Movie Segment

The Flying Machine Lab, ETH Zurich, 2011.

Interaction using a Kinect
@ the Flying Machine Arena

June 2011

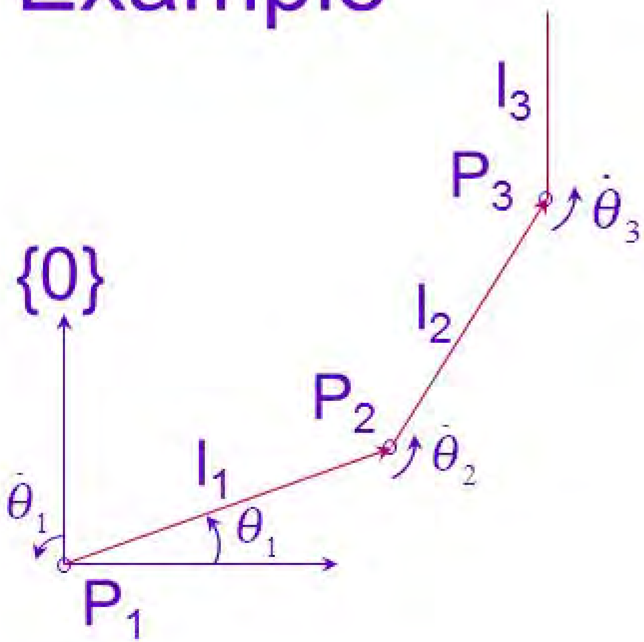
IDSC

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Jacobian

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

Example



$$v_{i+1} = v_i + \omega_i \times P_{i+1}$$

- $v_{P_1} = 0$ $\qquad {}^0\omega_1 = \dot{\theta}_1 . {}^0Z_1$
- $v_{P_2} = v_{P_1} + \omega_1 \times P_2$
- $v_{P_3} = v_{P_2} + \omega_2 \times P_3$

$$^0v_{P_2} = 0 + \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} l_1 . c_1 \\ l_1 . s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1 . s_1 \\ l_1 . c_1 \\ 0 \end{bmatrix} . \dot{\theta}_1$$

$$ {}^{0}v_{P_3} = {}^{0}v_{P_2} + {}^{0}\omega_2 \times {}^{0}P_3 $$

$$ {}^{0}v_{P_3} = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} .\dot{\theta}_1 + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .(\dot{\theta}_1 + \dot{\theta}_2).{}^{0}P_3 $$

$$ {}^{0}P_3 = \begin{bmatrix} l_2.c_{12} \\ l_2.s_{12} \\ 0 \end{bmatrix} $$

$$ = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} .\dot{\theta}_1 + \begin{bmatrix} -l_2.s_{12} \\ l_2.c_{12} \\ 0 \end{bmatrix} .(\dot{\theta}_1 + \dot{\theta}_2) $$

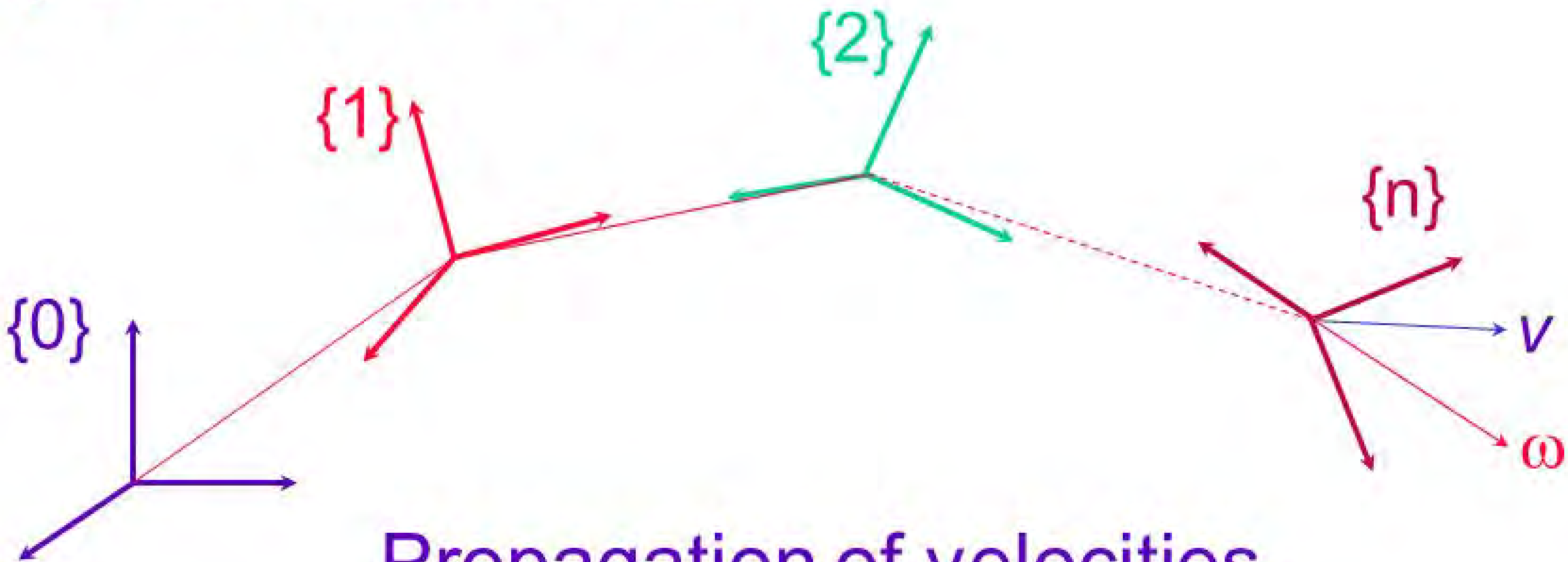
$$ {}^{0}\omega_3 = (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3).{}^{0}Z_0 $$

$$
{}^{0}v_{P_3} = \underbrace{\begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{J_v} . \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

$$
{}^{0}\omega_3 = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{J_\omega} . \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

$$
\begin{pmatrix} v \\ \omega \end{pmatrix} = J . \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix}
$$

Spatial Mechanisms

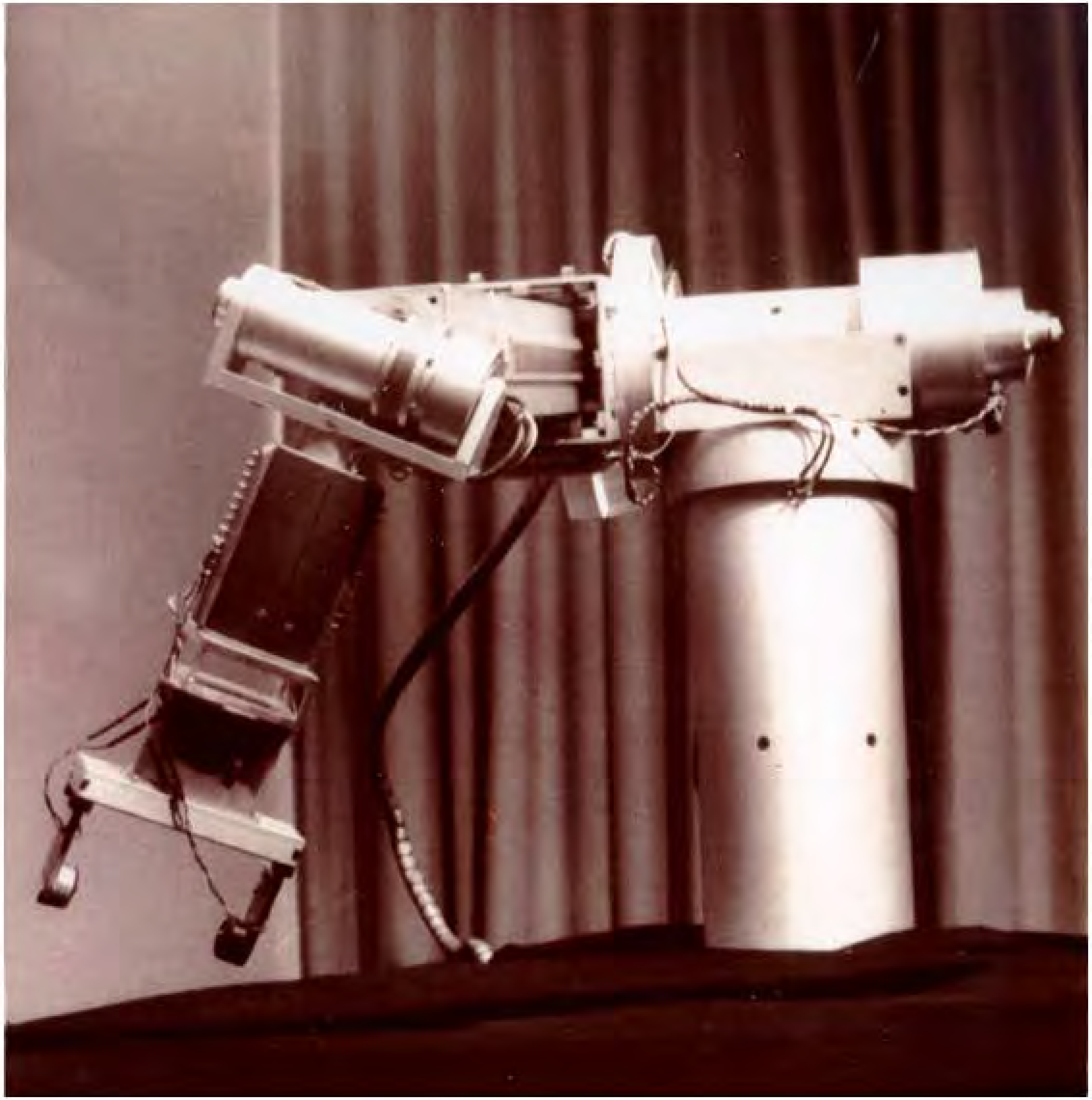


Propagation of velocities

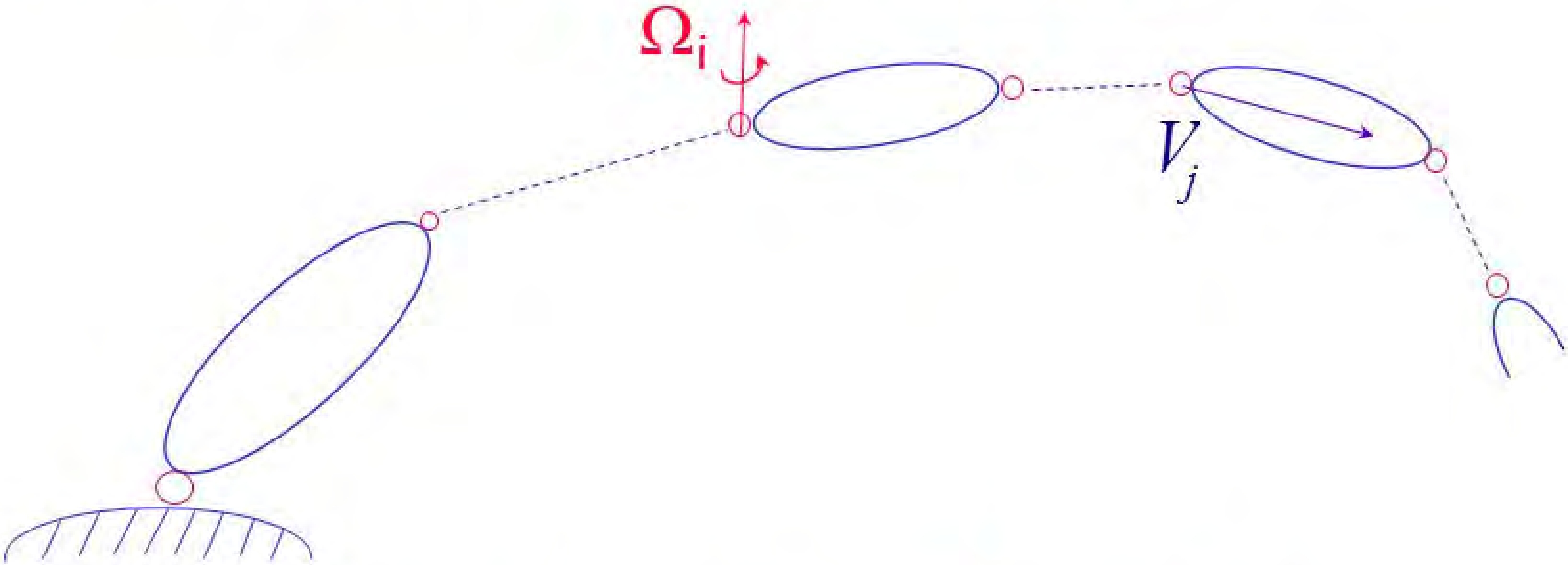
$\dot{x}$ → v : linear velocity
$\dot{x}$ → ω : angular velocity

$$\dot{x} = J(\theta).\dot{\theta}$$

Stanford Scheinman Arm



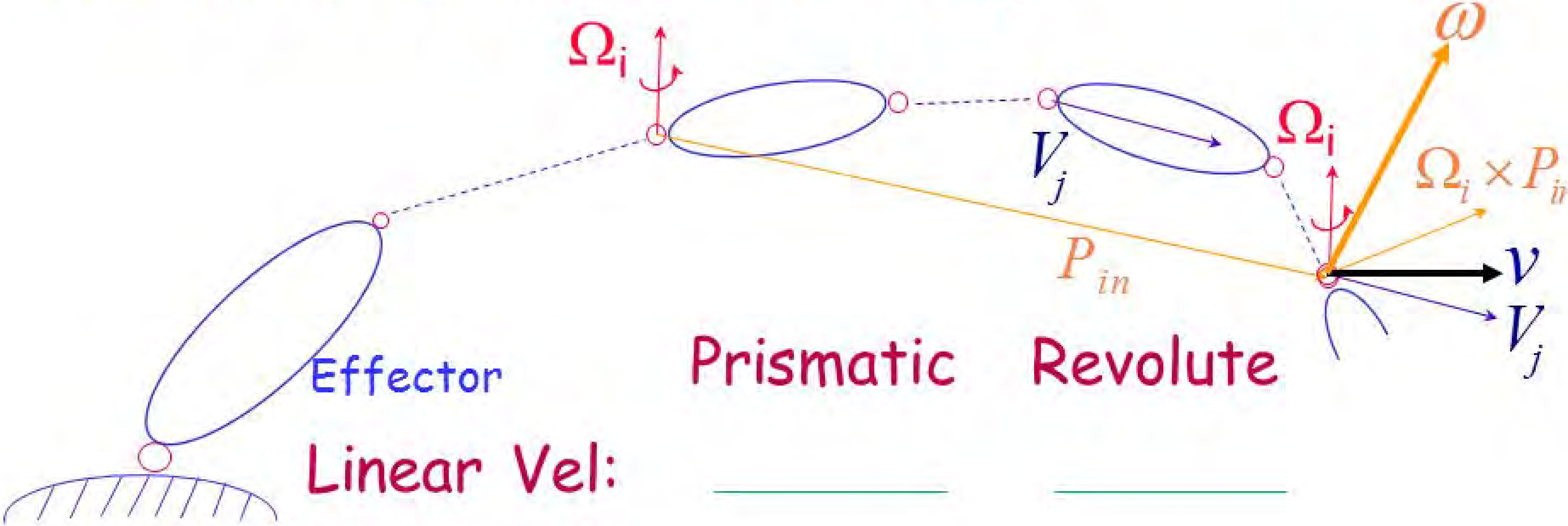
The Jacobian (EXPLICIT FORM)



Revolute Joint $\Omega_i = Z_i \dot{q}_i$

Prismatic Joint $V_i = Z_i \dot{q}_i$

The Jacobian (EXPLICIT FORM)



	Prismatic	Revolute
Linear Vel:	________	________
Angular Vel:	________	________

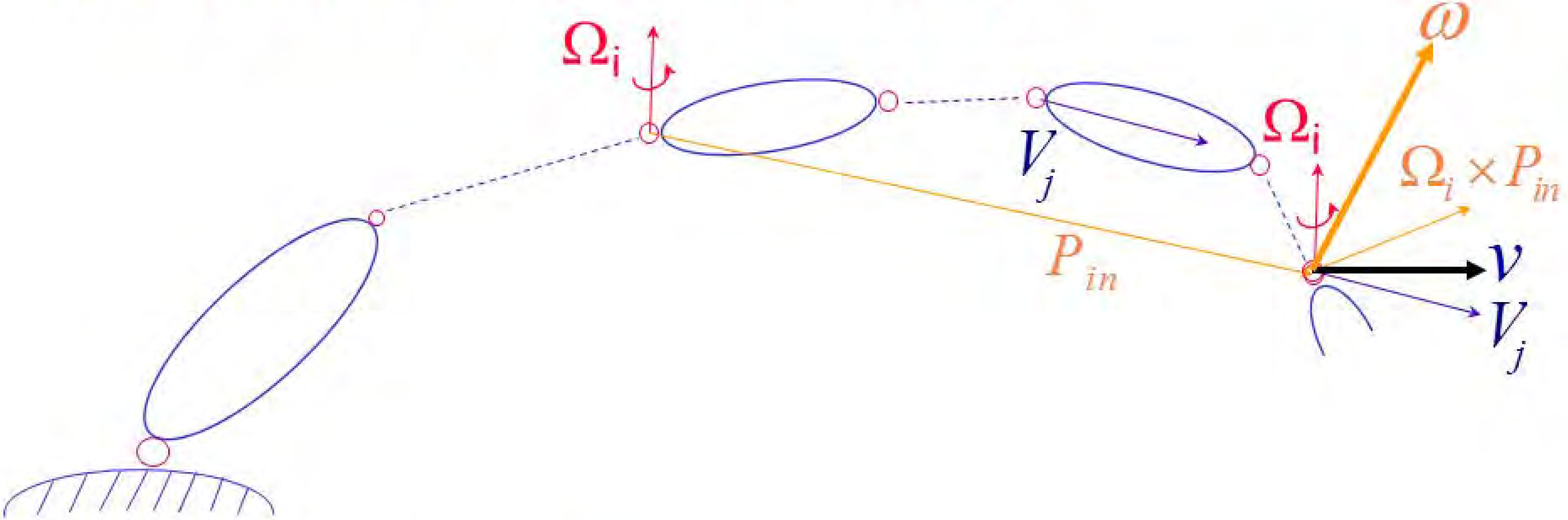
Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i V_i + \bar{\epsilon}_i (\Omega_i \times P_{in})] \quad \Leftarrow \quad V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i \Omega_i \quad \Leftarrow \quad \Omega_i = Z_i \dot{q}_i$$

The Jacobian (EXPLICIT FORM)



Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\in_i Z_i + \overline{\in}_i (Z_i \times P_{in})] \dot{q}_i \quad \Leftarrow \quad V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} (\overline{\in}_i Z_i) \dot{q}_i \quad \Leftarrow \quad \Omega_i = Z_i \dot{q}_i$$

$$v = [\in_1 Z_1 + \overline{\in}_1 (Z_1 \times P_{1n})]\dot{q}_1 + \cdots$$

$$+ [\in_{n-1} Z_{n-1} + \overline{\in}_{n-1} (Z_{n-1} \times P_{(n-1)n})]\dot{q}_{n-1} + \in_n Z_n \dot{q}_n$$

$$v = \begin{bmatrix} \in_1 Z_1 + \overline{\in}_1 (Z_1 \times P_{1n}) & \in_2 Z_2 + \overline{\in}_2 (Z_2 \times P_{2n}) & \cdots \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

$$\omega = \overline{\in}_1 Z_1 \dot{q}_1 + \overline{\in}_2 Z_2 \dot{q}_2 + \cdots + \overline{\in}_n Z_n \dot{q}_n$$

$$\omega = \begin{bmatrix} \overline{\in}_1 Z_1 & \overline{\in}_2 Z_2 & \cdots & \overline{\in}_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{\omega = J_\omega \dot{q}}$$

The Jacobian

$$J = \left(\frac{J_v}{J_w} \right)$$

Matrix J_v (direct differentiation)

$$v = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{x}_P = \frac{\partial x_P}{\partial q_1} . \dot{q}_1 + \frac{\partial x_P}{\partial q_2} . \dot{q}_2 + \cdots + \frac{\partial x_P}{\partial q_n} . \dot{q}_n$$

$$J_v = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \end{pmatrix}$$

Jacobian in a Frame

Vector Representation

$$J = \begin{pmatrix} \dfrac{\partial x_P}{\partial q_1} & \dfrac{\partial x_P}{\partial q_2} & \cdots & \dfrac{\partial x_P}{\partial q_n} \\ \overline{\epsilon}_1 . Z_1 & \overline{\epsilon}_2 . Z_2 & \cdots & \overline{\epsilon}_n . Z_n \end{pmatrix}$$

In {0}

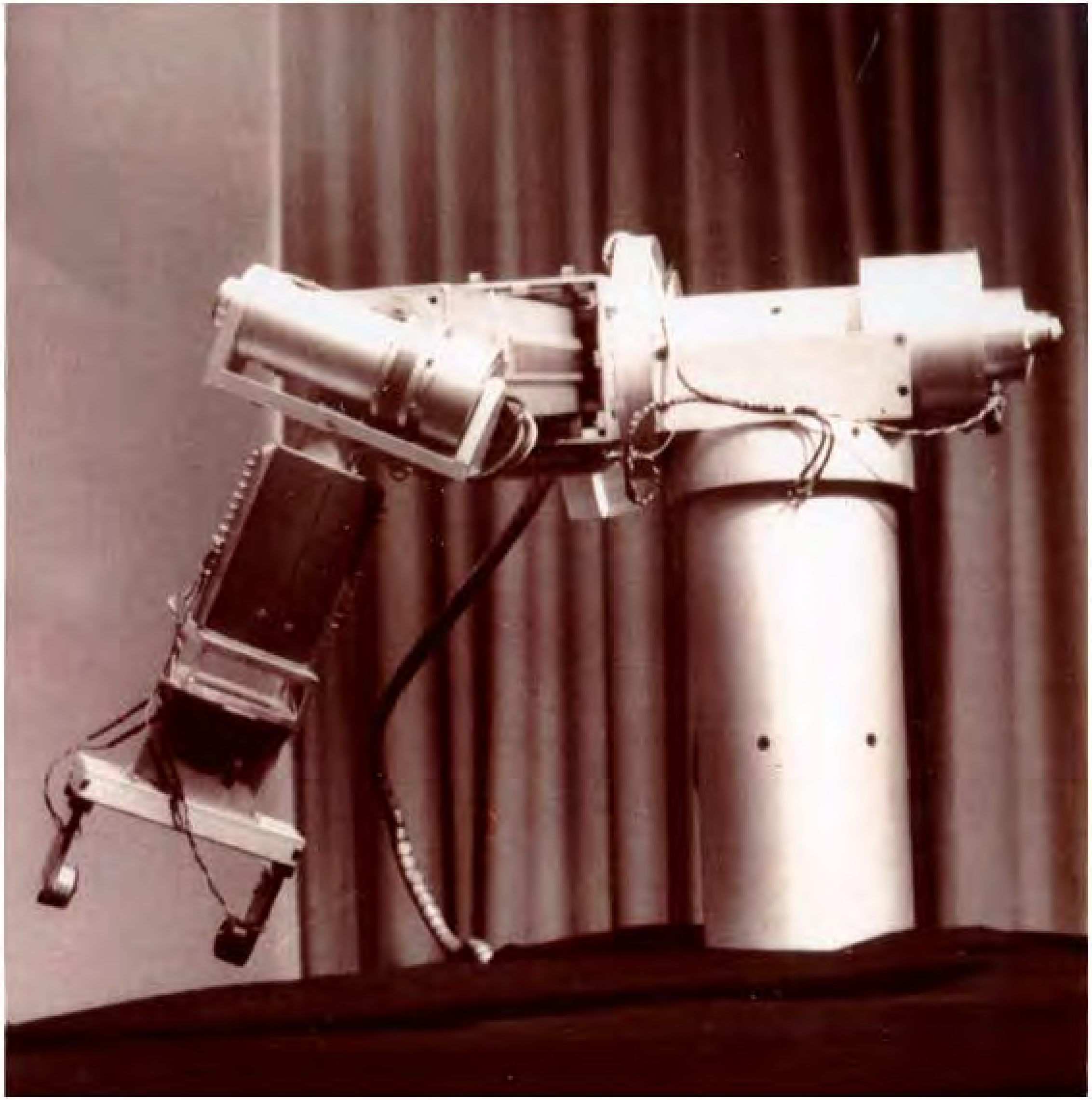
$$^0J = \begin{pmatrix} \dfrac{\partial\, ^0x_P}{\partial q_1} & \dfrac{\partial\, ^0x_P}{\partial q_2} & \cdots & \dfrac{\partial\, ^0x_P}{\partial q_n} \\ \overline{\epsilon}_1 . {}^0Z_1 & \overline{\epsilon}_2 . {}^0Z_2 & \cdots & \overline{\epsilon}_n . {}^0Z_n \end{pmatrix}$$

J in Frame {0}

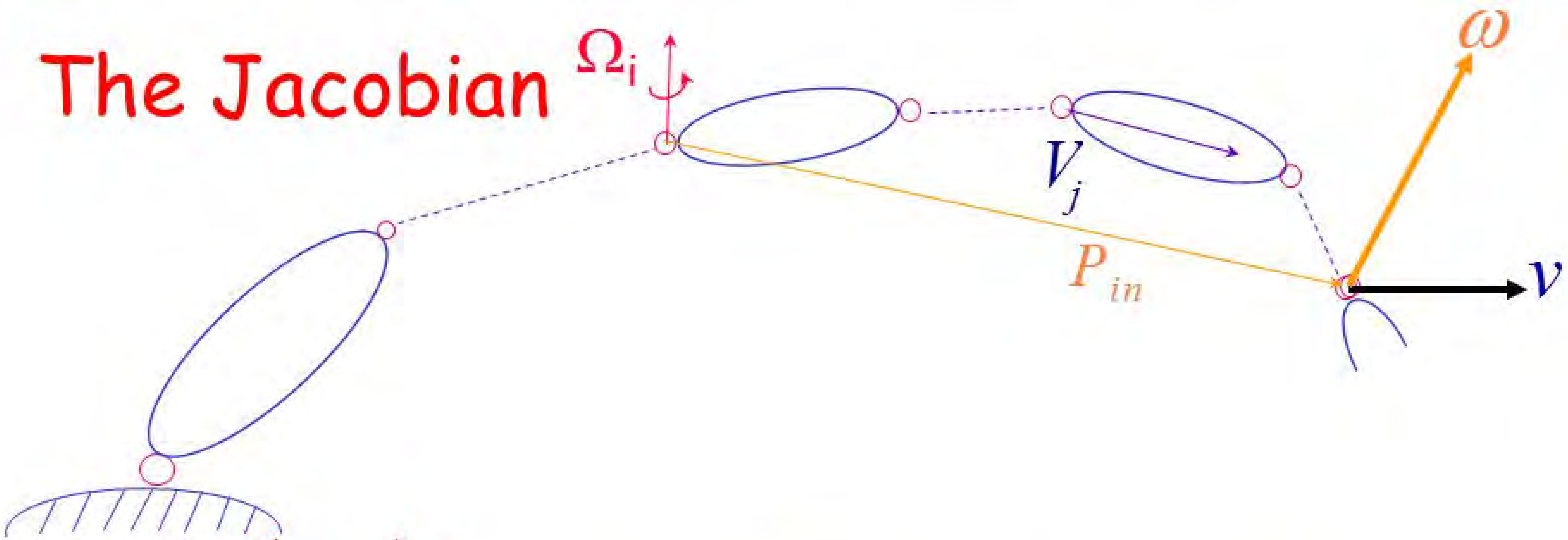
$$ {}^{0}Z_i = {}^{0}_{i}R \; {}^{i}Z_i; \quad {}^{i}Z_i = Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ {}^{0}J = \begin{pmatrix} \frac{\partial}{\partial q_1}({}^{0}x_P) & \frac{\partial}{\partial q_2}({}^{0}x_P) & \cdots & \frac{\partial}{\partial q_n}({}^{0}x_P) \\ \overline{\epsilon}_1 .({}^{0}_{1}R.Z) & \overline{\epsilon}_2 .({}^{0}_{2}R.Z) & \cdots & \overline{\epsilon}_n .({}^{0}_{n}R.Z) \end{pmatrix} $$

Stanford Scheinman Arm



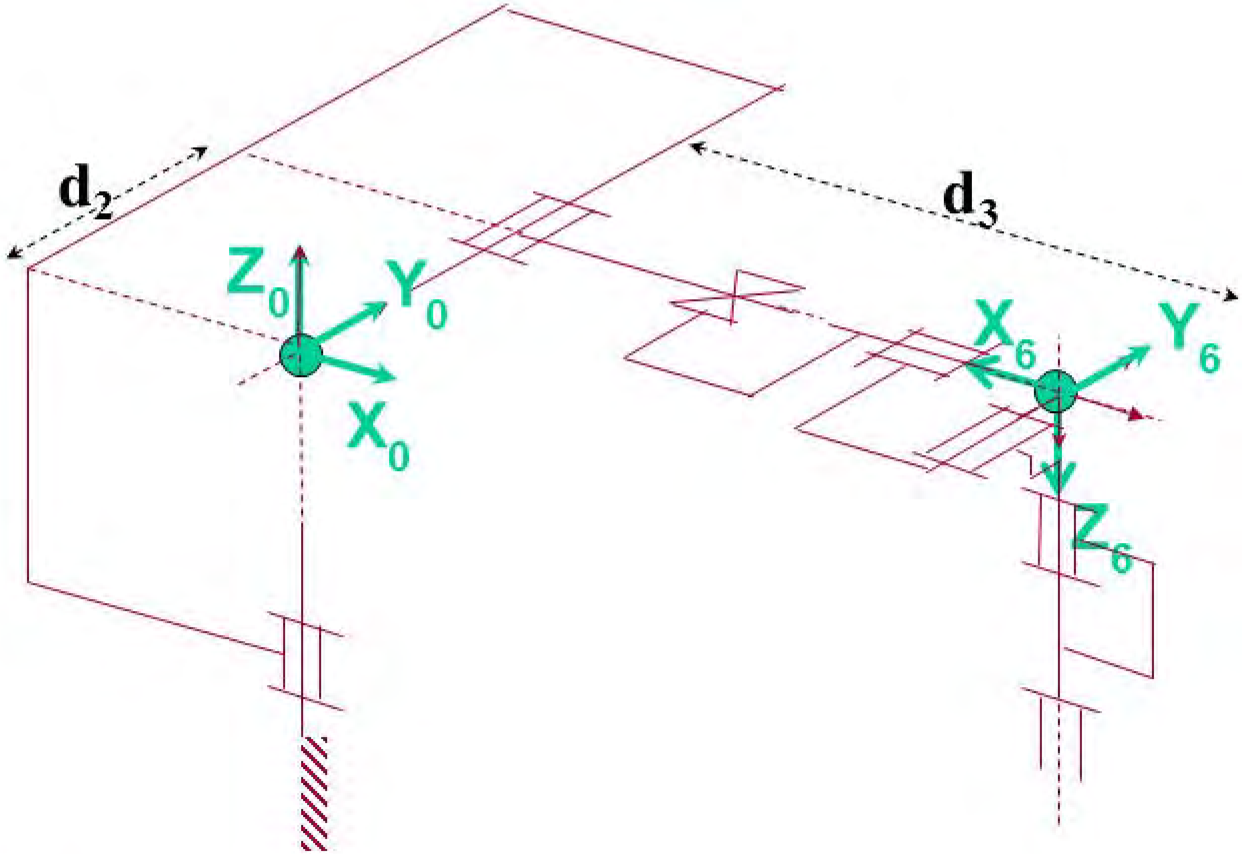
The Jacobian



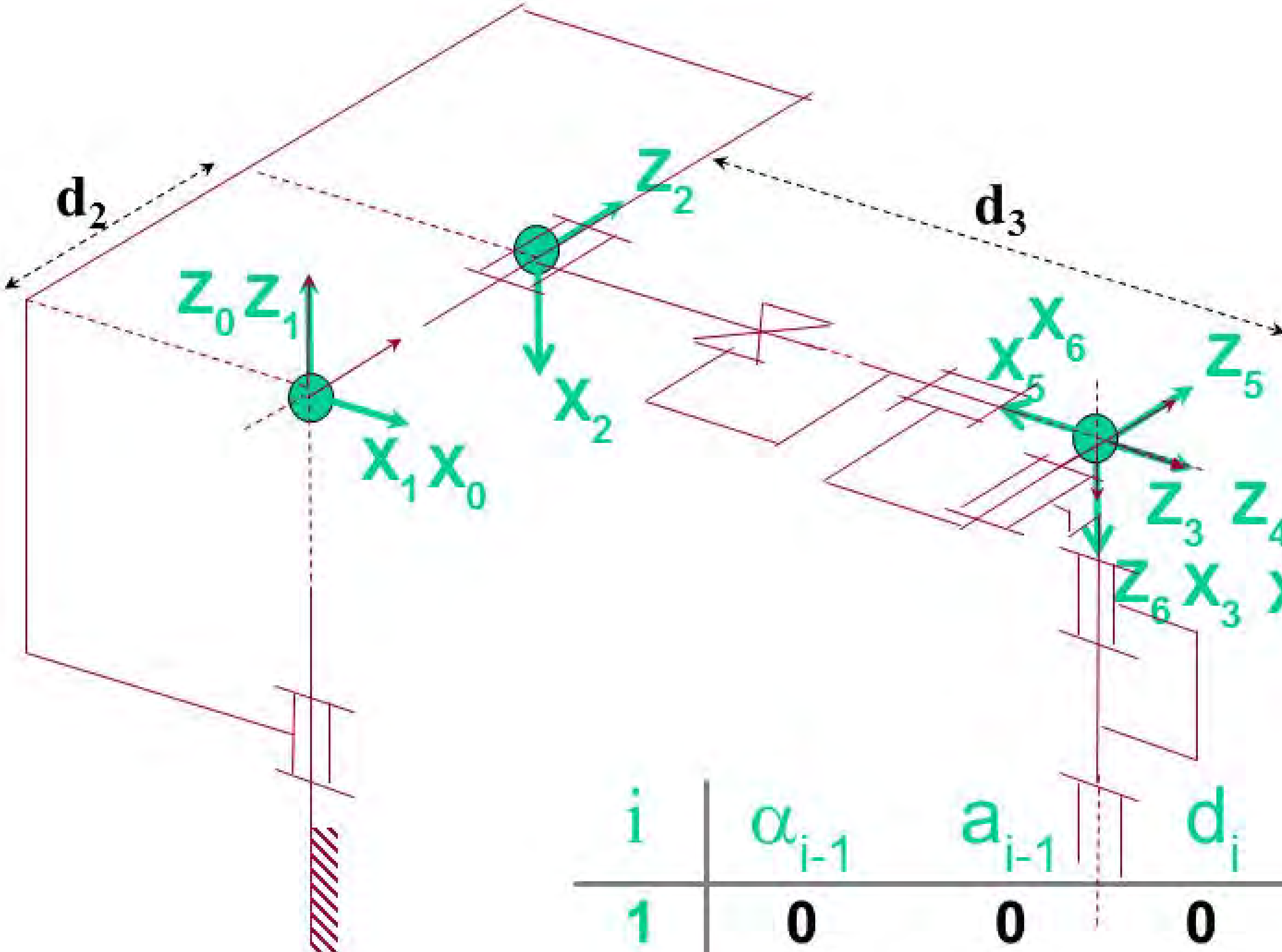
$$J = \begin{pmatrix} J_v \\ \hline J_w \end{pmatrix} \qquad v = J_v \dot{q} \qquad \omega = J_\omega \dot{q}$$

$$J_v = \left[\in_1 Z_1 + \overline{\in}_1 (Z_1 \times P_{1n}) \quad \in_2 Z_2 + \overline{\in}_2 (Z_2 \times P_{2n}) \quad \cdots \right]$$

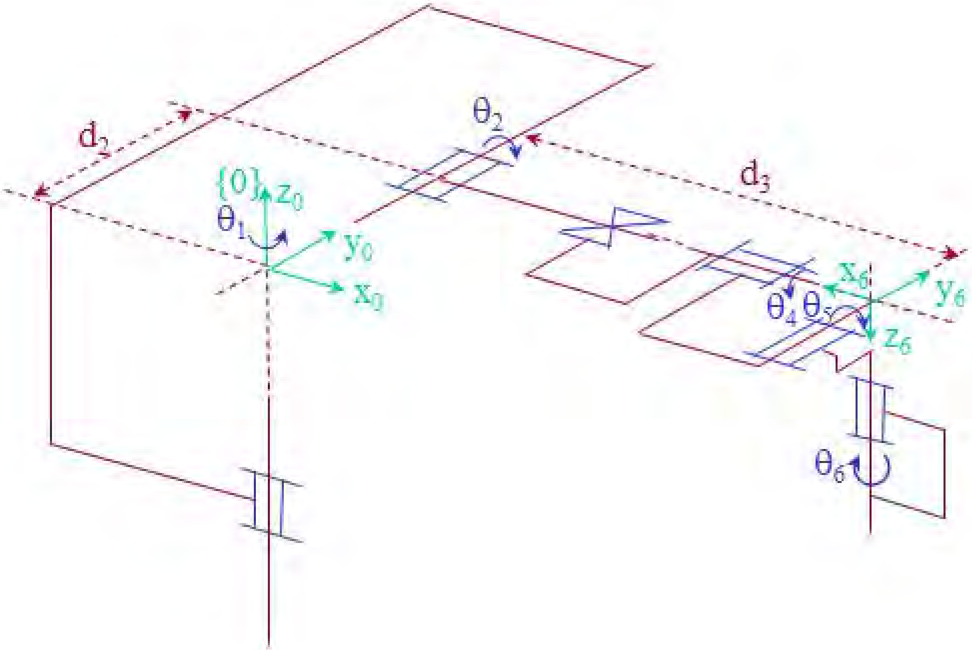
$$J_\omega = \left[\overline{\in}_1 Z_1 \quad \overline{\in}_2 Z_2 \quad \cdots \quad \overline{\in}_n Z_n \right]$$



$$J = \left(\begin{array}{c|c|c|c|c|c} \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \hline \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{array}\right)$$



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

$$ {}^{i-1}_{\ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Forward Kinematics: $ {}^{0}_{N}T = {}^{0}_{1}T \quad {}^{1}_{2}T \quad \ldots \quad {}^{N-1}_{\ \ N}T $

Stanford Scheinman Arm

$$ {}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ -s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$
{}^{3}_{4}T = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{4}_{5}T = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{5}_{6}T = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{2}T = \begin{bmatrix} c_1 c_2 & -c_1 s_2 & -s_1 & -s_1 d_2 \\ s_1 c_2 & -s_1 s_2 & c_1 & c_1 d_2 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

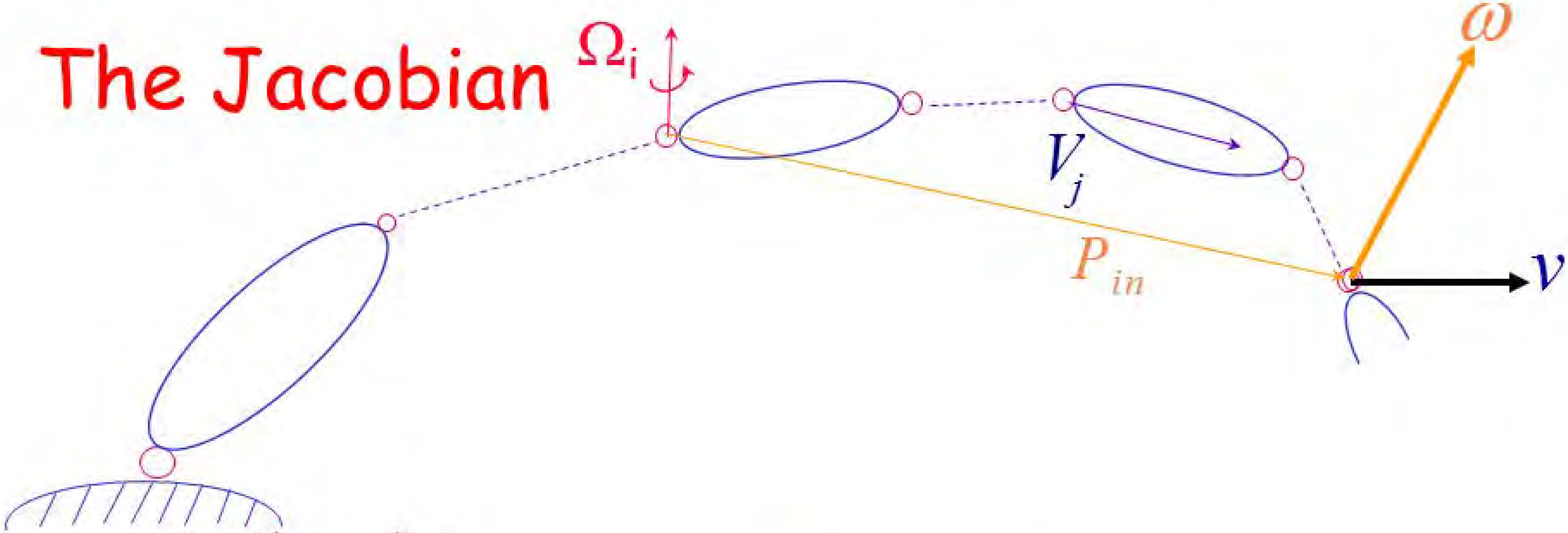
$$
{}^{0}_{3}T = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 d_3 s_2 - s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 d_3 s_2 + c_1 d_2 \\ -s_2 & 0 & c_2 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{4}T = \begin{bmatrix} c_1c_2c_4 - s_1s_4 & -c_1c_2s_4 - s_1c_4 & c_1s_2 & c_1d_3s_2 - s_1d_2 \\ s_1c_2c_4 + c_1s_4 & -s_1c_2s_4 + c_1c_4 & s_1s_2 & s_1d_3s_2 + c_1d_2 \\ -s_2c_4 & s_2s_4 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{5}T = \begin{bmatrix} X & X & -c_1c_2s_4 - s_1c_4 & c_1d_3s_2 - s_1d_2 \\ X & X & -s_1c_2s_4 + c_1c_4 & s_1d_3s_2 + c_1d_2 \\ X & X & s_2s_4 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{6}T = \begin{bmatrix} X & X & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2s_5 & c_1d_3s_2 - s_1d_2 \\ X & X & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 & s_1d_3s_2 + c_1d_2 \\ X & X & -s_2c_4s_5 + c_5c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The Jacobian

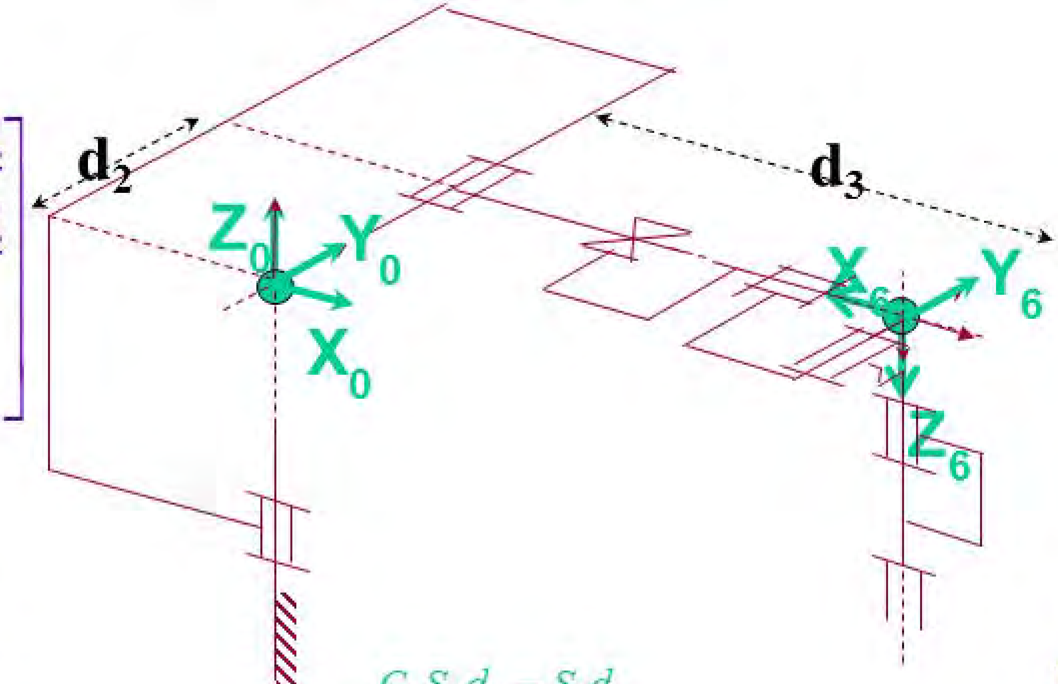


$$J = \begin{pmatrix} J_v \\ \hline J_w \end{pmatrix} \qquad v = J_v \dot{q} \qquad \omega = J_\omega \dot{q}$$

$$J_v = \left[\in_1 Z_1 + \overline{\in}_1 (Z_1 \times P_{1n}) \quad \in_2 Z_2 + \overline{\in}_2 (Z_2 \times P_{2n}) \quad \cdots \right]$$

$$J_\omega = \left[\overline{\in}_1 Z_1 \quad \overline{\in}_2 Z_2 \quad \cdots \quad \overline{\in}_n Z_n \right]$$

$$
{}^{0}_{6}T=\begin{bmatrix} X & X & c_1c_2c_4s_5-s_1s_4s_5+c_1s_2s_5 & c_1d_3s_2-s_1d_2 \\ X & X & s_1c_2c_4s_5+c_1s_4s_5+s_1s_2c_5 & s_1d_3s_2+c_1d_2 \\ X & X & -s_2c_4s_5+c_5c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



$$
x=\begin{pmatrix} x_P \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}=\left|\begin{array}{c} C_1S_2d_3-S_1d_2 \\ S_1S_2d_3+C_1d_2 \\ C_2d_3 \\ C_1[C_2(C_4C_5C_6-S_4S_6)-S_2S_5C_6]-S_1(S_4C_5C_6+C_4S_6) \\ S_1[C_2(C_4C_5C_6-S_4S_6)-S_2S_5C_6]+C_1(S_4C_5C_6+C_4S_6) \\ -S_2(C_4C_5C_6-S_4S_6)-C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6+S_4C_6)+S_2S_5S_6]-S_1(-S_4C_5S_6+C_4C_6) \\ S_1[-C_2(C_4C_5S_6+S_4C_6)+S_2S_5S_6]+C_1(-S_4C_5S_6+C_4C_6) \\ S_2(C_4C_5S_6+S_4C_6)+C_2S_5S_6 \\ C_1(C_2C_4S_5+S_2C_5)-S_1S_4S_5 \\ S_1(C_2C_4S_5+S_2C_5)+C_1S_4S_5 \\ -S_2C_4S_5+C_2C_5 \end{array}\right|
$$

Stanford Scheinman Arm Jacobian

$$ {}^0J = \begin{pmatrix} \dfrac{\partial {}^0x_P}{\partial q_1} & \dfrac{\partial {}^0x_P}{\partial q_2} & \dfrac{\partial {}^0x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^0Z_1 & {}^0Z_2 & 0 & {}^0Z_4 & {}^0Z_5 & {}^0Z_6 \end{pmatrix} $$

$$ x = \begin{pmatrix} x_P \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \left| \begin{array}{c} C_1S_2d_3 - S_1d_2 \\ S_1S_2d_3 + C_1d_2 \\ C_2d_3 \\ C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \\ S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6) \\ S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6 \\ C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4S_5 + C_2C_5 \end{array} \right| $$

Stanford Scheinman Arm Jacobian

$$
{}^{0}J = \begin{pmatrix} \dfrac{\partial\, {}^{0}x_P}{\partial q_1} & \dfrac{\partial\, {}^{0}x_P}{\partial q_2} & \dfrac{\partial\, {}^{0}x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^{0}Z_1 & {}^{0}Z_2 & 0 & {}^{0}Z_4 & {}^{0}Z_5 & {}^{0}Z_6 \end{pmatrix}
$$

$$
\begin{bmatrix}
-c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\
-s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\
0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\
0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\
0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\
1 & 0 & 0 & c_2 & s_2 s_4 & -s_2 c_4 s_5 + c_5 c_2
\end{bmatrix}
$$

Kinematic Singularity

The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$J = \left(J_1 \; J_2 \; \cdots \; J_n \right)$$

$$\det\left(J\right) = 0$$

$$\det\left({}^{i}J\right) = \det\left({}^{j}J\right)$$

Kinematic Singularity

$$ {}^{B}J = \begin{pmatrix} {}^{B}_{A}R & 0 \\ 0 & {}^{B}_{A}R \end{pmatrix} {}^{A}J $$

$$ \det[{}^{B}J] \equiv \det[{}^{A}J] $$

$$ \boxed{\det\left({}^{i}J\right) = \det\left({}^{j}J\right)} $$

Singular Configurations

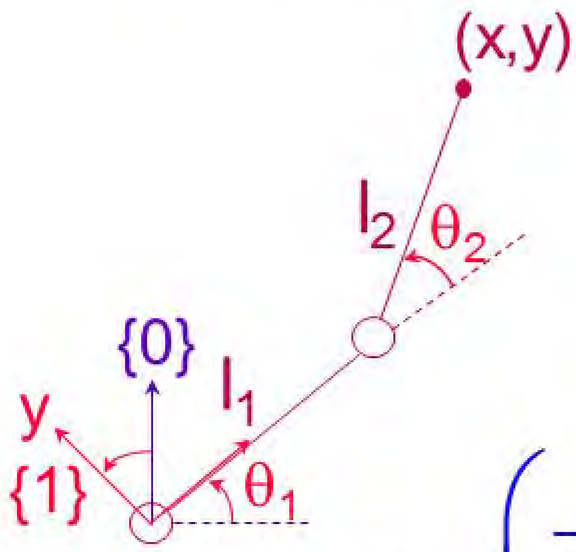
$$\det[J(q)] = 0$$

$\Longrightarrow$ Singular Configurations

$$\det[J(q)] = S_1(q)S_2(q)\ldots S_s(q) = 0$$

$$\Rightarrow \quad \begin{array}{|l|} S_1(q) = 0 \\ S_2(q) = 0 \\ \vdots \\ S_s(q) = 0 \end{array}$$

Example (Kinematic Singularities)



$$x = l_1 C1 + l_2 C12$$

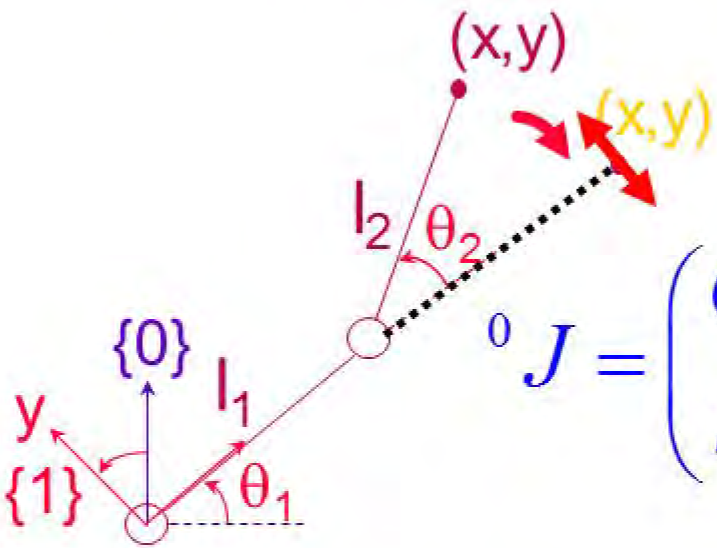
$$y = l_1 S1 + l_2 S12$$

$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \end{pmatrix}$$

$$\det(J) = l_1 l_2 S2$$

Singularity at $q_2 = k\pi$

Example (Kinematic Singularities)



$$^{1}J = {}^{1}_{0}R\ {}^{0}J$$

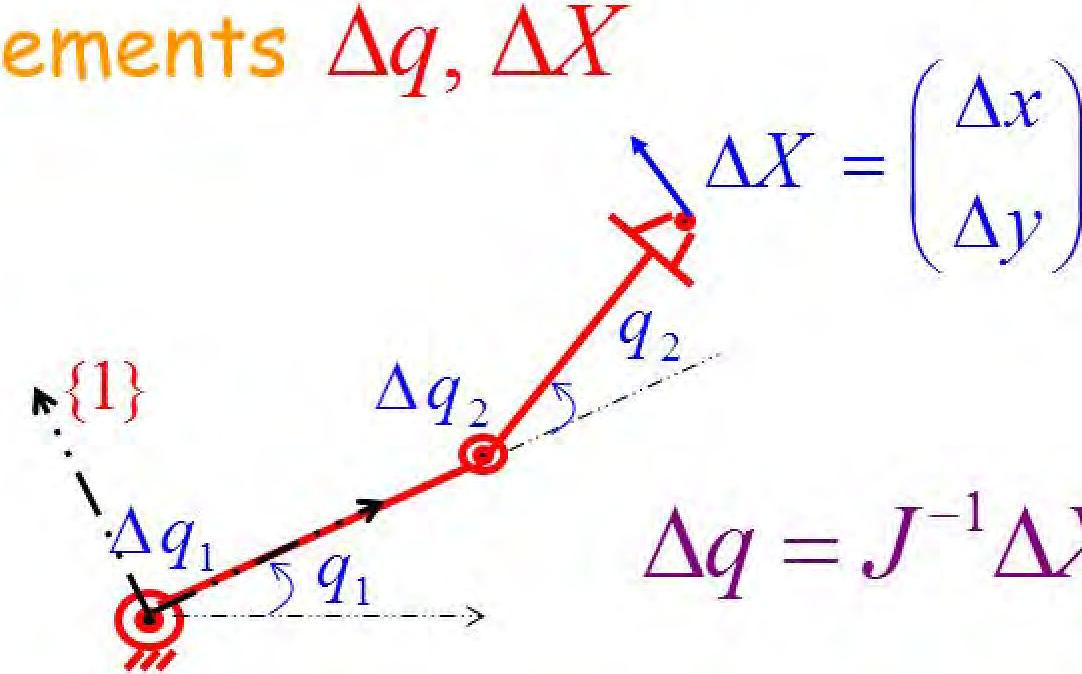
$$^{0}J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \begin{pmatrix} -l_2 S2 & -l_2 S2 \\ l_1 + l_2 C2 & l_2 C2 \end{pmatrix}$$

At Singularity

$$^{1}J = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$$

$$\begin{cases} {}^{1}\delta x = 0 \\ {}^{1}\delta y = (l_1 + l_2)\delta\theta_1 + l_2 \delta\theta_2 \end{cases}$$

Small Displacements $\Delta q, \Delta X$

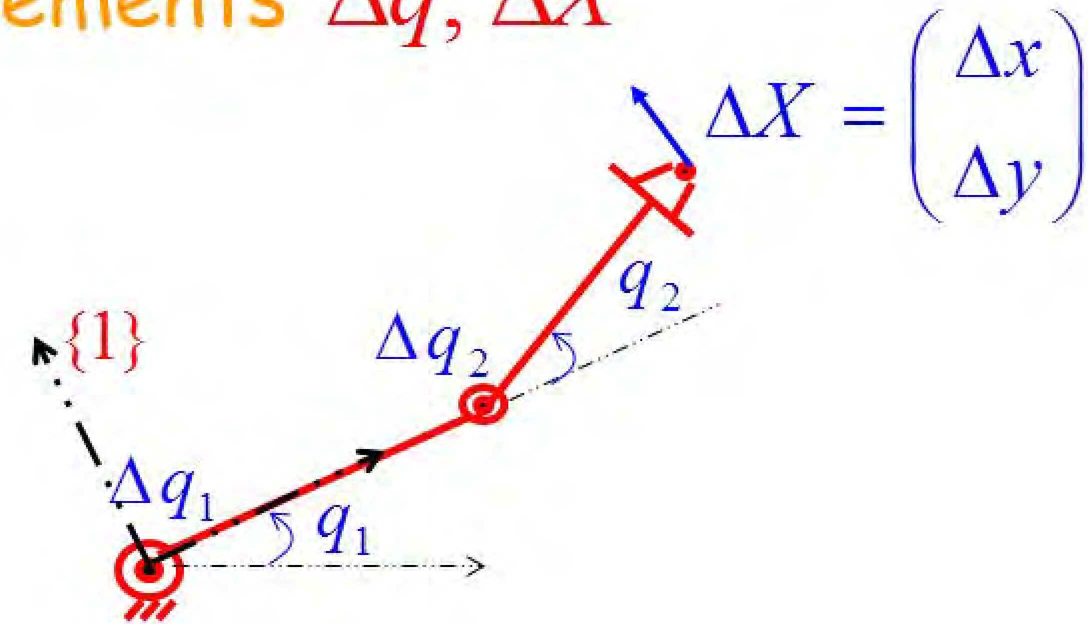


$$\Delta q = J^{-1} \Delta X$$

small θ_2

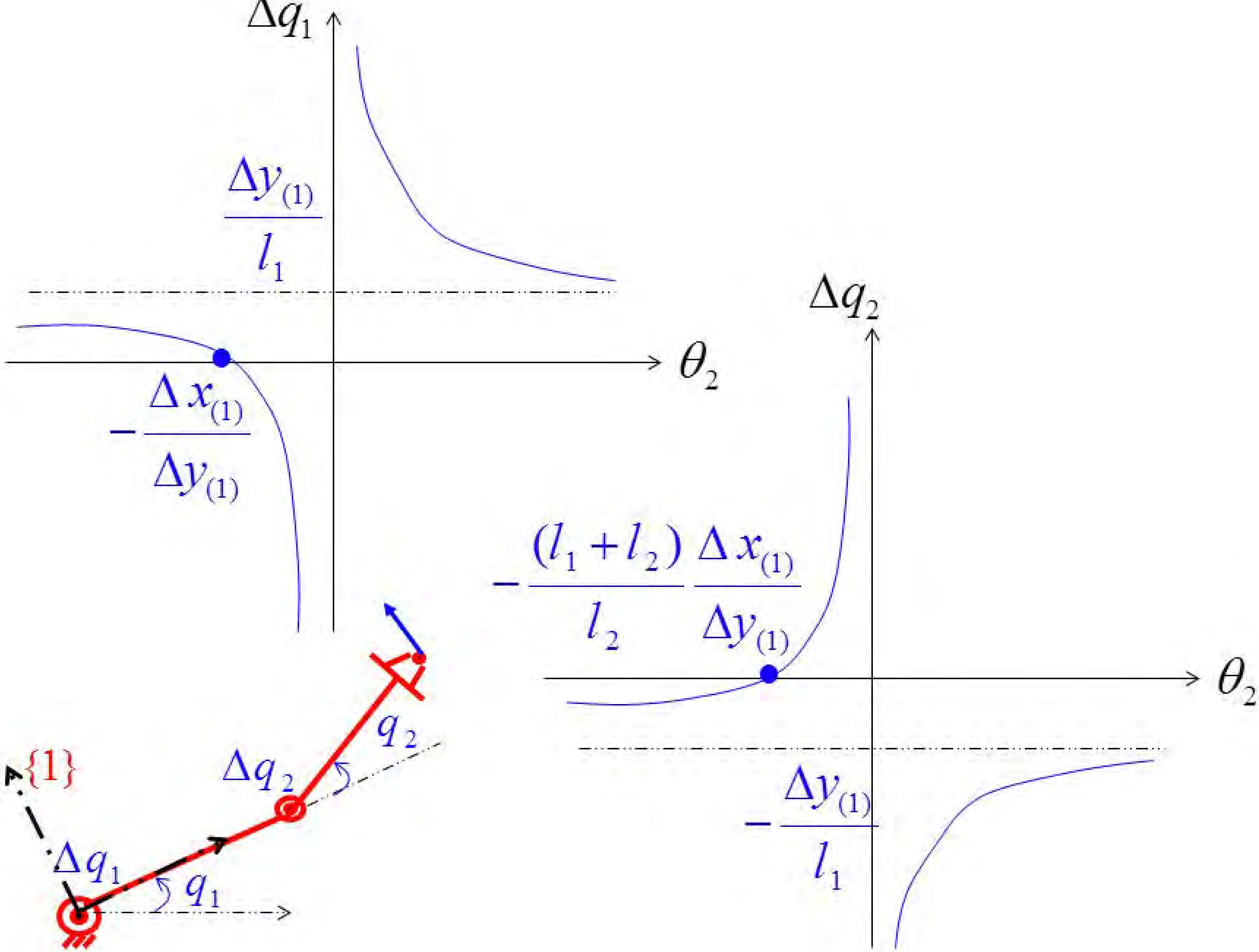
$$J_{(1)}^{-1} \cong \begin{pmatrix} \frac{1}{l_1\theta_2} & \frac{1}{l_1} \\ -\frac{l_1 + l_2}{l_1 l_2 \theta_2} & -\frac{1}{l_1} \end{pmatrix}$$

Small Displacements $\Delta q, \Delta X$



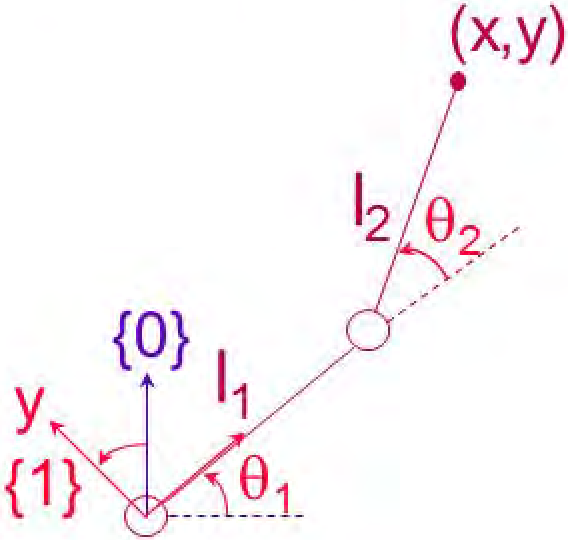
$$\Delta q_1 = \frac{\Delta x_{(1)}}{l_1} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$

$$\Delta q_2 = \frac{\left(l_1 + l_2\right) \Delta x_{(1)}}{l_1 l_2} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$



Δq_1
$\frac{\Delta y_{(1)}}{l_1}$
θ_2
$-\frac{\Delta x_{(1)}}{\Delta y_{(1)}}$
Δq_2
$-\frac{(l_1+l_2)}{l_2}\frac{\Delta x_{(1)}}{\Delta y_{(1)}}$
θ_2
$-\frac{\Delta y_{(1)}}{l_1}$
$\{1\}$
Δq_2
q_2
Δq_1
q_1

Kinematic Singularities (reduced matrix)



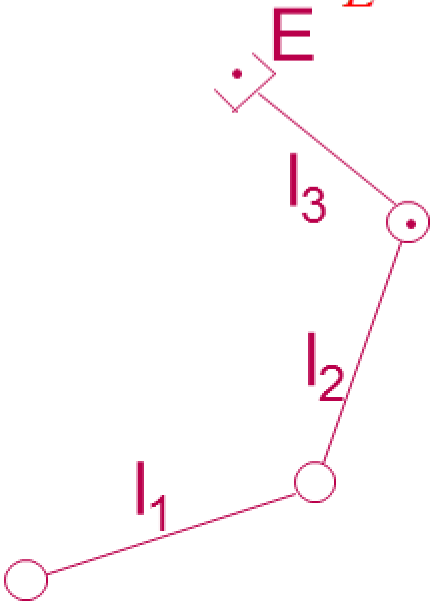
$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\det(J) = l_1 l_2 S2$$

$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \end{pmatrix}$$

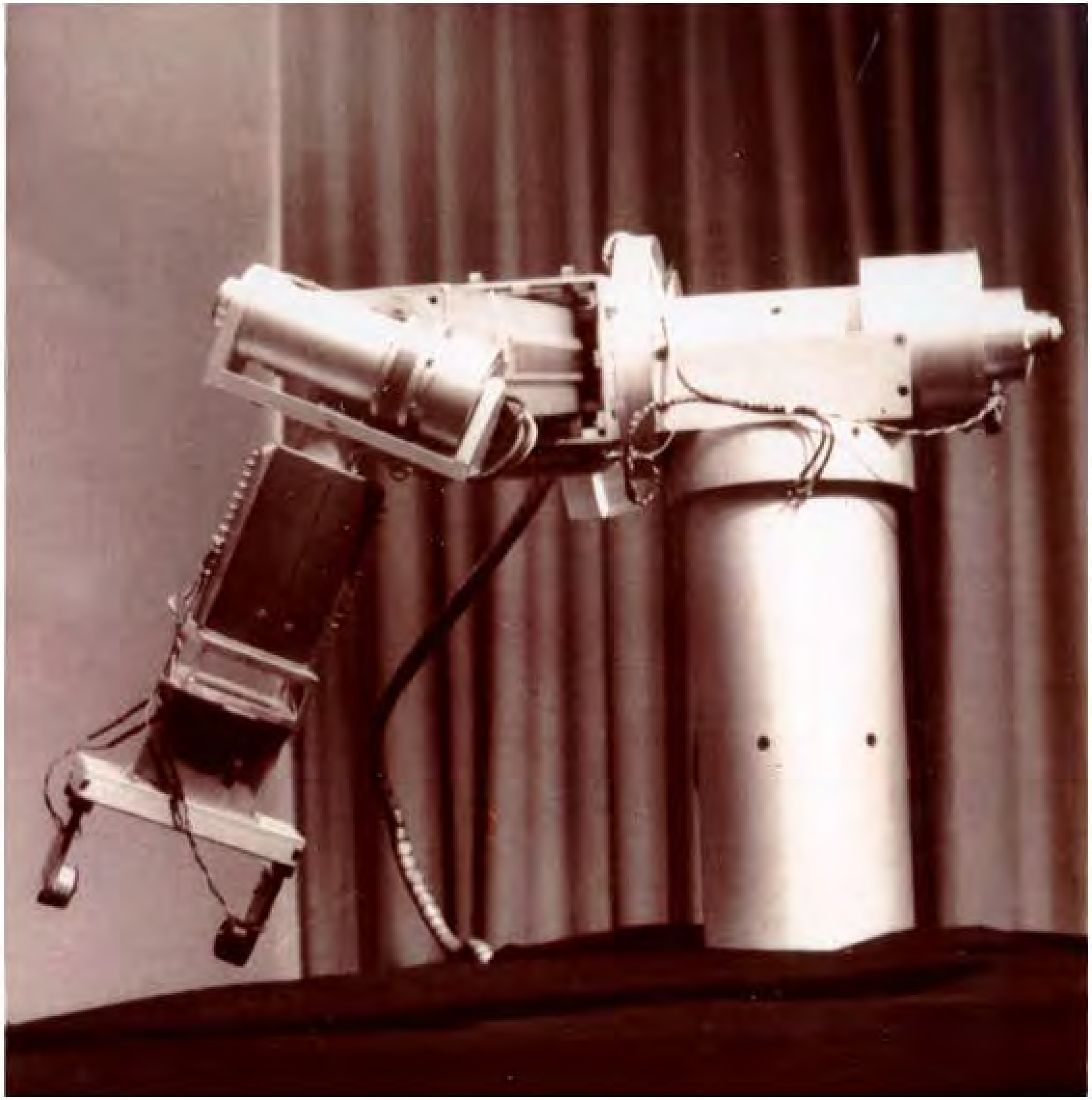
Singularity at $q_2 = k\pi$

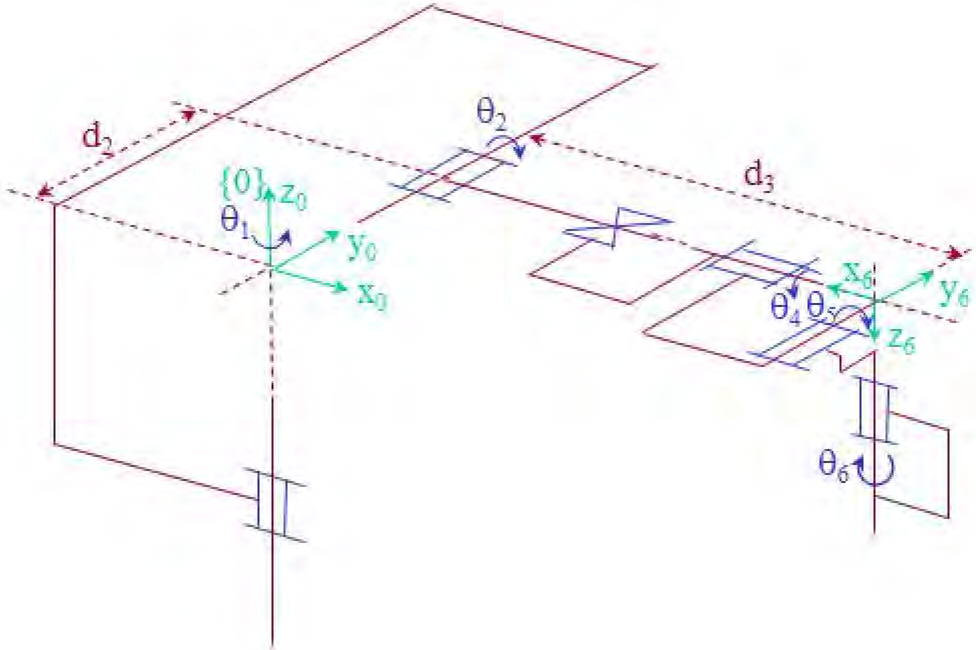
$$
{}^{0}J_E = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 1 & 1 & 1 \end{pmatrix}
$$



$$
{}^{0}J_E = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

Stanford Scheinman Arm





i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

$$ {}^{i-1}_{1}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Forward Kinematics: $ {}^{0}_{N}T = {}^{0}_{1}T \;\; {}^{1}_{2}T \;\; \ldots \;\; {}^{N-1}_{N}T $

Stanford Scheinman Arm Jacobian

$$
{}^{0}J = \begin{pmatrix} \dfrac{\partial {}^{0}x_P}{\partial q_1} & \dfrac{\partial {}^{0}x_P}{\partial q_2} & \dfrac{\partial {}^{0}x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^{0}Z_1 & {}^{0}Z_2 & 0 & {}^{0}Z_4 & {}^{0}Z_5 & {}^{0}Z_6 \end{pmatrix}
$$

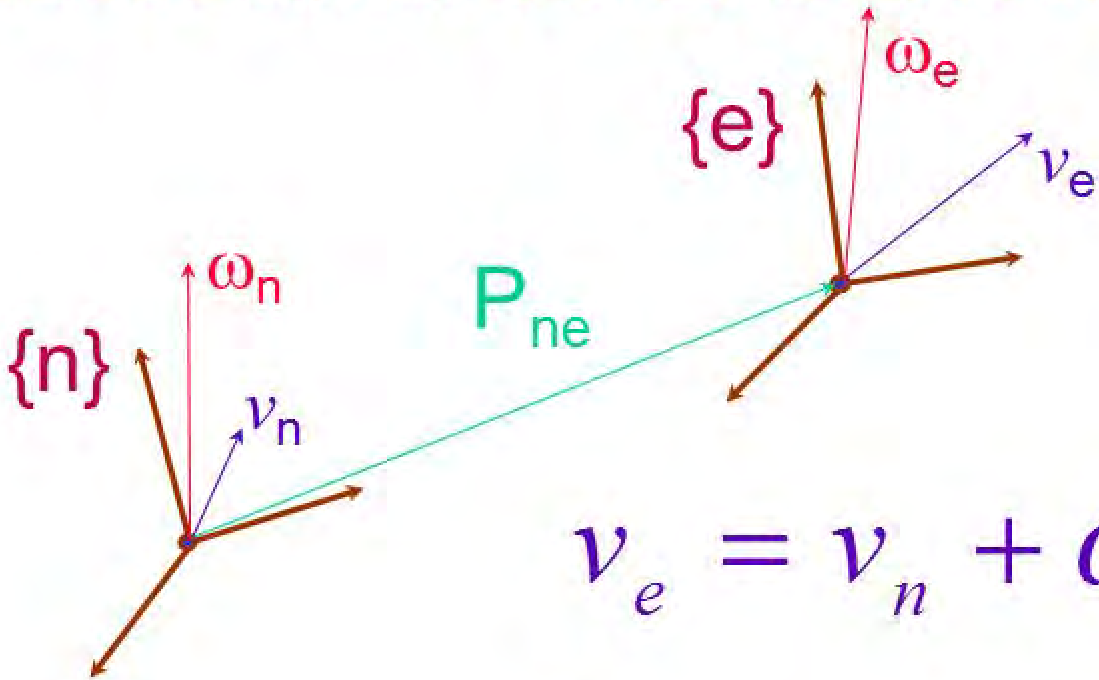
$$
\begin{bmatrix}
-c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\
-s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\
0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\
0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\
0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\
1 & 0 & 0 & c_2 & s_2 s_4 & -s_2 c_4 s_5 + c_5 c_2
\end{bmatrix}
$$

Stanford Scheinman Arm Jacobian

$$\theta_5 = k\pi$$

$$J = \begin{bmatrix} -c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ -s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 s_2 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 s_2 \\ 1 & 0 & 0 & c_2 & s_2 s_4 & c_2 \end{bmatrix}$$

Jacobian at the End-Effector



$$v_e = v_n + \omega_n \times P_{ne}$$

$$\begin{cases} v_e = v_n - P_{ne} \times \omega_n \\ \omega_e = \omega_n \end{cases}$$

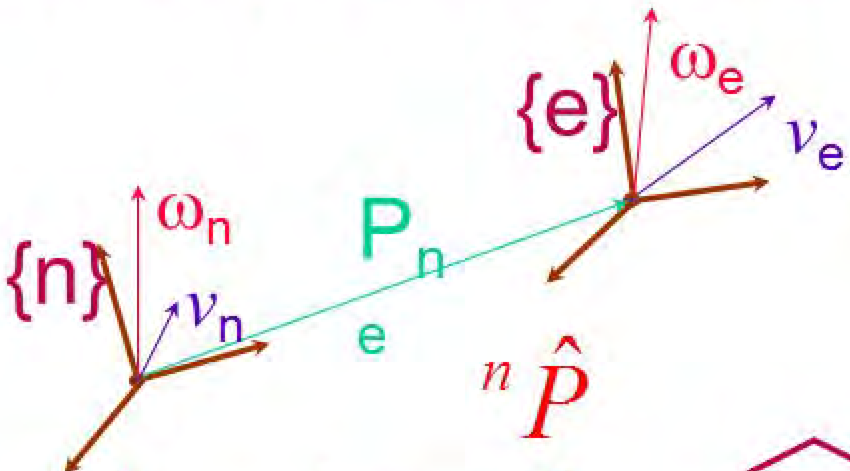
$$\begin{cases} v_e = v_n - P_{ne} \times \omega_n \\ \omega_e = \omega_n \end{cases}$$

$$\begin{pmatrix} v_e \\ \omega_e \end{pmatrix} = \left(\begin{array}{c|c} \mathrm{I} & -\hat{P}_{ne} \\ \hline \mathrm{O} & \mathrm{I} \end{array}\right) \begin{pmatrix} v_n \\ \omega_n \end{pmatrix}$$

$$J_e \dot{q} = \left(\begin{array}{c|c} \mathrm{I} & -\hat{P}_{ne} \\ \hline \mathrm{O} & \mathrm{I} \end{array}\right) J_n \dot{q}$$

$$J_e = \left(\begin{array}{c|c} \mathrm{I} & -\hat{P}_{ne} \\ \hline \mathrm{O} & \mathrm{I} \end{array}\right) J_n$$

Cross Product Operator (in diff. frames)



$$^{0}J_{e} = \begin{pmatrix} I & -{}^{0}\hat{P}_{ne} \\ 0 & I \end{pmatrix} {}^{0}J_{n}$$

$${}^{n}\hat{P}$$

$${}^{0}\hat{P} \neq {}^{0}_{n}R \; {}^{n}\hat{P}; \qquad \widehat{{}^{0}P} = \widehat{({}^{0}_{n}R.{}^{n}P)} \neq {}^{0}_{n}R.\widehat{{}^{n}P}$$

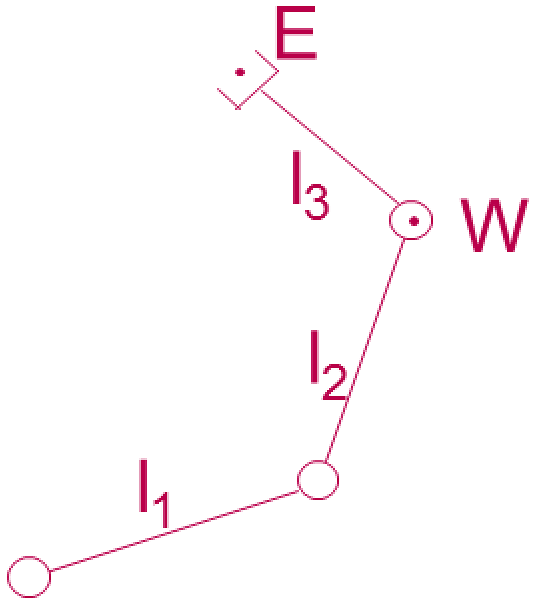
$${}^{0}P \times {}^{0}\omega = {}^{0}_{n}R.({}^{n}P \times {}^{n}\omega)$$

$${}^{0}\hat{P}.{}^{0}\omega = {}^{0}_{n}R.({}^{n}\hat{P}.{}^{n}\omega) = {}^{0}_{n}R.({}^{n}\hat{P}.{}^{0}_{n}R^{T}.{}^{0}\omega)$$

$$\boxed{{}^{0}\hat{P} = {}^{0}_{n}R \; {}^{n}\hat{P} \; {}^{0}_{n}R^{T}}$$

$$ {}^{i}J = \begin{pmatrix} {}^{i}_{j}R & 0 \\ 0 & {}^{i}_{j}R \end{pmatrix} {}^{j}J $$

$$ {}^{0}J_{e} = \begin{pmatrix} {}^{0}_{n}R & -{}^{0}_{n}R \, {}^{n}\hat{P}_{ne} \, {}^{0}_{n}R^{T} \\ 0 & {}^{0}_{n}R \end{pmatrix} {}^{n}J_{n} $$



Wrist Point

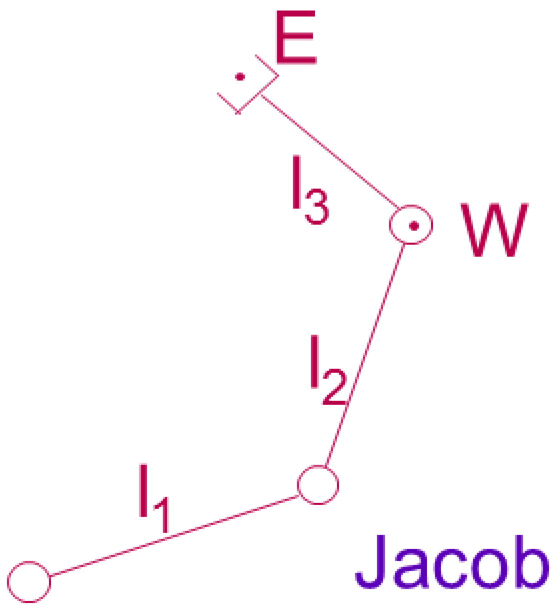
$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$



Wrist Point

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

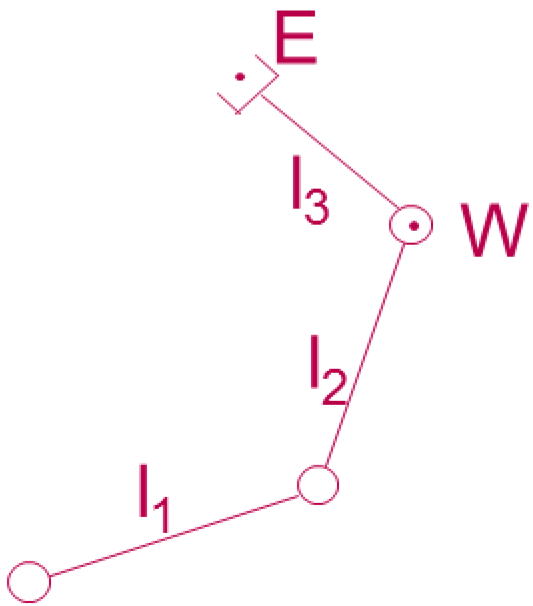
End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

Jacobian (W)

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} ; \; {}^0 J_E = \begin{pmatrix} I & -{}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0 J_W$$



Wrist Point

$$x = l_1 c_1 + l_2 c_{12}$$

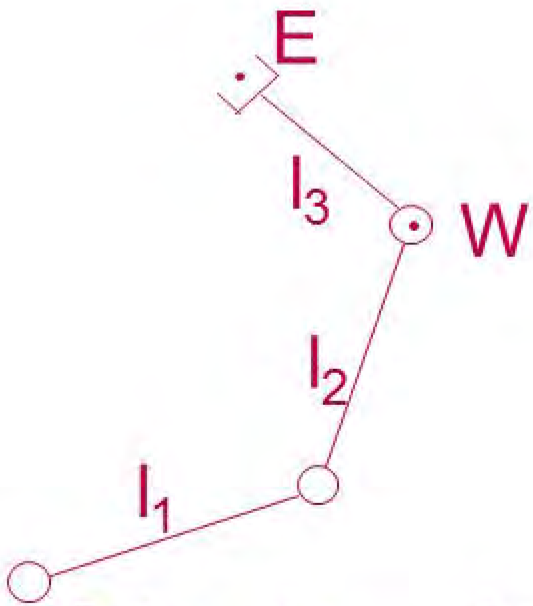
$$y = l_1 s_1 + l_2 s_{12}$$

End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad {}^0 J_E = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$



Wrist Point

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$^0J_E = \begin{pmatrix} I & -{}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0J_W$$

$$^0P_{WE} = \begin{bmatrix} l_3 c_{123} \\ l_3 s_{123} \\ 0 \end{bmatrix} \Rightarrow {}^0\hat{P}_{WE} = \begin{pmatrix} 0 & 0 & l_3 s_{123} \\ 0 & 0 & -l_3 c_{123} \\ -l_3 s_{123} & l_3 c_{123} & 0 \end{pmatrix}$$

Resolved Motion Rate Control (Whitney 72)

$$\delta x = J(\theta)\delta\theta$$

Outside singularities

$$\delta\theta = J^{-1}(\theta)\delta x$$

Arm at Configuration θ

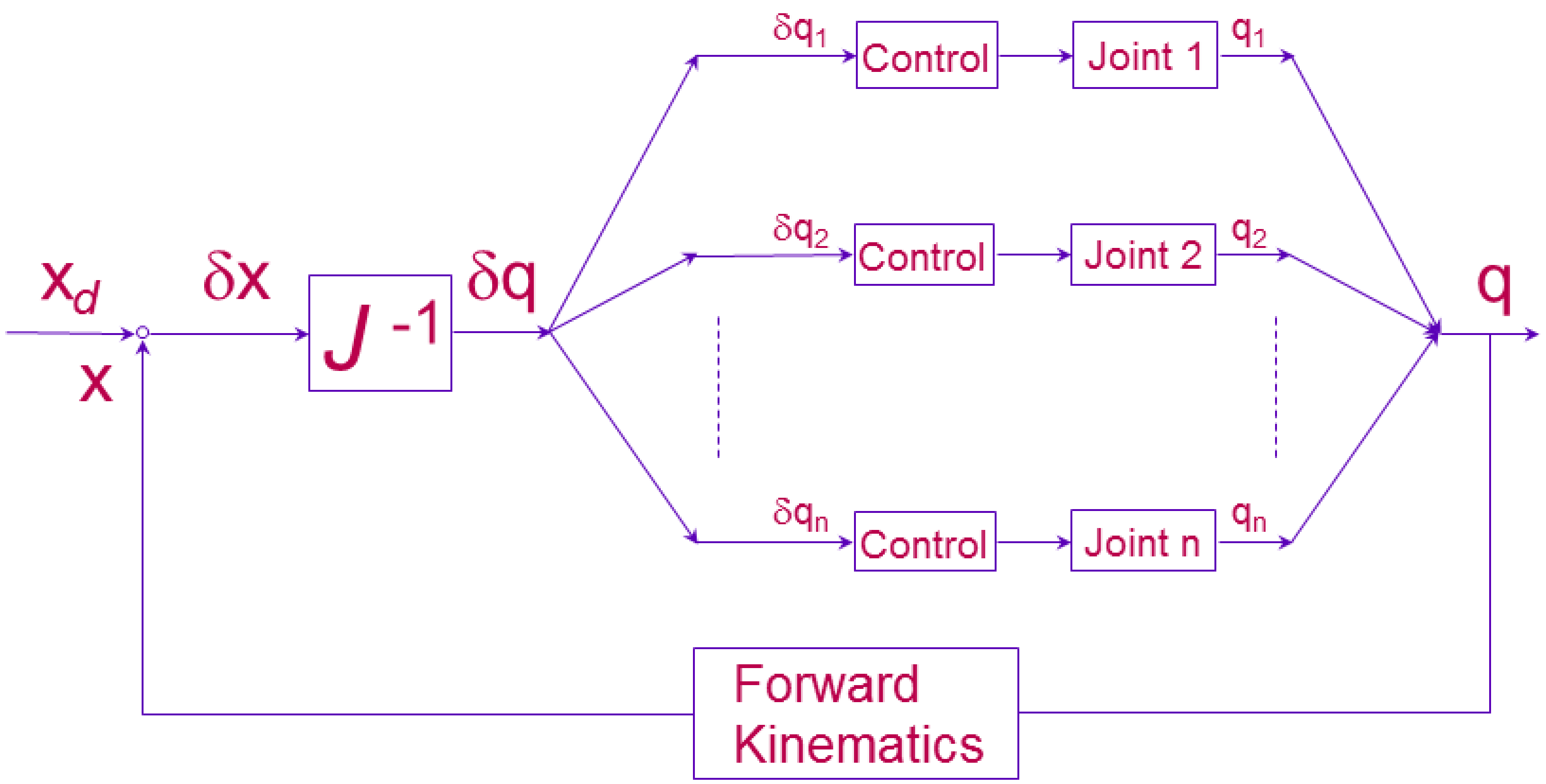
$$x = f(\theta)$$

$$\delta x = x_d - x$$

$$\delta\theta = J^{-1}\delta x$$

$$\boxed{\theta^+ = \theta + \delta\theta}$$

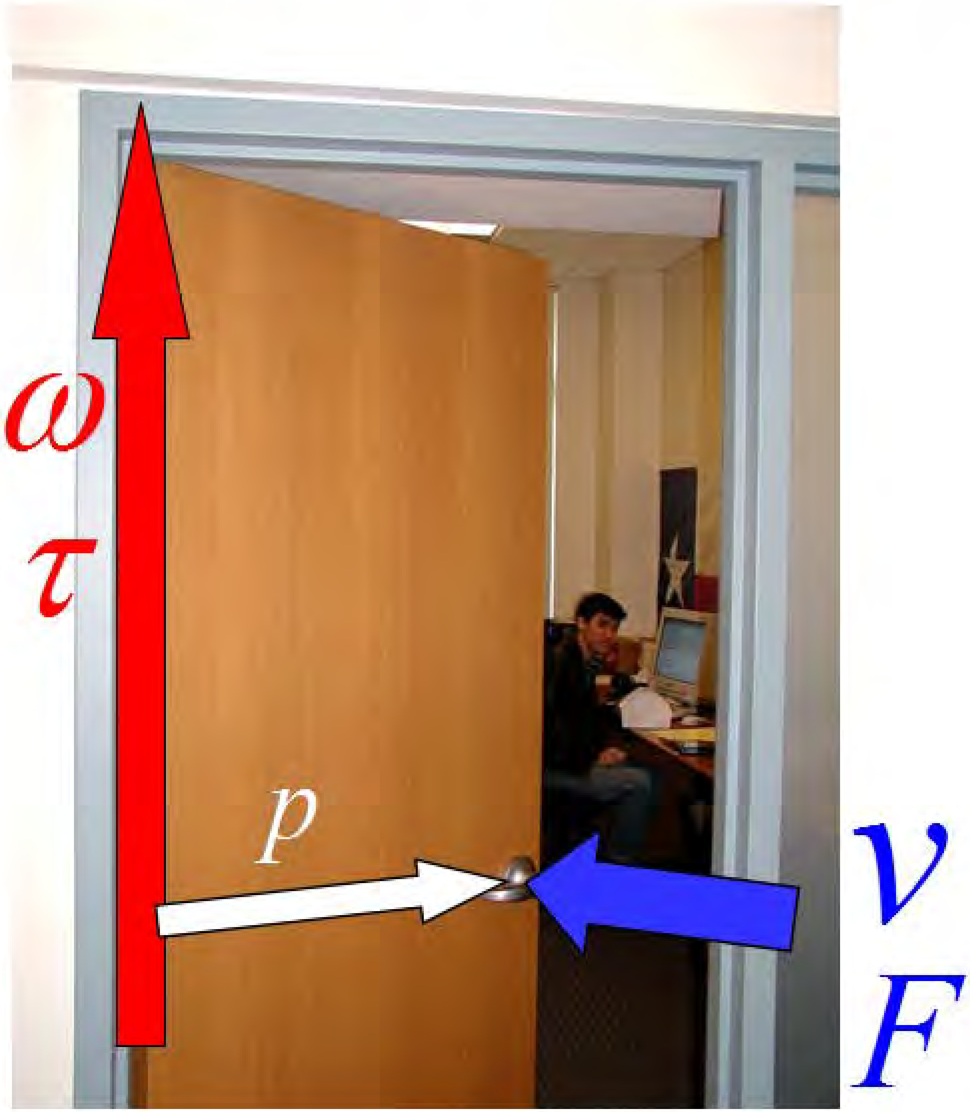
Resolved Motion Rate Control



Jacobian

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces ⇦

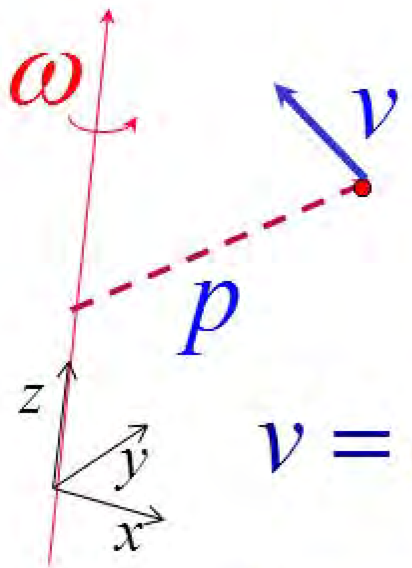
Angular/Linear – Velocities/Forces



$$v = \omega \times p$$

$$\tau = p \times F$$

Angular/Linear – Velocities/Forces

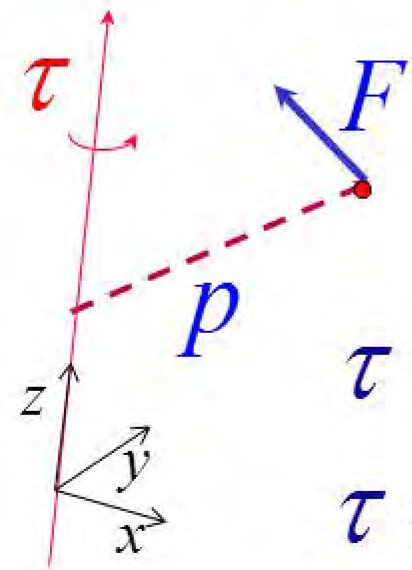


$$v = \omega \times p$$

$$v = -\hat{p}\,\omega$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -p_y \\ p_x \end{pmatrix} \dot{\theta}$$

$$\boxed{v = J\,\dot{\theta}}$$



$$\tau = p \times F$$

$$\tau = \hat{p}\;F$$

$$\tau = (-\hat{p})^T\;F$$

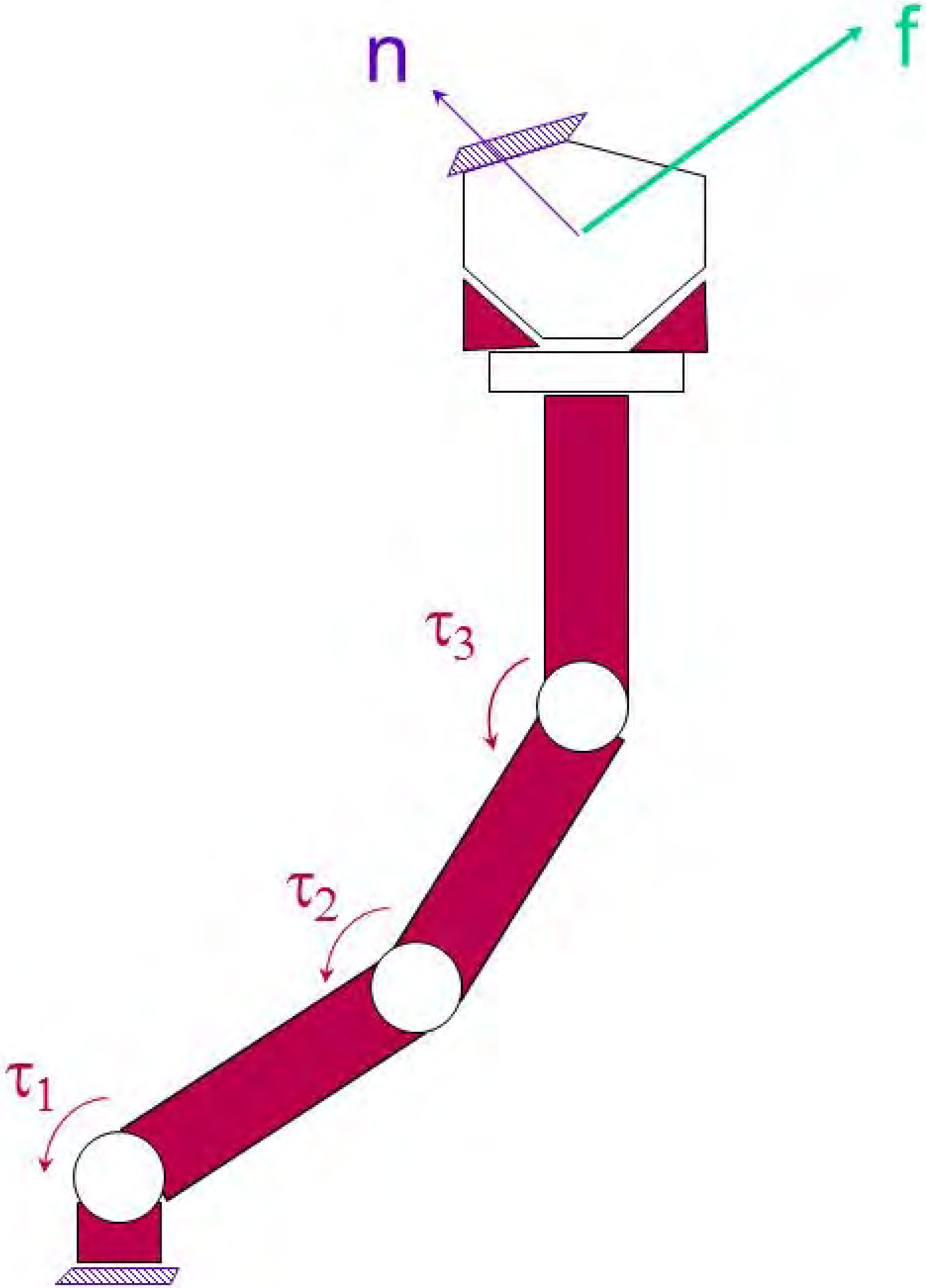
$$\tau = \begin{pmatrix} -p_y & p_x \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

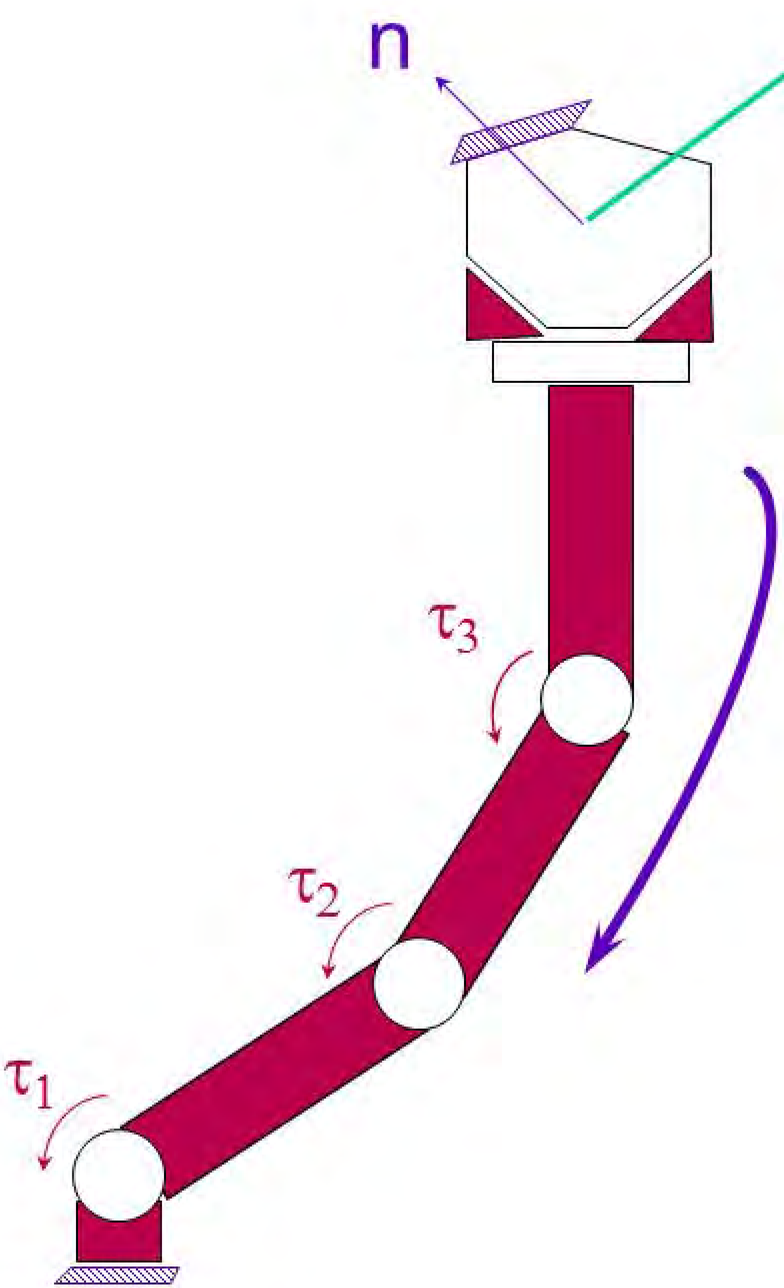
$$\boxed{\tau = J^T F}$$

Velocity/Force Duality

$$\dot{x} = J\,\dot{\theta}$$

$$\tau = J^T F$$


n
f
τ_3
τ_2
τ_1



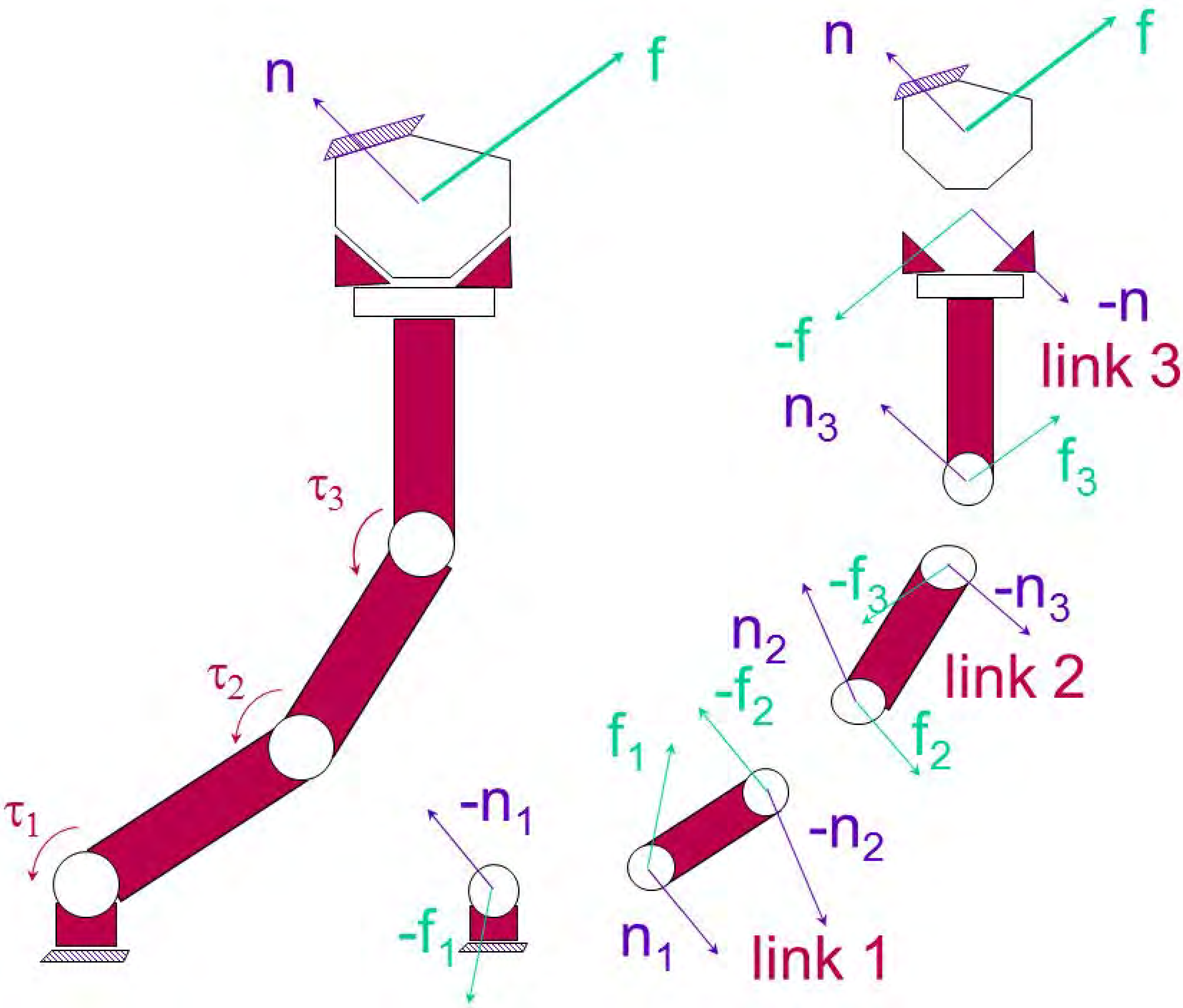
Propagation

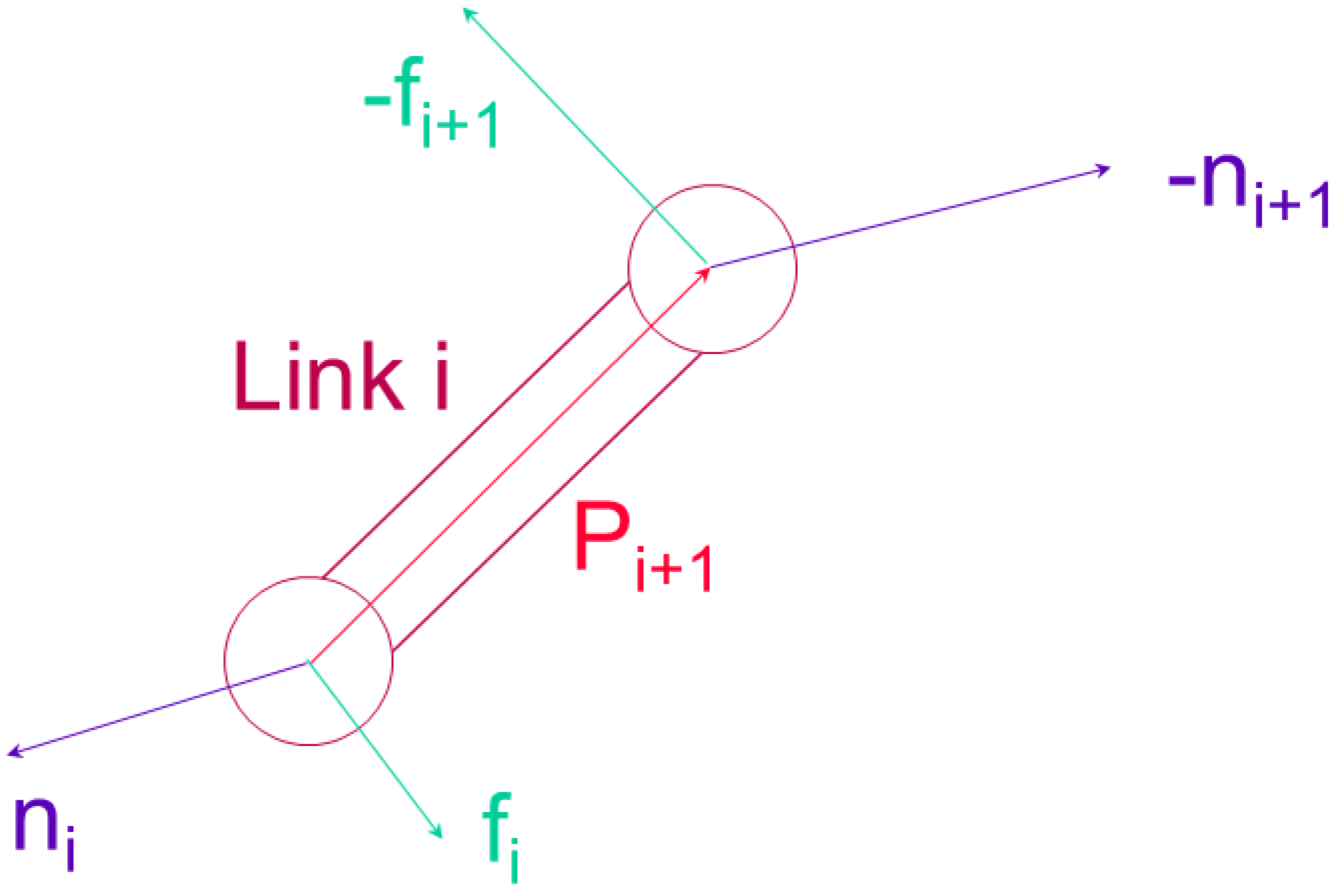
Elimination of Internal forces

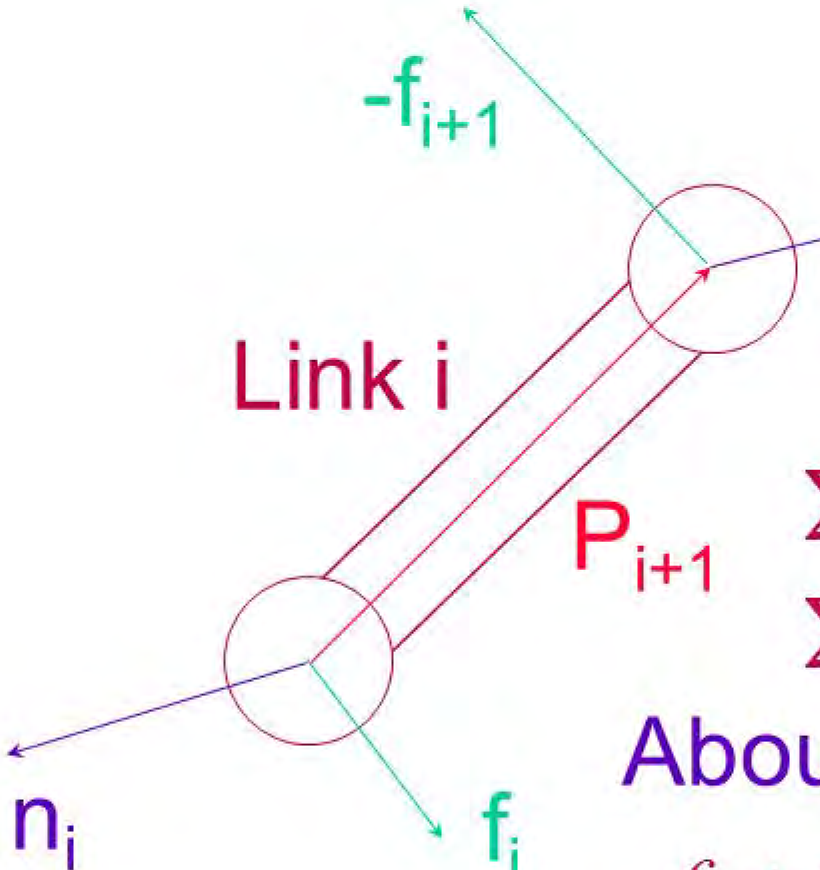
Energy Analysis

Virtual Work

Static Equilibrium


n
f
n
f
-f
-n
link 3
n3
f3
τ3
-f3
-n3
n2
link 2
τ2
-f2
f1
f2
τ1
-n1
-n2
-f1
n1
link 1


-f$_{i+1}$
-n$_{i+1}$
Link i
P$_{i+1}$
n$_i$
f$_i$



Static Equilibrium

Σ forces = 0

Σ moments / a point = 0

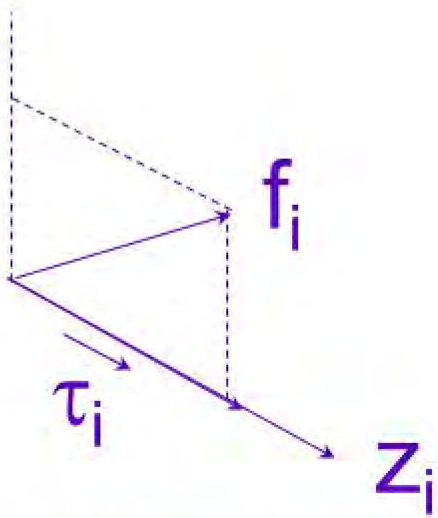
About origin {i}

$$f_i + (-f_{i+1}) = 0$$

$$n_i + (-n_{i+1}) + P_{i+1} \times (-f_{i+1}) = 0$$

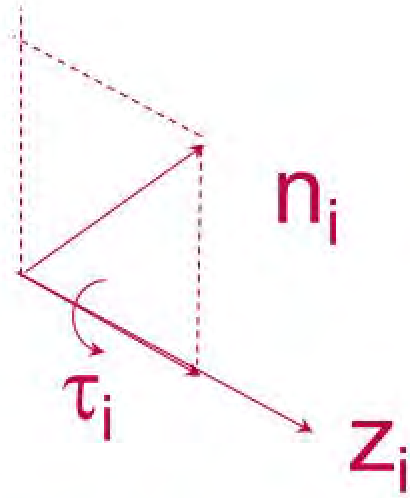
$$f_i = f_{i+1}$$

$$n_i = n_{i+1} + P_{i+1} \times f_{i+1}$$



Prismatic Joint

$$\tau_i = f_i^T Z_i$$



Revolute Joint

$$\tau_i = n_i^T Z_i$$

Algorithm

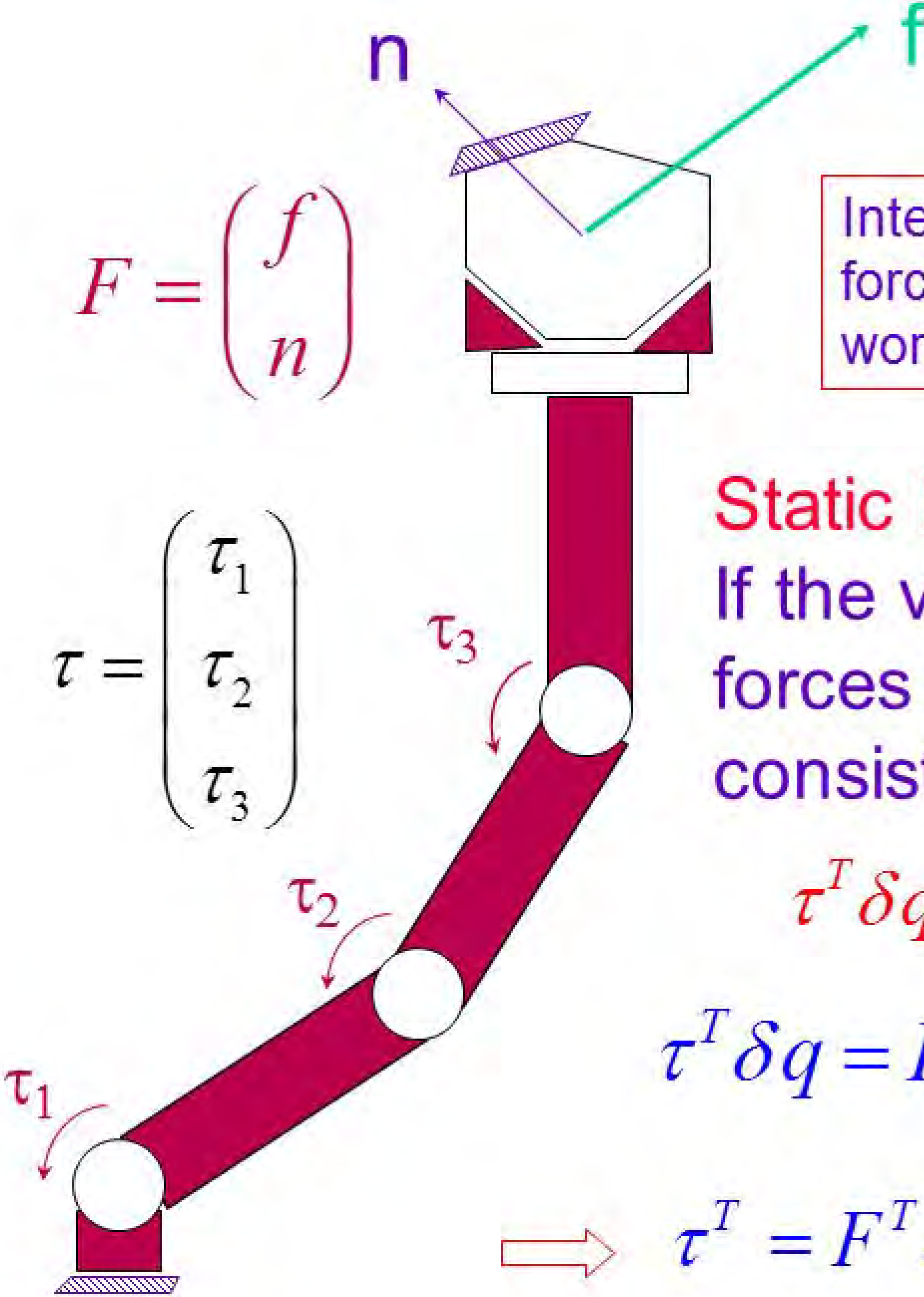
$$^{n}f_{n} = {}^{n}f$$

$$^{n}n_{n} = {}^{n}n + {}^{n}P_{n+1} \times {}^{n}f$$

$$^{i}f_{i} = {}^{i}_{i+1}R.\,{}^{i+1}f_{i+1}$$

$$^{i}n_{i} = {}^{i}_{i+1}R.\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i}$$

Virtual Work Principal



$$F = \begin{pmatrix} f \\ n \end{pmatrix}$$

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$$

Internal forces are workless

$$\delta w = \sum_i f_i \, \delta x_i$$

f_i: applied forces; δx_i: virtual displacements

Static Equilibrium:
If the virtual work done by applied forces is zero in displacements consistent with constraints

$$\tau^T \delta q + (-F)^T \delta x = 0$$

$$\tau^T \delta q = F^T \delta x \quad \text{using} \quad \delta x = J \, \delta q$$

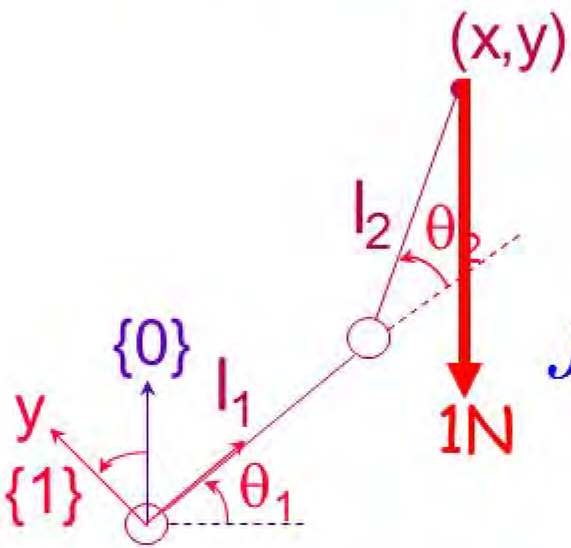
$$\Rightarrow \quad \tau^T = F^T J \quad \Rightarrow \quad \boxed{\tau = J^T F}$$

Velocity/Force Duality

$$\dot{x} = J\,\dot{\theta}$$

$$\tau = J^{T} F$$

Example (Static Forces)



$$J = \begin{pmatrix} -(l_1S1 + l_2S12) & -l_2S12 \\ l_1C1 + l_2C12 & l_2C12 \end{pmatrix}$$

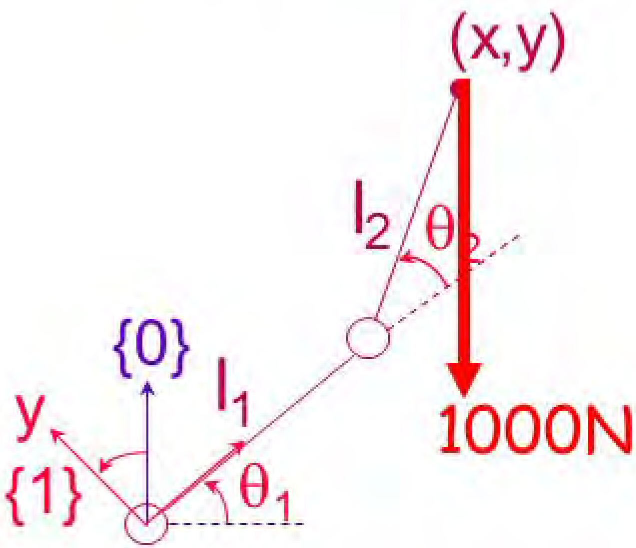
$$J^T = \begin{pmatrix} -(l_1S1 + l_2S12) & l_1C1 + l_2C12 \\ -l_2S12 & l_2C12 \end{pmatrix}$$

$$\tau = J^T F$$

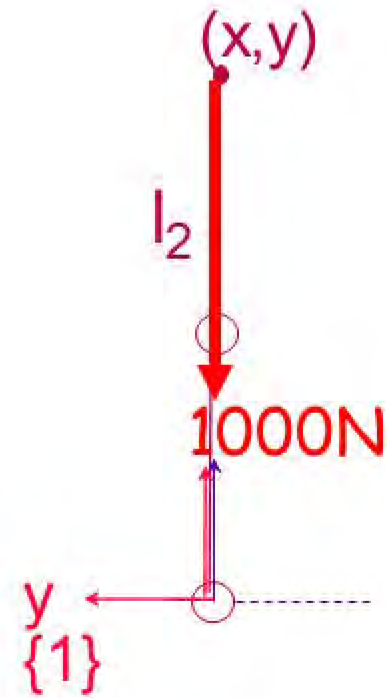
$$l_1 = l_2 = 1; \quad \theta_1 = 0; \; \theta_2 = 60^\circ$$

$$\tau = \begin{pmatrix} -(l_1S1 + l_2S12) & l_1C1 + l_2C12 \\ -l_2S12 & l_2C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} l_1C1 + l_2C12 \\ l_2C12 \end{bmatrix} = -\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Example (Static Forces)



$$\tau = J^T F$$



$$\tau = \begin{pmatrix} -\left(l_1 S1 + l_2 S12\right) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1K \end{bmatrix} = \begin{bmatrix} l_1 C1 + l_2 C12 \\ l_2 C12 \end{bmatrix} (-1K) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$l_1 = l_2 = 1; \quad \theta_1 = 90; \theta_2 = 0^\circ$$