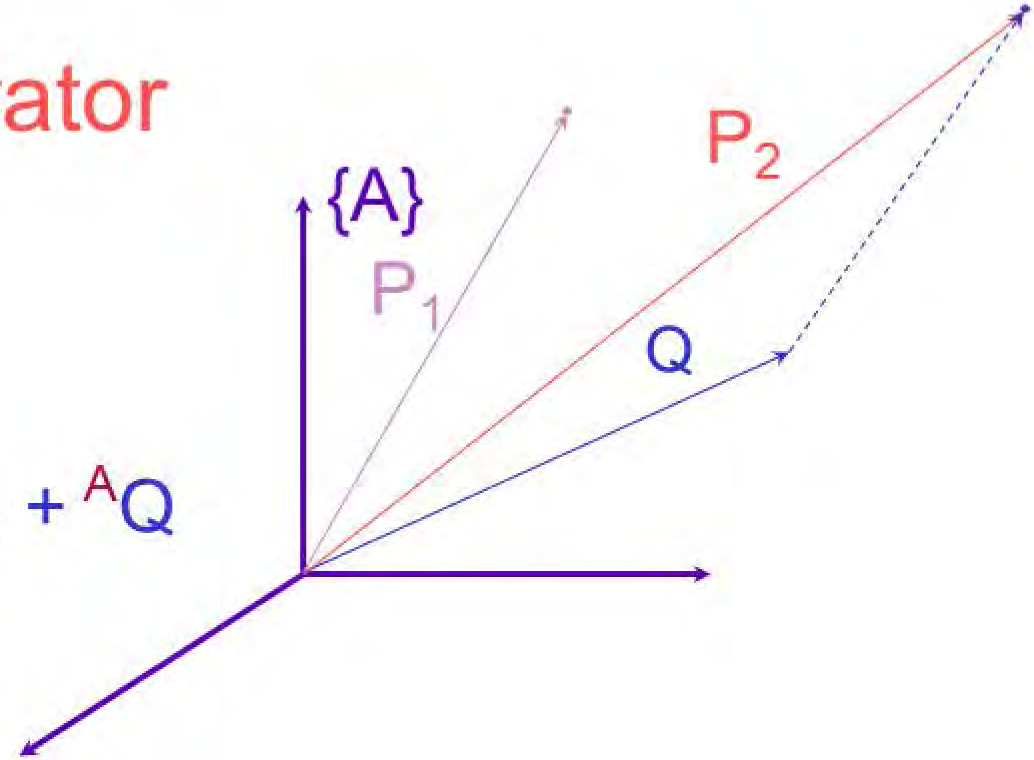


Translational Operator:

$Q: \quad P_1 \longrightarrow P_2$ (2 points, 2 diff vectors)

$$P_2 = P_1 + Q$$

Translation Operator

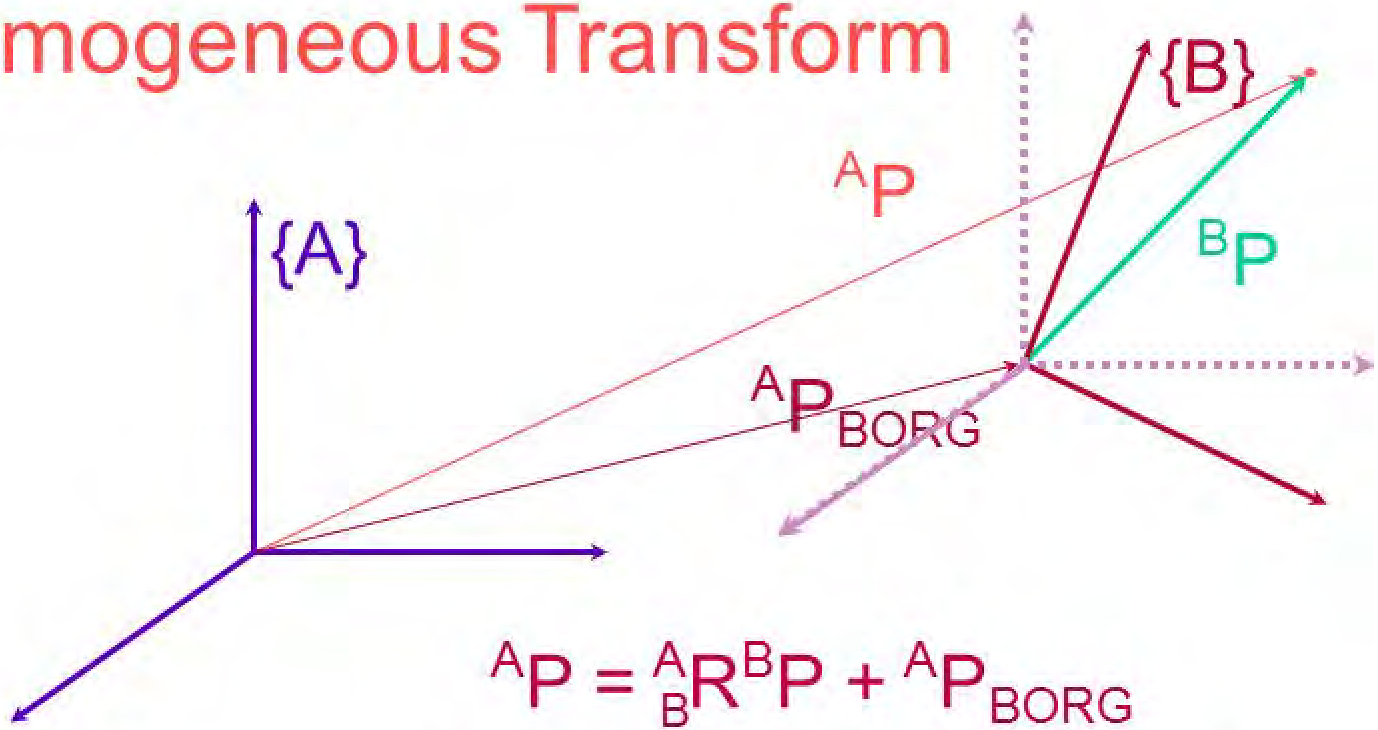


Operator: ${}^{A}P_2 = {}^{A}P_1 + {}^{A}Q$

Homogeneous Transform:

$$D_Q = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow {}^{A}P_2 = {}^{A}D_Q\,{}^{A}P_1$$

Homogeneous Transform



$$^AP = {}^A_BR\,{}^BP + {}^AP_{BORG}$$

$$\begin{bmatrix} ^AP \\ \hline 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^A_BR & & {}^AP_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} ^BP \\ \hline 1 \end{bmatrix}$$

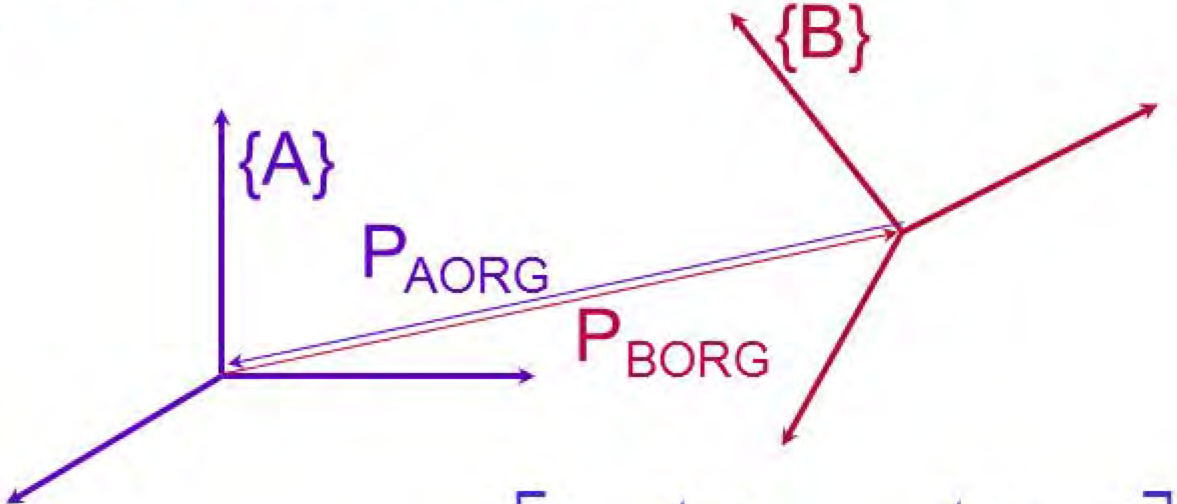
$$\underset{(4\times1)}{^AP} = \underset{(4\times4)}{{}^A_BT}\;\underset{(4\times1)}{^BP}$$

General Operators

$$P_2 = \left(\begin{array}{ccc|c} & R_K(\theta) & & Q \\ \hline 0 & 0 & 0 & 1 \end{array}\right) P_1$$

$$\boxed{P_2 = T\,P_1}$$

Inverse Transform



$$ {}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{Borg} \\ 0\quad 0\quad 0 & 1 \end{bmatrix} $$

$$ R^{-1} = R^{T} \quad (T^{-1} \neq T^{T}) $$

$$ {}^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T} \cdot {}^{A}P_{Borg} \\ 0\quad 0\quad 0 & 1 \end{bmatrix} $$

$^{B}P_{AORG}$

Announcement: Computer Forum Career Fair

Wednesday, Jan. 15
Computer Forum Career Fair
11am - 4pm

Lawn behind Gates & Mudd Chemistry Buildings

a9
Accel Partners
Adap.tv
Addepar
Adobe
Amazon Web Services
Andreessen Horowitz
Apple
Apportable
Arista Networks
Bloomberg
Box
Brightroll
Broadcom
Cash Dynamics
C3 Energy
Chopper Trading
Cisco
Counsyl
Coursera
Cutler Group
D.E. Shaw
Dropbox
eBay
EMC
Ericsson
Evernote
Facebook
GE
Google
Groupon
HealthTap
Hewlett Packard
IBM
Intuit
IXL
Juniper Networks
Lab 126
Lenovo
Lightspeed
LinkedIn
LiveRamp
Marin Software
Microsoft
Mobile Iron
Nissan
Nvidia
Oracle
Palantir Technologies
Pocket Gems
Quantcast
Rocket Fuel
Salesforce
Samsung
SAP Americas
Sequoia Capital
Serendipity
Shape Security
Snapchat
Splunk
Spokeo
Square
Storm8
Symantec
Tableau Software
Technicolor
Teradata
Texas Instruments
Tower Research
Turn
Twitter
Two Sigma
VMware
WhatsApp
Workday
Yahoo!
Yelp
Zazzle
Zynga

Kinematics

Movie Segment

LittleDog

Learning Locomotion with LittleDog

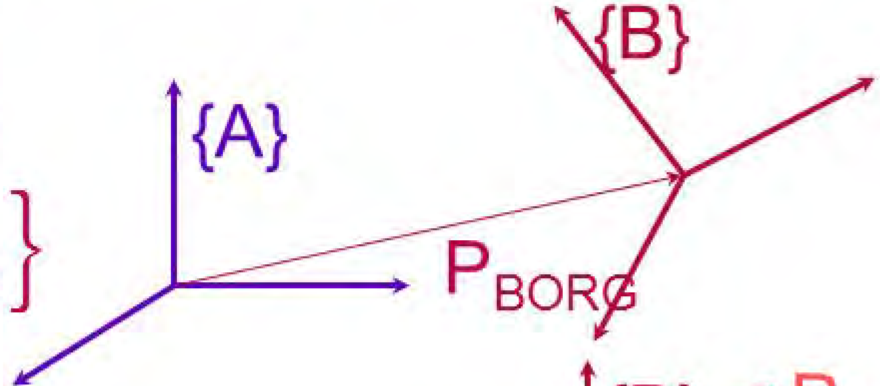
http://www-clmc.usc.edu

Mrinal Kalakrishnan, Jonas Buchli, Peter Pastor, Michael Mistry, and Stefan Schaal

Homogeneous Transform Interpretations

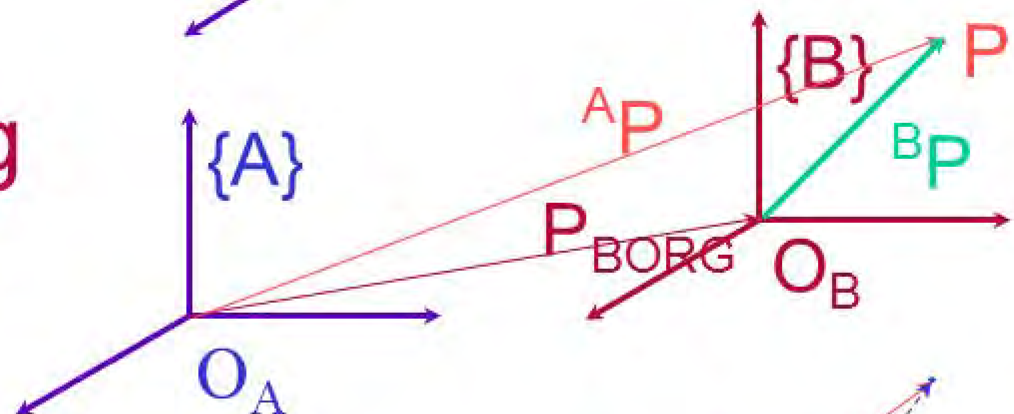
Description of a frame

$^{A}_{B}T$: $\{B\} = \{ {}^{A}_{B}R \quad {}^{A}P_{Borg} \}$



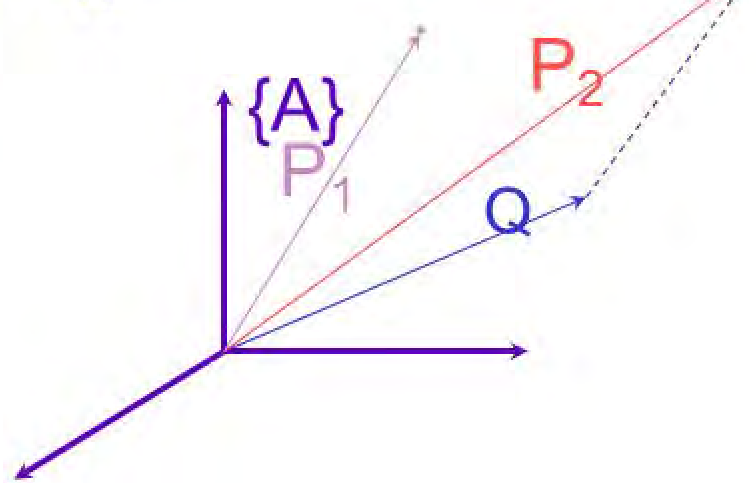
Transform mapping

$^{A}_{B}T$: $^{B}P \rightarrow {}^{A}P$

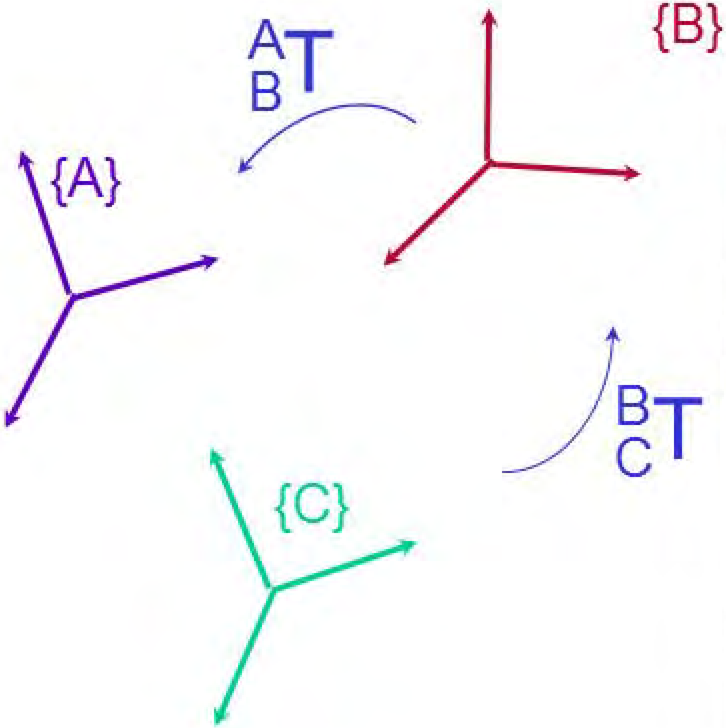


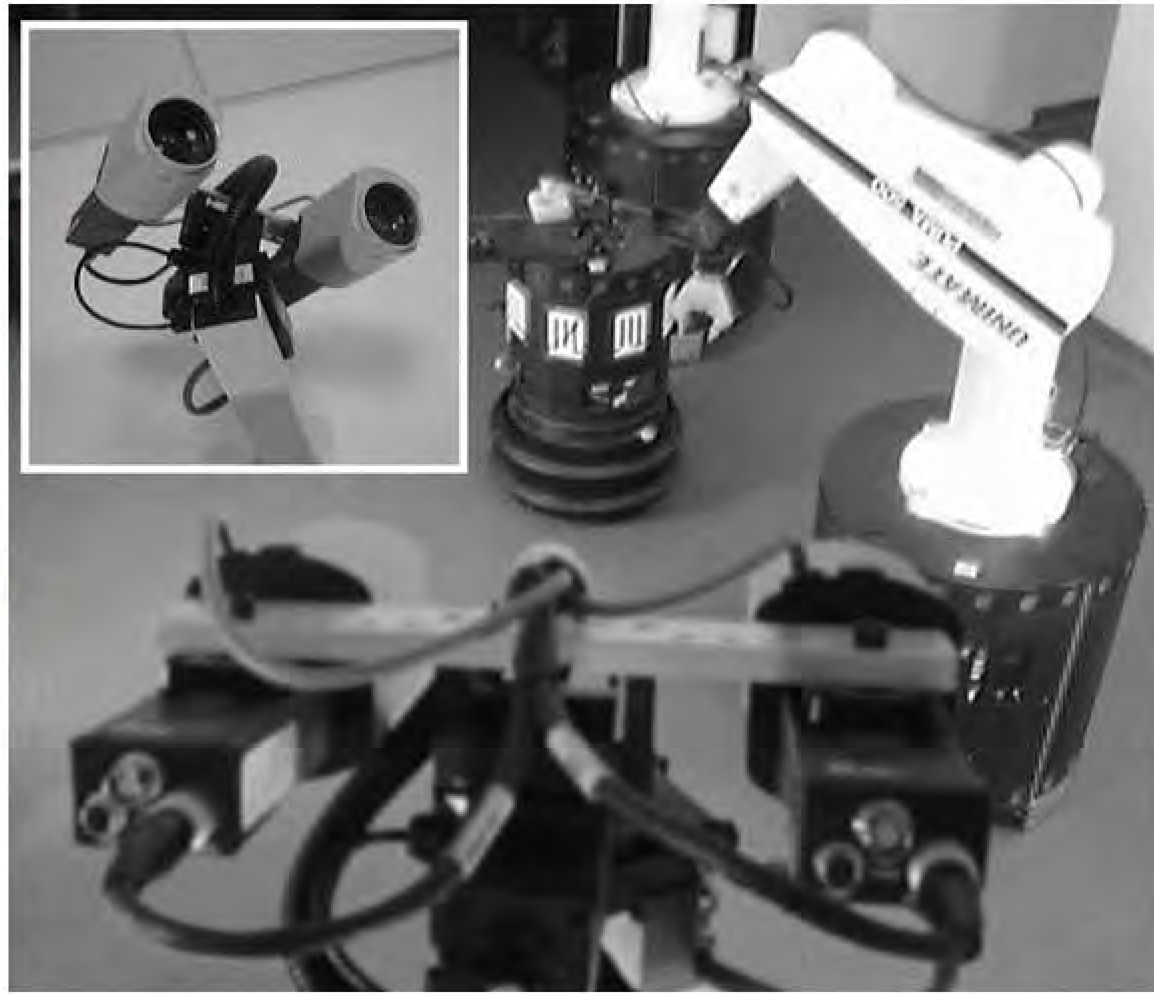
Transform operator

T : $P_1 \rightarrow P_2$

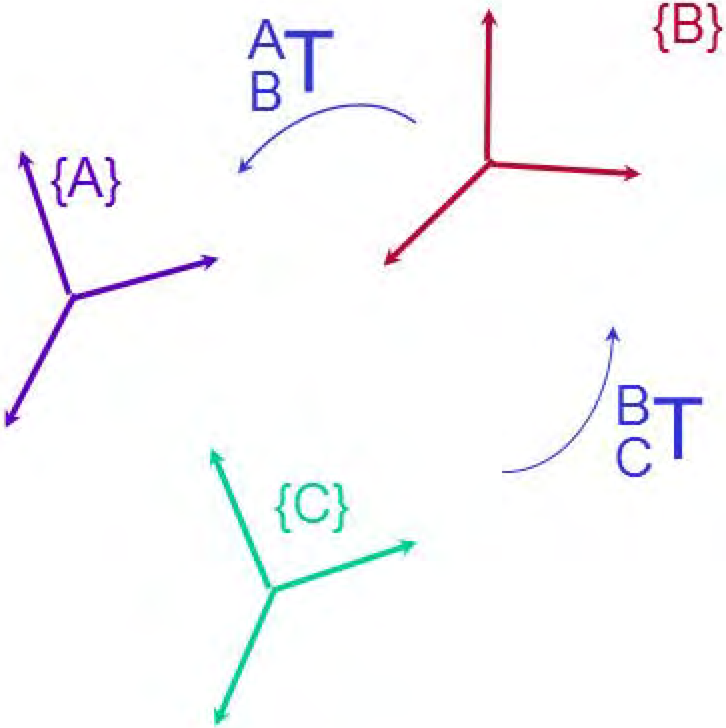


Transform Equation

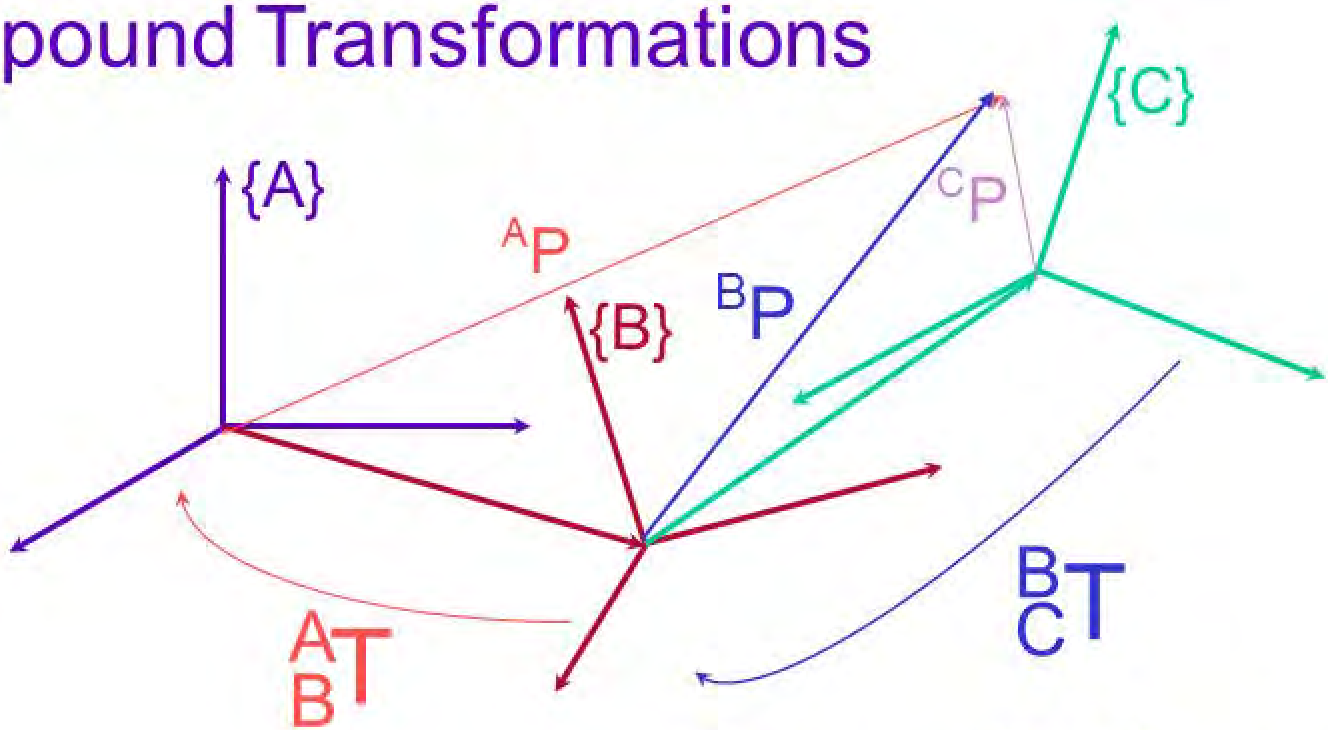




Transform Equation



Compound Transformations



$$^{B}P = {}^{B}_{C}T \; {}^{C}P$$

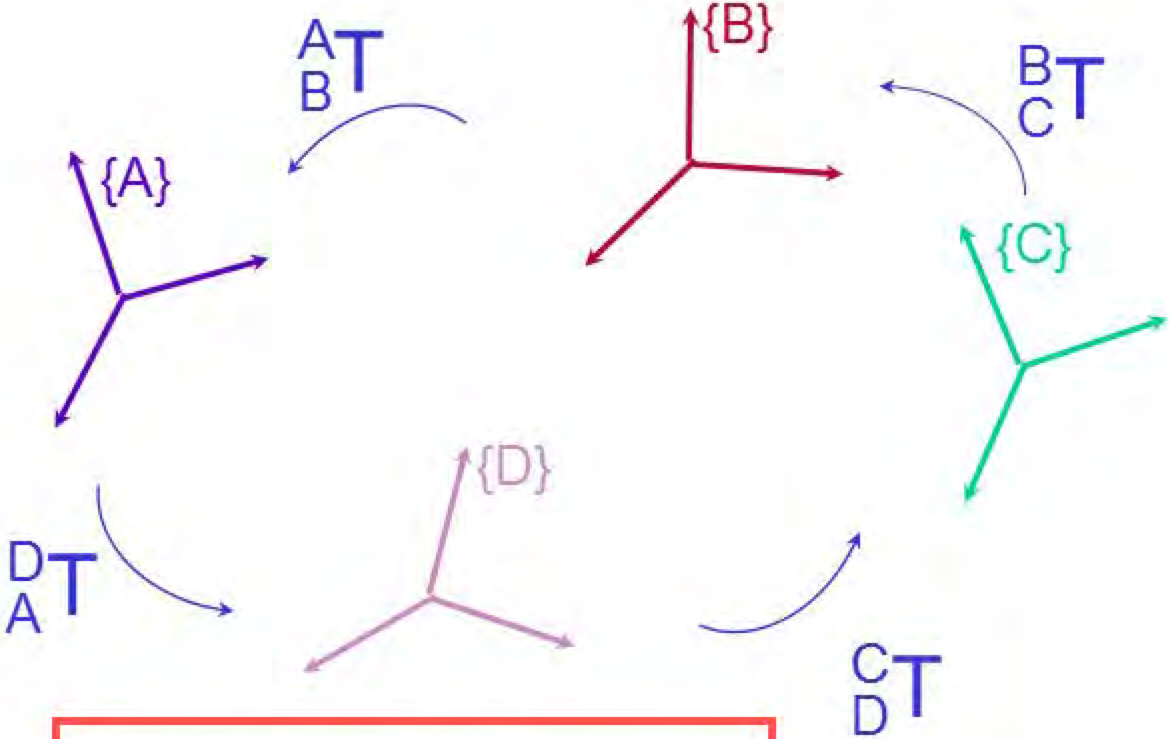
$$^{A}P = {}^{A}_{B}T \; {}^{B}P$$

$$^{A}P = {}^{A}_{B}T \; {}^{B}_{C}T \; {}^{C}P \quad \Rightarrow \quad {}^{A}_{C}T = {}^{A}_{B}T \; {}^{B}_{C}T$$

$$ {}^{A}_{C}T = {}^{A}_{B}T \, {}^{B}_{C}T $$

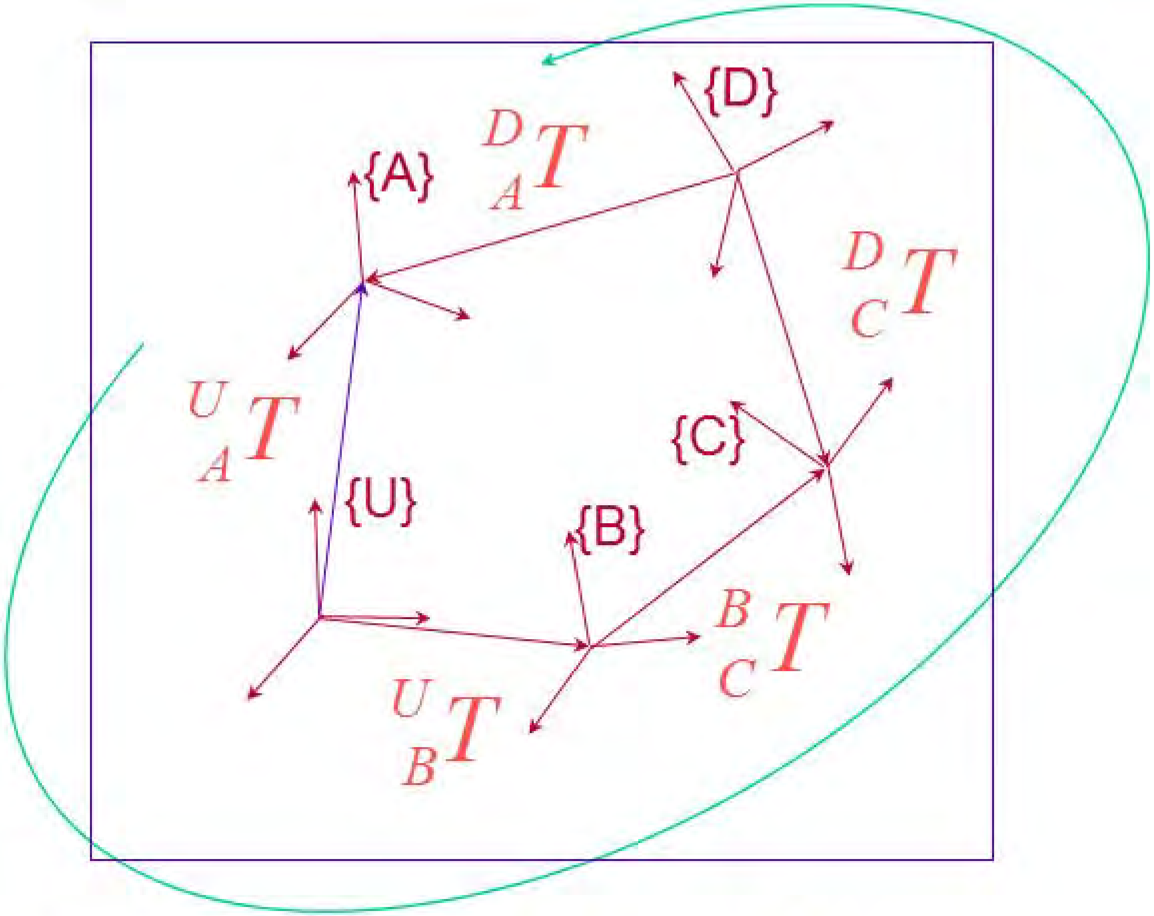
$$ {}^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R \, {}^{B}_{C}R & {}^{A}_{B}R \, {}^{B}P_{Corg} + {}^{A}P_{Borg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

Transform Equation



$$^{A}_{B}T\,^{B}_{C}T\,^{C}_{D}T\,^{D}_{A}T = I$$

$$\Rightarrow \quad ^{B}_{A}T = {}^{B}_{C}T\,^{C}_{D}T\,^{D}_{A}T$$



$$ {}^{D}_{A}T^{-1} \cdot {}^{D}_{C}T \cdot {}^{B}_{C}T^{-1} \cdot {}^{U}_{B}T^{-1} \cdot {}^{U}_{A}T \equiv I $$

$$ {}^{U}_{A}T = {}^{U}_{B}T \cdot {}^{B}_{C}T \cdot {}^{D}_{C}T^{-1} \cdot {}^{D}_{A}T $$

Spatial Descriptions

- Task Description
- Transformations
- Representations

Spatial Descriptions

- Task Description
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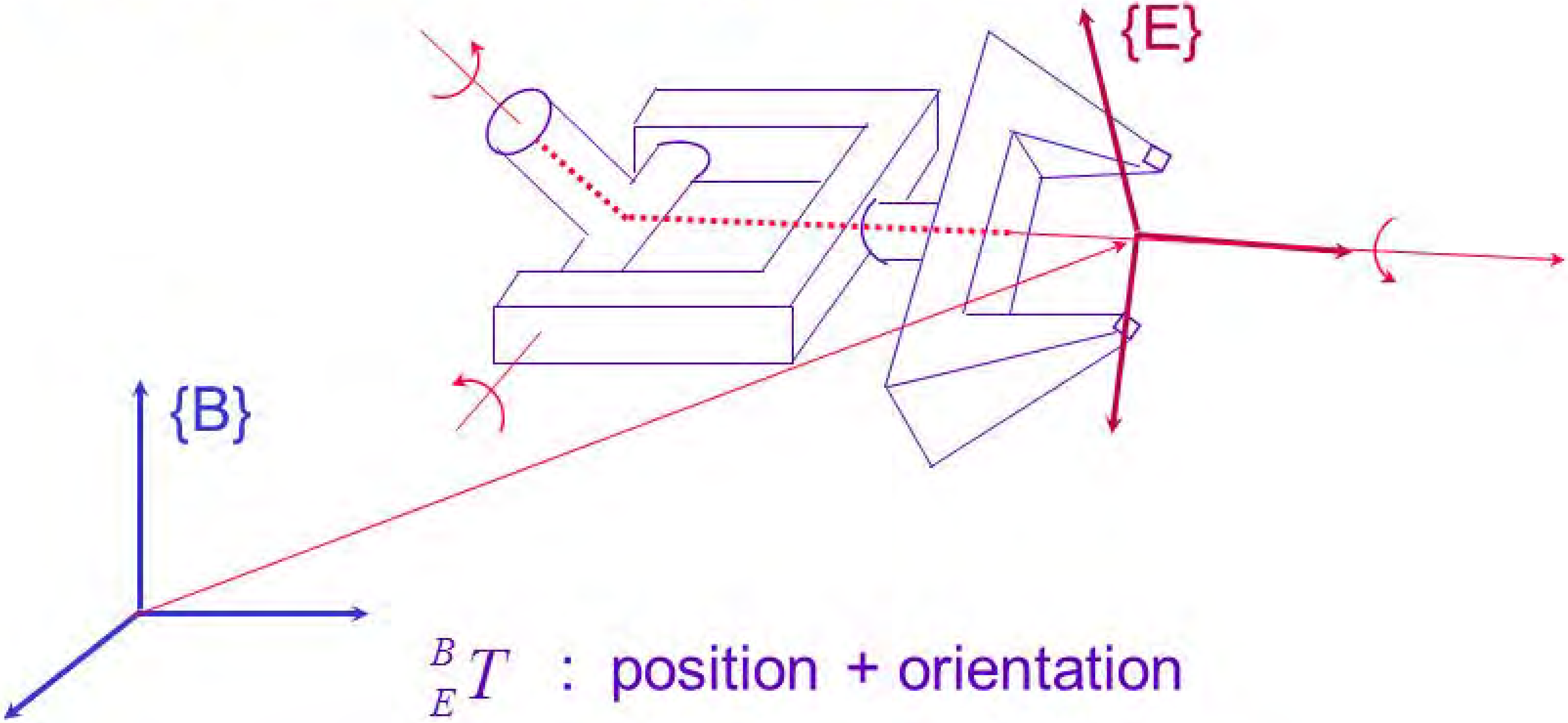
End-Effector Configuration



$^{B}_{E}T$: position + orientation

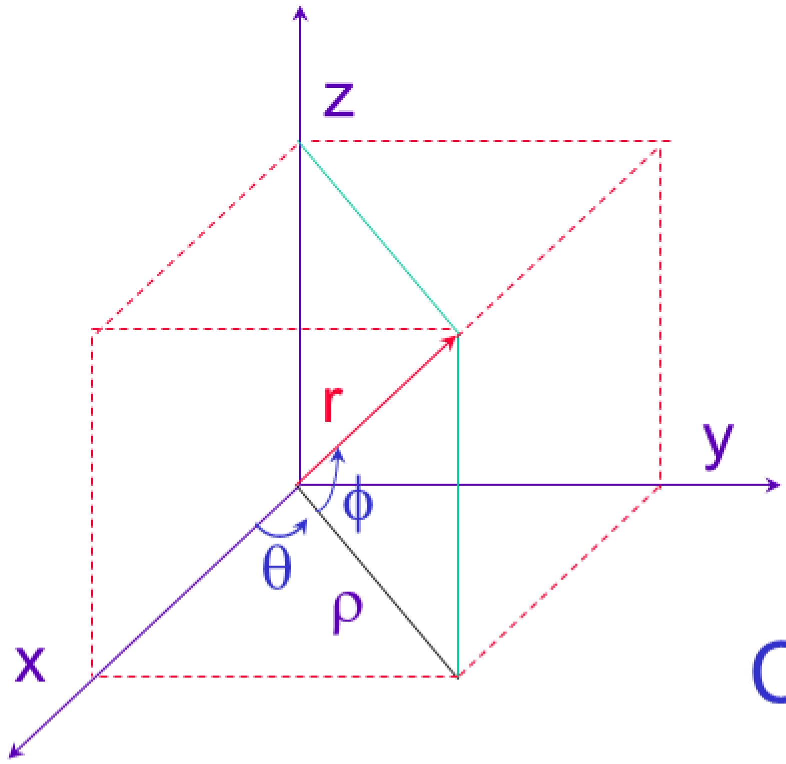
End-Effector Configuration Parameters

$$X = \begin{bmatrix} X_P \\ X_R \end{bmatrix}$$

X_P ← position

X_R ← orientation

Position Representations



Cartesian: (x, y, z)

Cylindrical: (ρ, θ, z)

Spherical: (r, θ, ϕ)

Rotation Representations

Rotation Matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix}$$

Direction Cosines

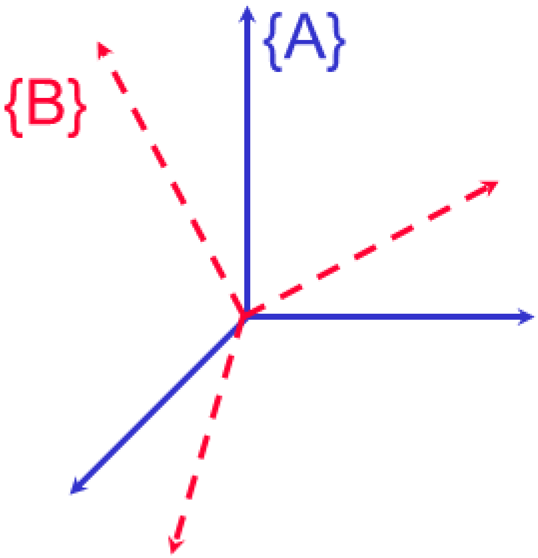
$$x_r = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}_{(9x1)}$$

Constraints

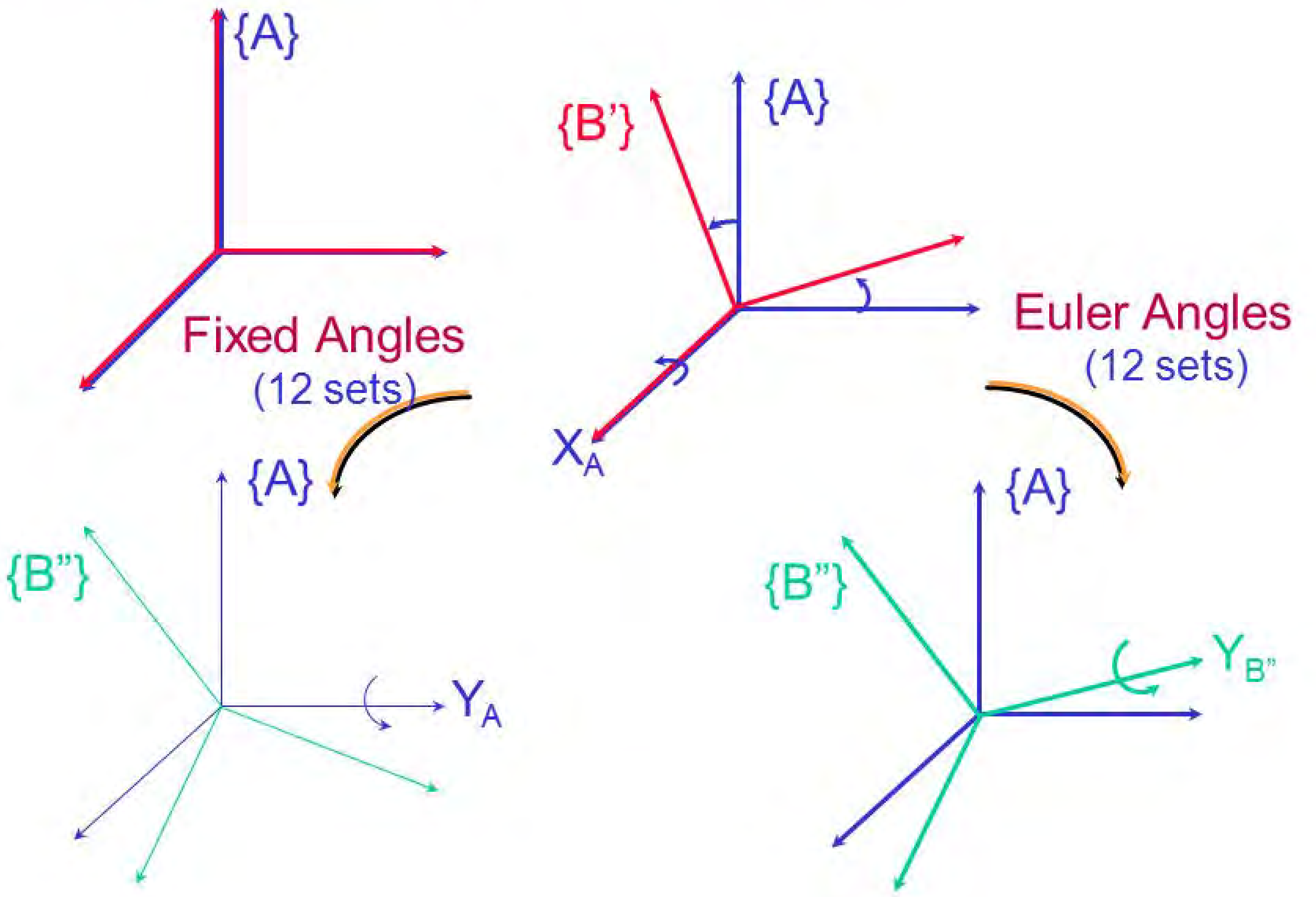
$$|\mathbf{r}_1| = |\mathbf{r}_2| = |\mathbf{r}_3| = 1$$

$$\mathbf{r}_1 . \mathbf{r}_2 = \mathbf{r}_1 . \mathbf{r}_3 = \mathbf{r}_2 . \mathbf{r}_3 = 0$$

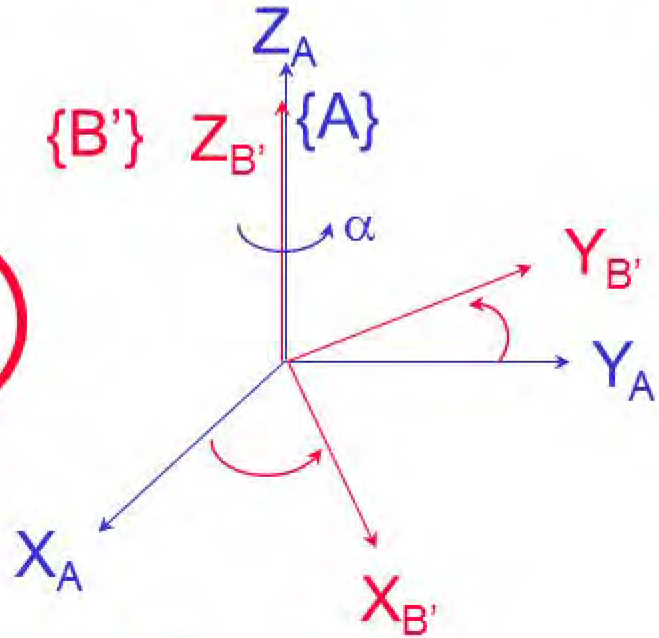
Three Angle Representations



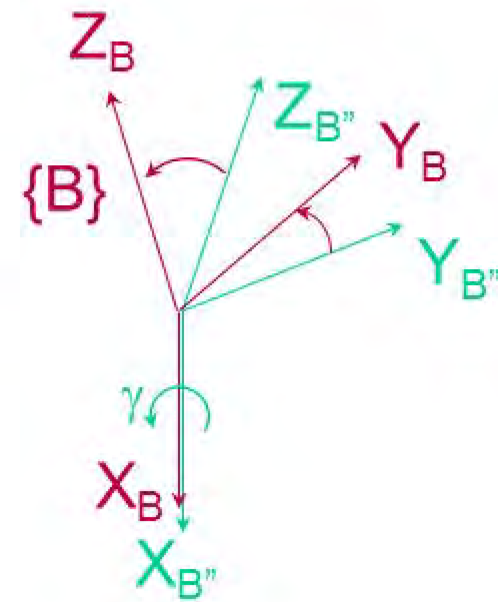
Three Angle Representations



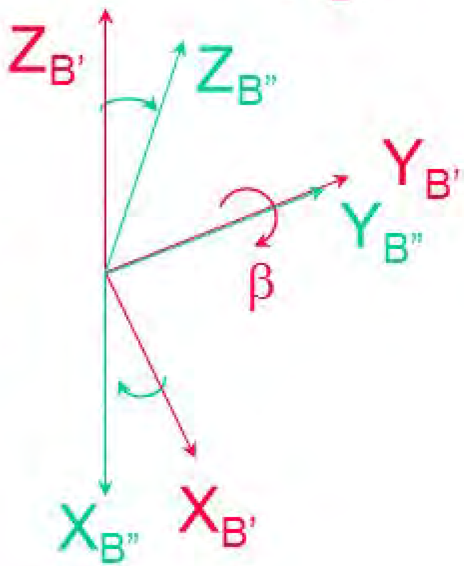
Euler Angles (Z-Y-X)



$${}^{A}_{B'}R$$



$${}^{B''}_{B}R$$

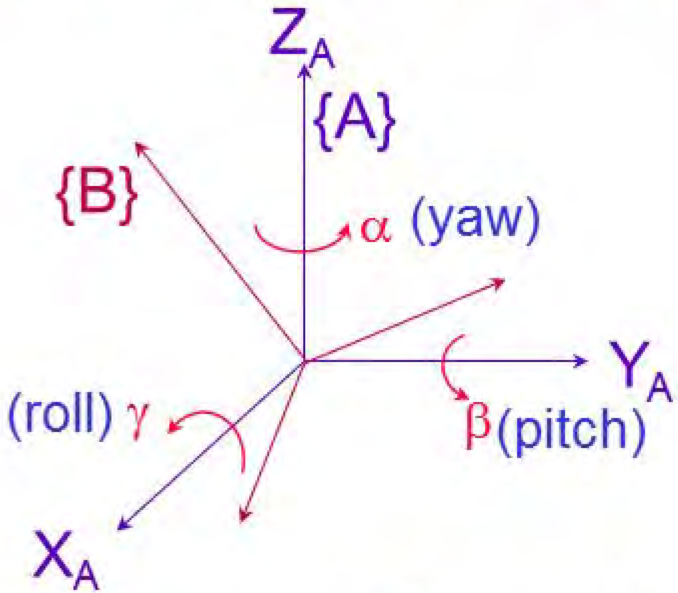


$${}^{B'}_{B''}R$$

$${}^{A}_{B}R = {}^{A}_{B'}R.\,{}^{B'}_{B''}R.\,{}^{B''}_{B}R$$

$${}^{A}_{B}R = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

X-Y-Z Fixed Angles



$$R_X(\gamma):\ v \rightarrow R_X(\gamma).v$$

$$R_Y(\beta):\ (R_X(\gamma).v) \rightarrow R_Y(\beta).(R_X(\gamma).v)$$

$$R_Z(\alpha):\ (R_Y(\beta).R_X(\gamma).v) \rightarrow R_Z(\alpha).(R_Y(\beta).R_X(\gamma).v)$$

$$\boxed{{}^A_B R = {}^A_B R_{XYZ}(\gamma, \beta, \alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)}$$

Z-Y-X Euler Angles

$$ {}^A_B R = R_{Z'}(\alpha).R_{Y'}(\beta).R_{X'}(\gamma) $$

$$ \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

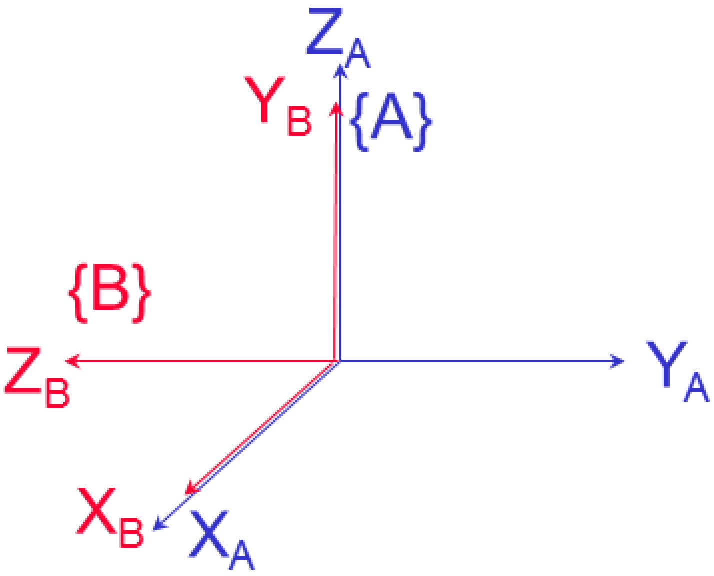
$$ {}^A_B R = {}^A_B R_{Z'Y'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha.c\beta & X & X \\ s\alpha.c\beta & X & X \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix} $$

Z-Y-Z Euler Angles

$$ {}_{B}^{A}R = R_{Z'}(\alpha).R_{Y'}(\beta).R_{Z'}(\gamma) $$

$$ {}_{B}^{A}R = {}_{B}^{A}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} X & X & c\alpha.s\beta \\ X & X & s\alpha.s\beta \\ -s\beta.c\gamma & s\beta.s\gamma & c\beta \end{bmatrix} $$

Example



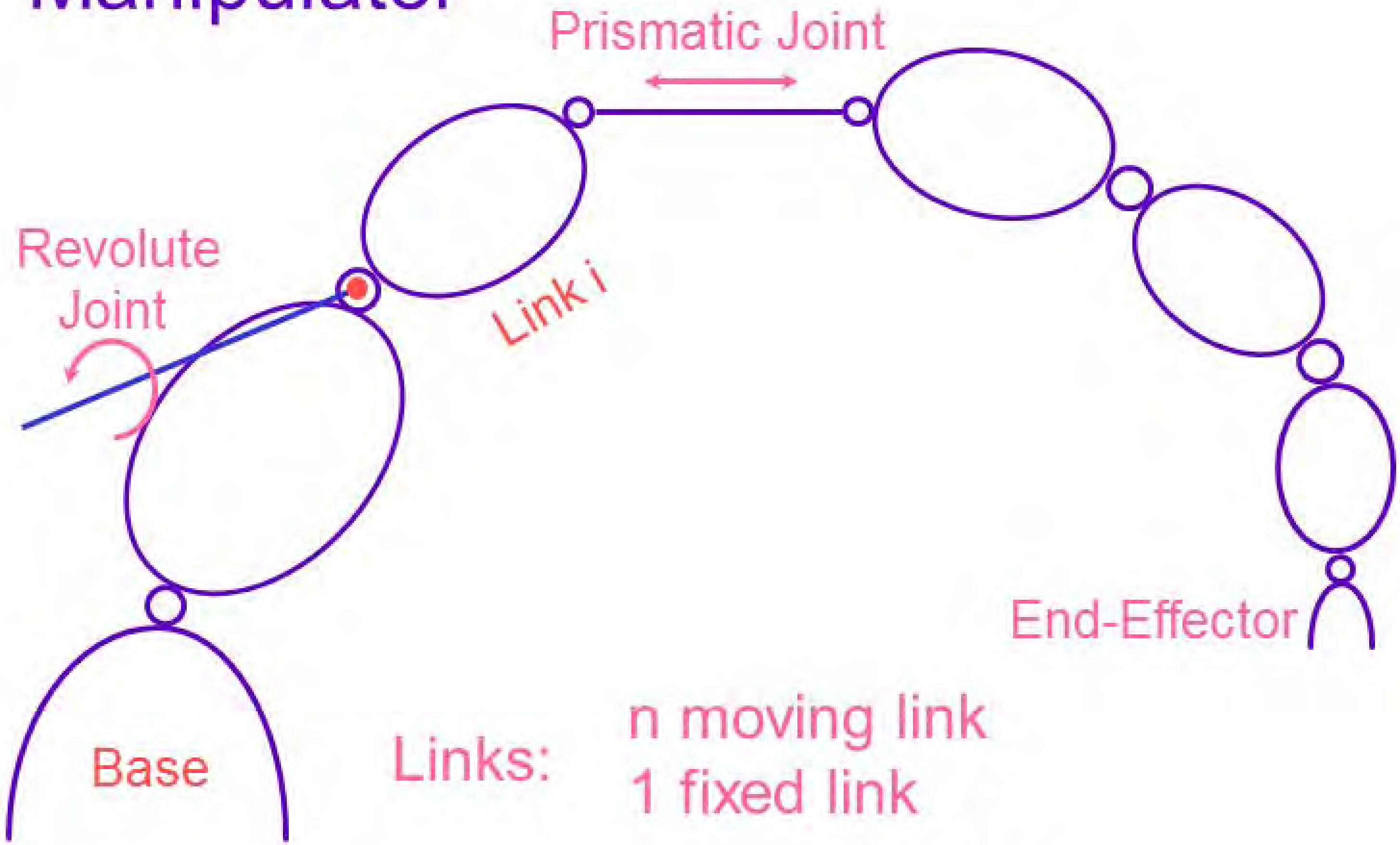
$R_{Z'Y'X'}(\alpha, \beta, \gamma)$:

$$\alpha = 0$$

$$\beta = 0$$

$$\gamma = 90^o$$

Manipulator



Links: n moving link
1 fixed link

Joints: Revolute (1 DOF)
Prismatic (1 DOF)

Fixed & Euler Angles

X-Y-Z Fixed Angles

$$R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

Z-Y-X Euler Angles

$$R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

$$\boxed{R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_{XYZ}(\gamma,\beta,\alpha)}$$

Inverse Problem

Given ${}^A_B R$ find (α, β, γ)

$R_{Z'Y'X'}$

$$ {}^A_B R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha.c\beta & c\alpha.s\beta.s\gamma - s\alpha.c\gamma & c\alpha.s\beta.c\gamma + s\alpha.s\gamma \\ s\alpha.c\beta & s\alpha.s\beta.s\gamma + c\alpha.c\gamma & s\alpha.s\beta.c\gamma - c\alpha.s\gamma \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix} $$

$$ \left.\begin{aligned} \cos\beta = c\beta &= \sqrt{r_{11}^2 + r_{21}^2} \\ \sin\beta = s\beta &= -r_{31} \end{aligned}\right\} \rightarrow \beta = A\tan 2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}) $$

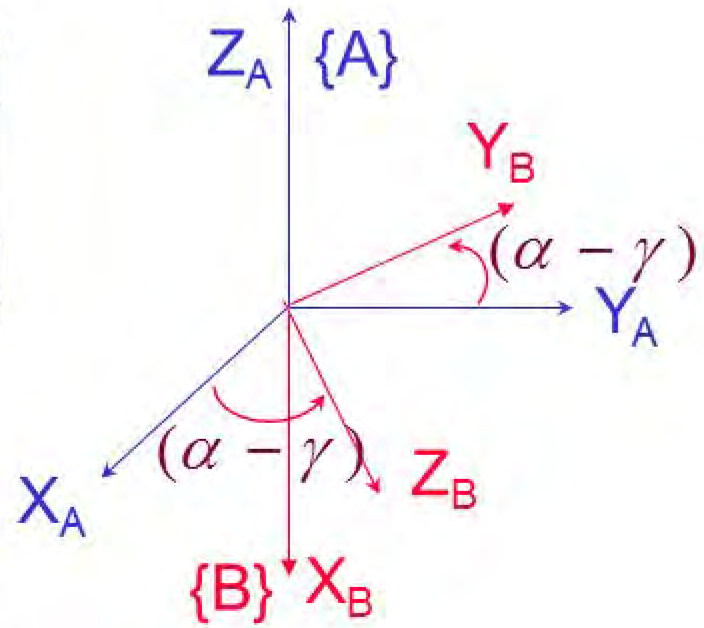
if $c\beta = 0$ $(\beta = \pm 90^\circ)$ ⇨ Singularity of the representation

⇨ Only $(\alpha + \gamma)$ or $(\alpha - \gamma)$ is defined

Singularities - Example ($R_{Z'Y'X'}$)

$c\beta = 0,\ s\beta = +1$

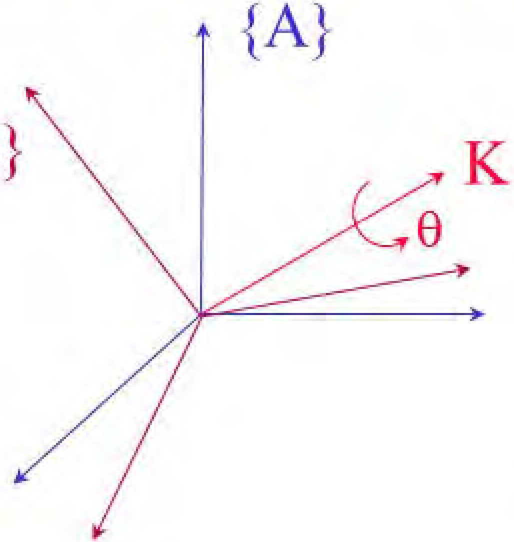
$$
{}^A_B R = \begin{pmatrix} 0 & -s(\alpha-\gamma) & c(\alpha-\gamma) \\ 0 & c(\alpha-\gamma) & s(\alpha-\gamma) \\ -1 & 0 & 0 \end{pmatrix}
$$



$c\beta = 0,\ s\beta = -1$

$$
{}^A_B R = \begin{pmatrix} 0 & -s(\alpha+\gamma) & -c(\alpha+\gamma) \\ 0 & c(\alpha+\gamma) & -s(\alpha+\gamma) \\ 1 & 0 & 0 \end{pmatrix}
$$

Equivalent angle-axis representation, $R_K(\theta)$



$$X_r = \theta.K = \begin{bmatrix} \theta.k_X \\ \theta.k_Y \\ \theta.k_Z \end{bmatrix}$$

$$R_K(\theta) = \begin{bmatrix} k_x.k_x.v\theta + c\theta & k_x.k_y.v\theta - k_z.s\theta & k_x.k_z.v\theta + k_y.s\theta \\ k_x.k_y.v\theta + k_z.s\theta & k_y.k_y.v\theta + c\theta & k_y.k_z.v\theta - k_x.s\theta \\ k_x.k_z.v\theta - k_y.s\theta & k_y.k_z.v\theta + k_x.s\theta & k_z.k_z.v\theta + c\theta \end{bmatrix}$$

with $v\theta = 1 - c\,\theta$

$$R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = Ar\cos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$${}^{A}K = \frac{1}{2.\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix},$$

singularity for $\sin\theta = 0$

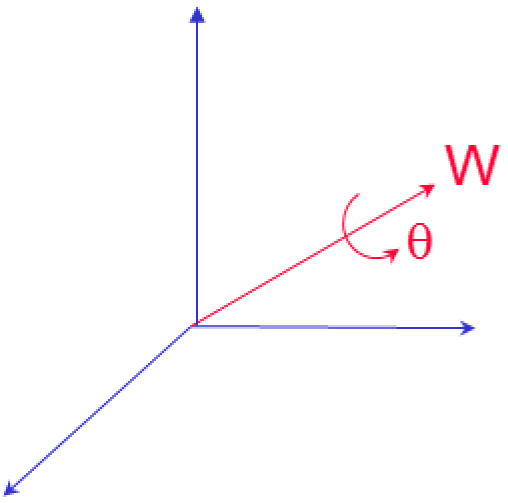
Euler Parameters

$$\varepsilon_1 = W_x . \sin\frac{\theta}{2}$$

$$\varepsilon_2 = W_y . \sin\frac{\theta}{2}$$

$$\varepsilon_3 = W_z . \sin\frac{\theta}{2}$$

$$\varepsilon_4 = \cos\frac{\theta}{2}$$



Normality Condition

$$|W| = 1, \quad \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$$

ε : point on a unit hypershpere in four-dimensional space

Inverse Problem   Given ${}^{A}_{B}R$ find ε

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \equiv \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}$$

$$r_{11}+r_{22}+r_{33} = 3-4(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2)$$

$$(1-\varepsilon_4^2)$$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$

$$\varepsilon_1 = \frac{r_{32}-r_{23}}{4\varepsilon_4}, \quad \varepsilon_2 = \frac{r_{13}-r_{31}}{4\varepsilon_4}, \quad \varepsilon_3 = \frac{r_{21}-r_{12}}{4\varepsilon_4}$$

$\varepsilon_4 = 0$?

Lemma For all rotations one of the Euler Parameters is greater than or equal to 1/2

$$\left(\sum_{1}^{4} \varepsilon_i^2 = 1\right)$$

Algorithm Solve with respect to $\max_i\{\varepsilon_i\}$

- $\varepsilon_1 = \max_i\{\varepsilon_i\}$

$$\varepsilon_1 = \frac{1}{2}\sqrt{r_{11} - r_{22} - r_{33} + 1}$$

$$\varepsilon_2 = \frac{(r_{21} + r_{12})}{4\varepsilon_1}, \quad \varepsilon_3 = \frac{(r_{31} + r_{13})}{4\varepsilon_1}, \quad \varepsilon_4 = \frac{(r_{32} - r_{23})}{4\varepsilon_1}$$

- $\varepsilon_1 = \max_i \{\varepsilon_i\}$

$$\varepsilon_1 = \frac{1}{2}\sqrt{r_{11} - r_{22} - r_{33} + 1}$$

- $\varepsilon_2 = \max_i \{\varepsilon_i\}$

$$\varepsilon_2 = \frac{1}{2}\sqrt{-r_{11} + r_{22} - r_{33} + 1}$$

- $\varepsilon_3 = \max_i \{\varepsilon_i\}$

$$\varepsilon_3 = \frac{1}{2}\sqrt{-r_{11} - r_{22} + r_{33} + 1}$$

- $\varepsilon_4 = \max_i \{\varepsilon_i\}$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

Euler Parameters / Euler Angles

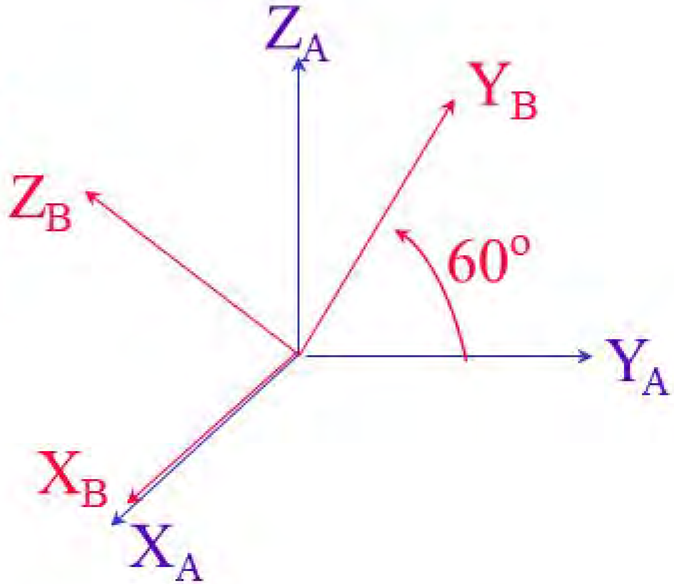
$$\varepsilon_1 = \sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2}$$

$$\varepsilon_2 = \sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}$$

$$\varepsilon_3 = \cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}$$

$$\varepsilon_4 = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}$$

Quiz



Direction Cosines

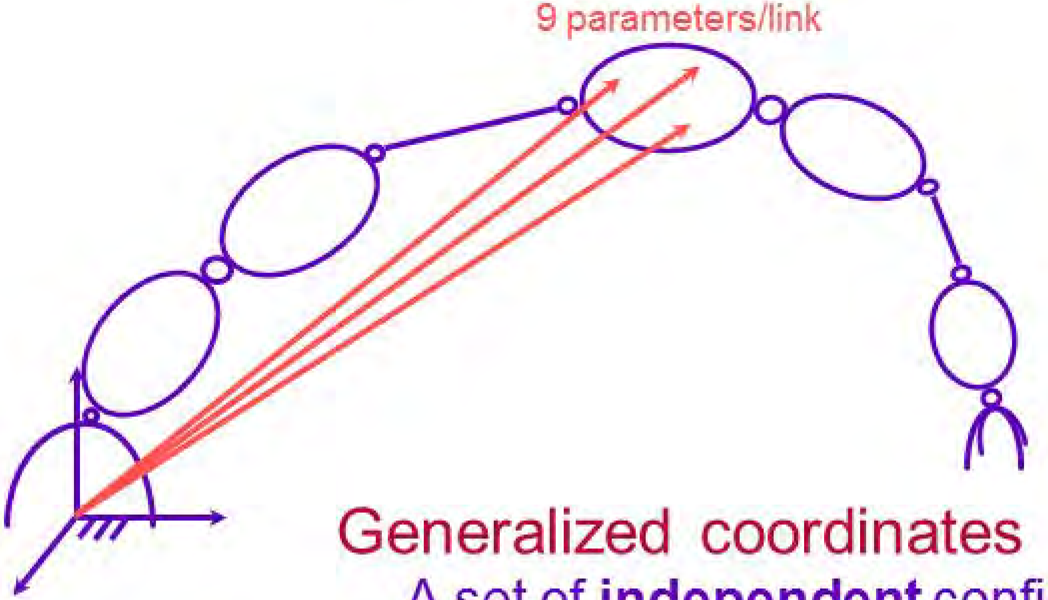
$$x_r = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ \sqrt{3}/2 \\ 0 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \begin{matrix} r_1 \\ \\ \\ r_2 \\ \\ \\ r_3 \end{matrix}$$

Euler Parameters

$$x_r = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ \sqrt{3}/2 \end{bmatrix}$$

Configuration Parameters

A set of position parameters that describes the full configuration of the system.



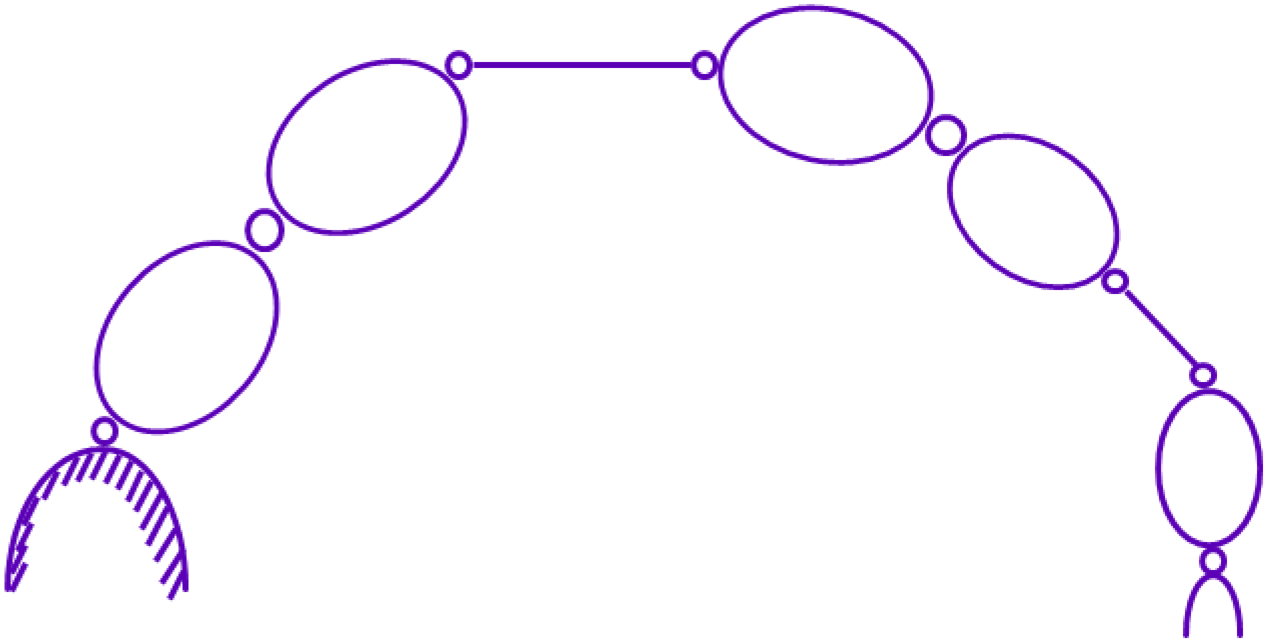
Generalized coordinates

A set of **independent** configuration parameters

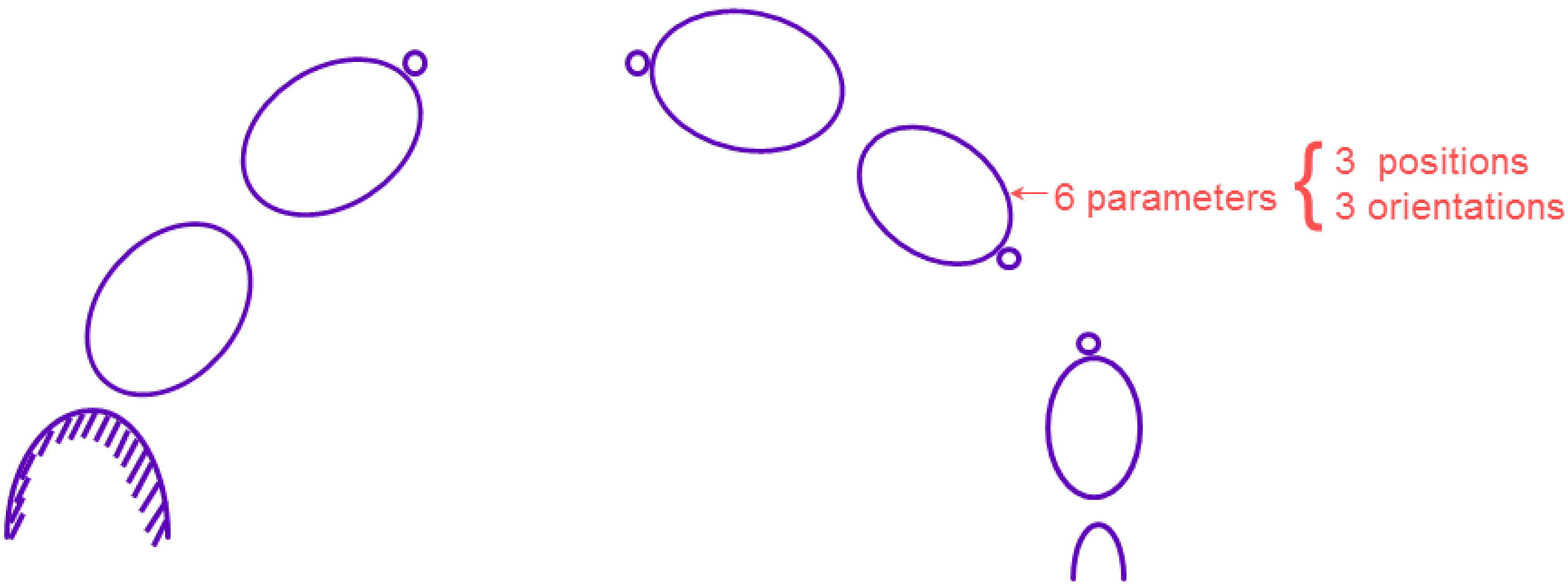
Degrees of Freedom

Number of generalized coordinates

Generalized Coordinates



Generalized Coordinates



n moving links: 6n parameters