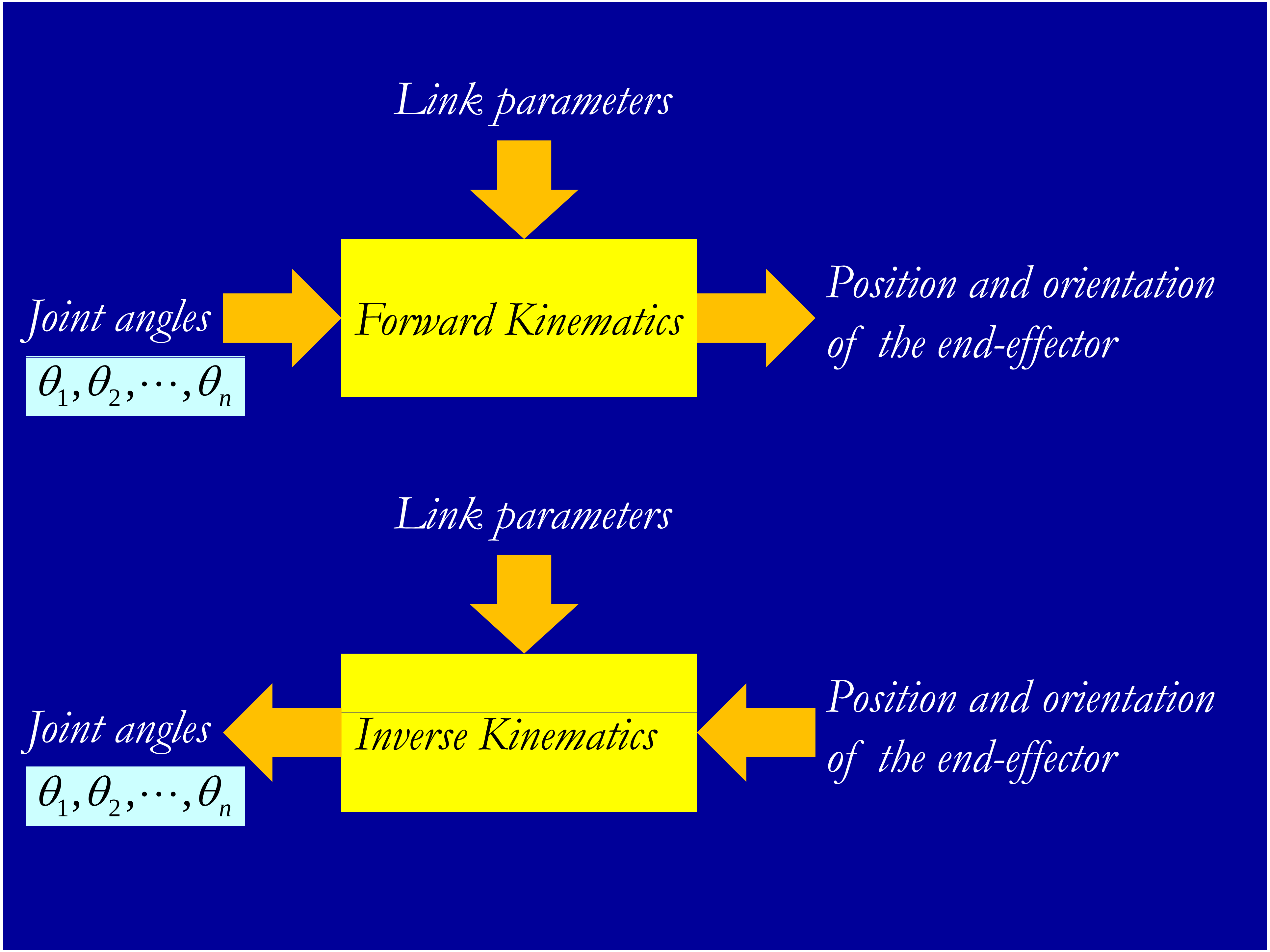
Inverse Manipulator Kinematics

December 21, 2009


Link parameters
Joint angles
$\theta_1, \theta_2, \cdots, \theta_n$
Forward Kinematics
Position and orientation of the end-effector
Link parameters
Joint angles
$\theta_1, \theta_2, \cdots, \theta_n$
Inverse Kinematics
Position and orientation of the end-effector

Inverse Kinematics

- Given the desired position and orientation of the tool relative to the station, how do we compute the set of joint angles which will achieve this desired result?
- First, frame transformations are performed to find the wrist frame, $\{W\}$, relative to the base frame, $\{B\}$, and then the inverse kinematics are used to solve for the joint angles.

Solvability

Given the numerical value of $^{0}_{N}T$ we attempt to find values of $\theta_1, \theta_2, \cdots, \theta_n$.

The PUMA 560:
Given $^{0}_{6}T$ as 16 numerical values, solve (3.14) for 6 joint angles, $\theta_1, \theta_2, \cdots, \theta_6$.

↓

12 equations and 6 unknowns

↓

6 equations and 6 unknowns
(*nonlinear, transcendental equations*)

Not algebraic,
Exponential, trigonometric functions

Solvability

$$ {}^{0}_{6}T = \underbrace{{}^{0}_{1}T(\theta_1)\,{}^{1}_{2}T(\theta_2)\,{}^{2}_{3}T(\theta_3)\,{}^{3}_{4}T(\theta_4)\,{}^{4}_{5}T(\theta_5)\,{}^{5}_{6}T(\theta_6)}_{\text{Forward Kinematics}} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Goal}} $$

Forward Kinematics

Goal

$$ r_{11} = c_1\left[c_{23}\left(c_4 c_5 c_6 - s_4 s_5\right) - s_{23} s_5 c_5\right] + s_1\left(s_4 c_5 c_6 + c_4 s_6\right), $$

$$ r_{21} = \cdots $$

$$ \vdots $$

$$ p_y = \cdots $$

$$ p_z = -a_3 s_{23} - a_2 s_2 - d_4 c_{23}. $$

Independent Equation:
3 - *Rotation Matrix*
3 - *Position Vector*

Existence of solutions

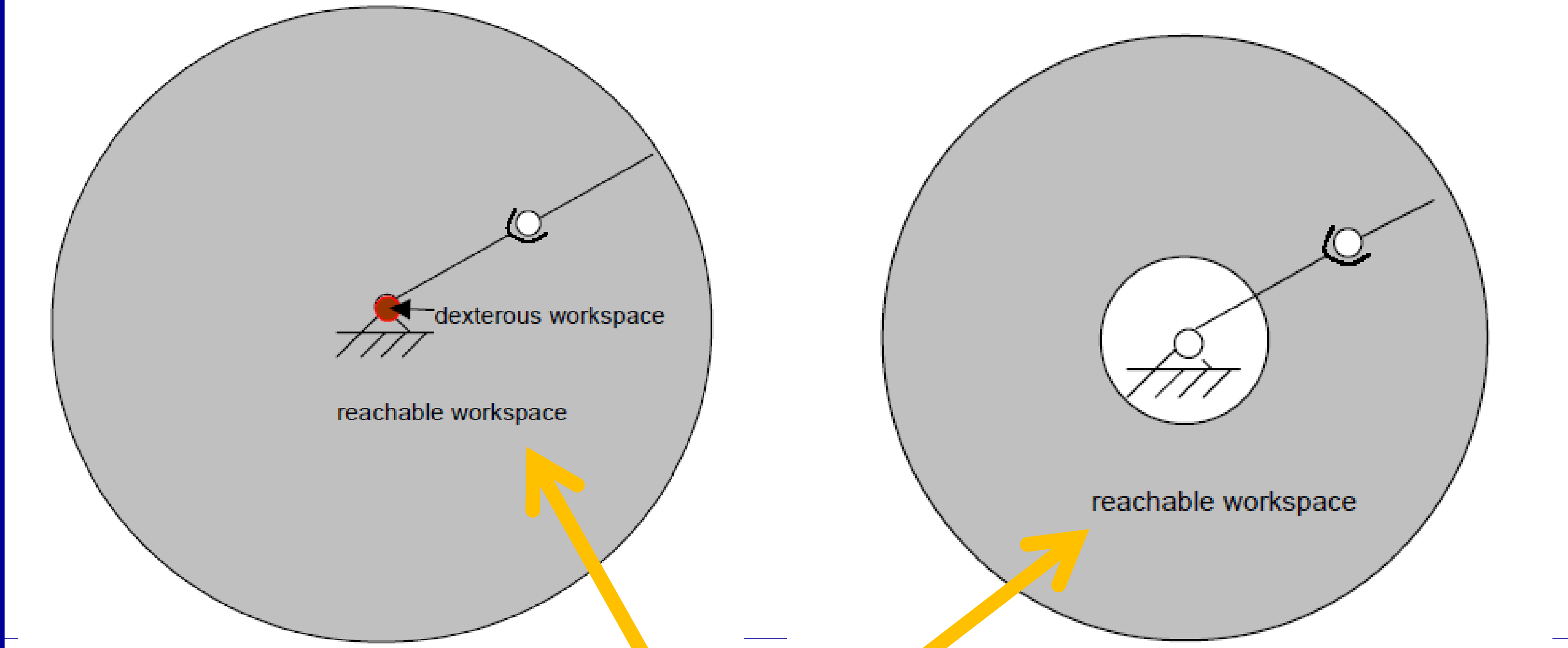
- In the forward kinematics problem, each set of input joint parameters gave a unique output pose. However, in the inverse kinematics, a given pose may be satisfied with several different sets of input angles.

${}_{N}^{0}T$ *must be in the workspace of the manipulator.*

Workspace

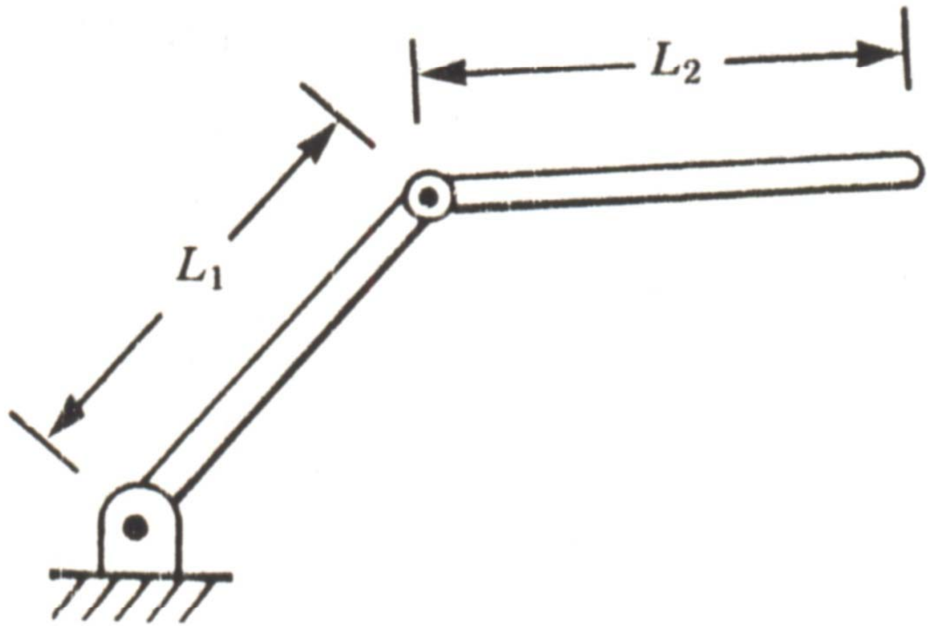
- The volume of the space which the end-effector of the manipulator can reach.
- *Dextrous* workspace (*DW*): reachable with all orientations
- *Reachable* workspace (*RW*): reachable in at least one orientation

How to maximize the dexterous workspace?
DW is a subset of RW



the set of all possible positions of the end-effector:
the total volume swept out by the end-effector
as the manipulator executes all possible motions

Two-link manipulator



$l_1 = l_2$

RW a disk of radius $2l_1$

DW the origin

$l_1 \neq l_2$

RW

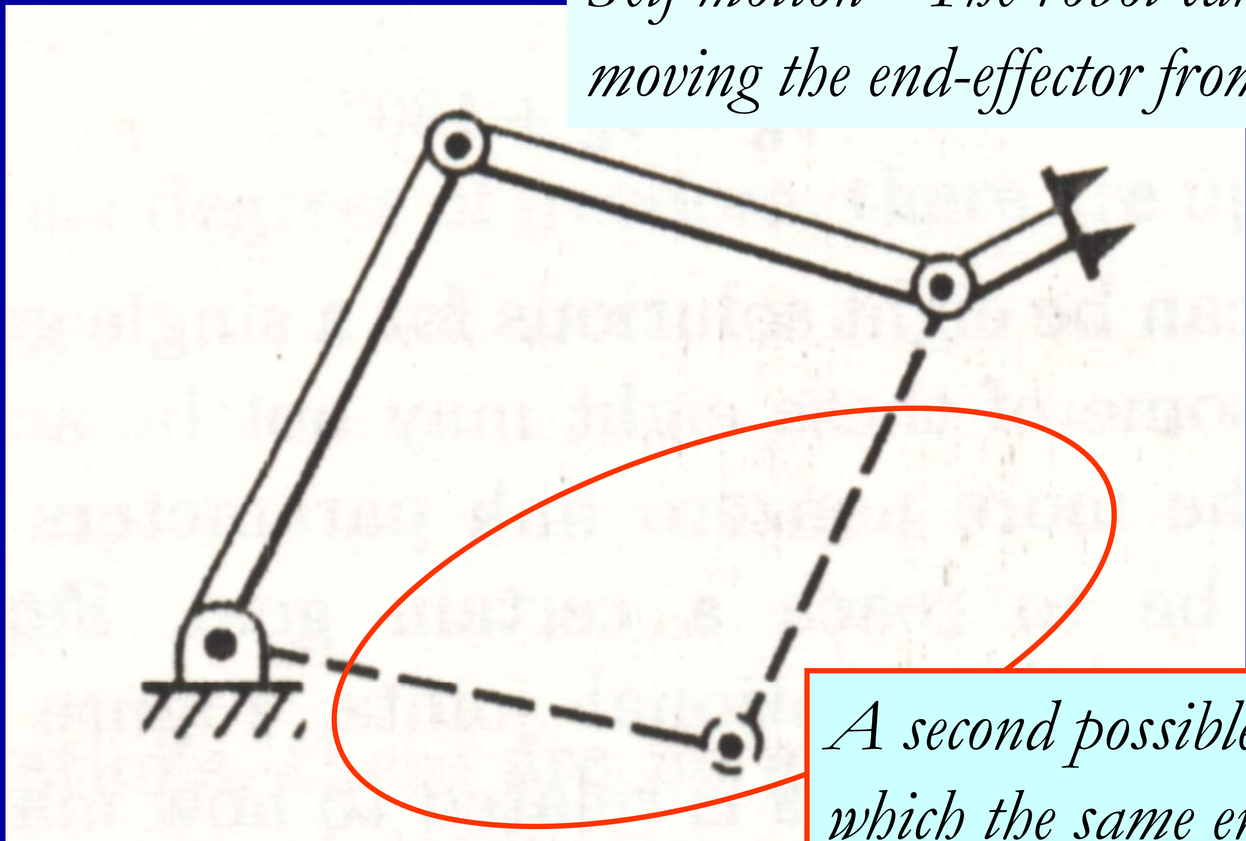
a ring of outer radius $l_1 + l_2$

inner radius $|l_1 - l_2|$

When joint limits are a subset of the full 360 degrees, then the workspace is correspondingly reduced.

Three-link manipulator: Multiple solutions

Self-motion - The robot can be moved without moving the end-effector from the goal



A second possible configuration in which the same end-effector position and orientation are achieved.

Possible Problems

- Multiple solutions
- Infinitely many solutions
- No solutions
- No closed-form (analytical) solutions

This only works if the number of kinematic constraints is the same as the number of degrees-of-freedom of the robot.

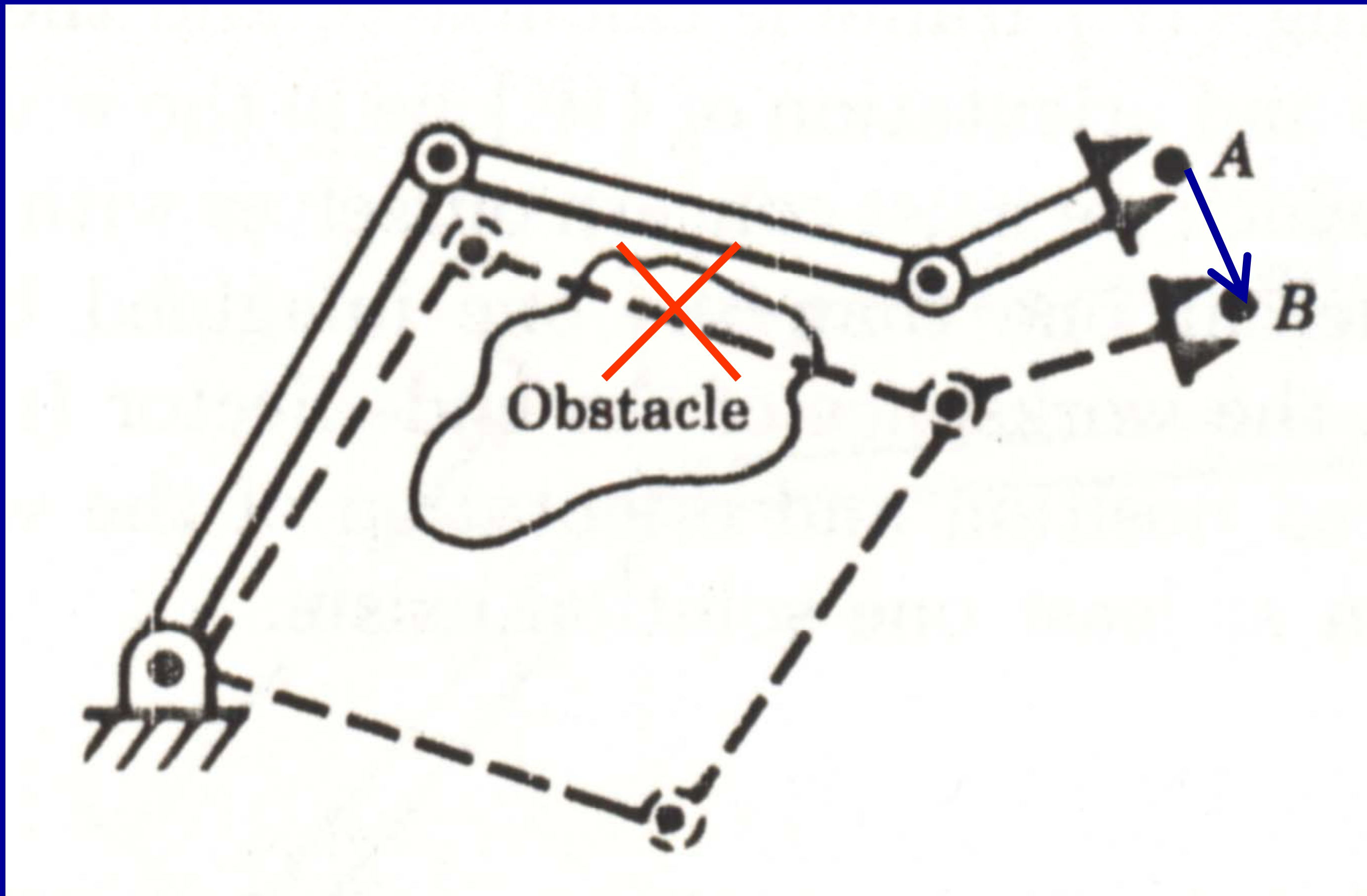
Multiple solutions

- We need to able to calculate all the possible solutions.
- The system has to be able to choose one.
- The closest solution: the solution which minimizes the amount that each joint is required to move.

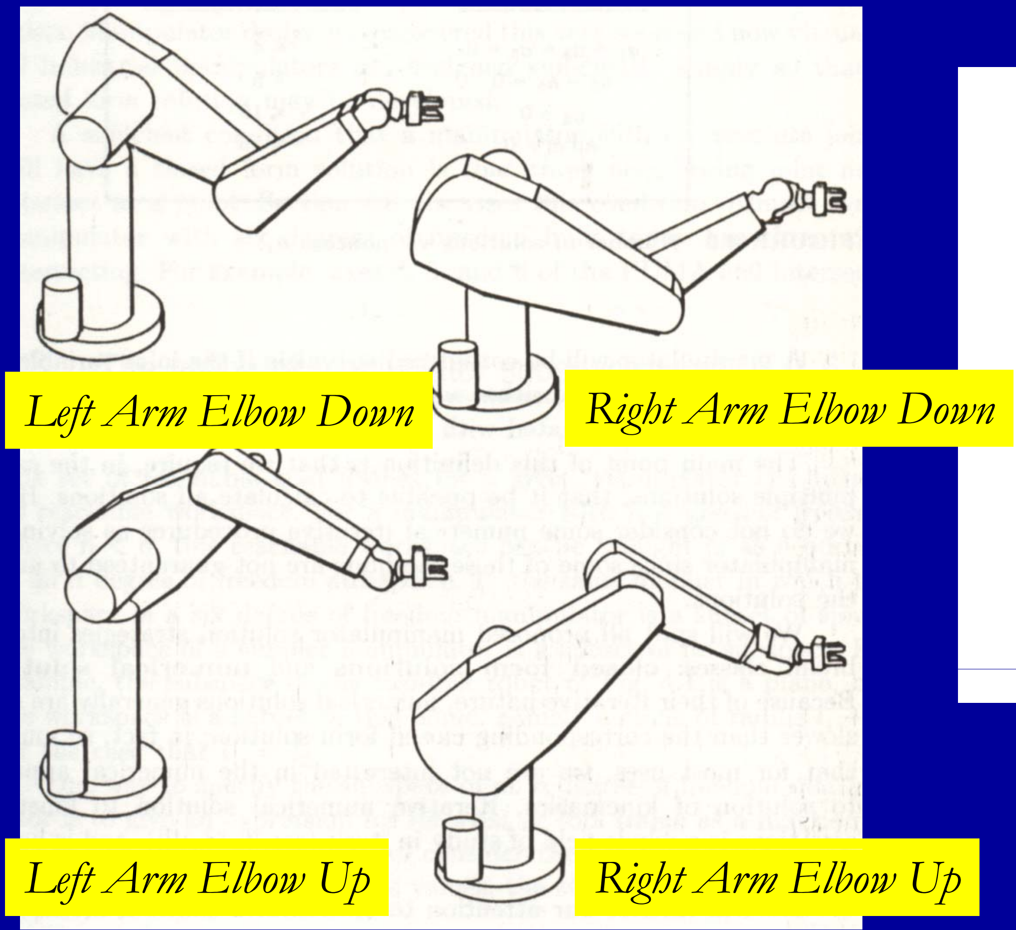
The closest solution in joint space

- Weights might be applied: moving small joints (*wrist*) instead of moving large joints (*Shoulder/ Elbow*)
- The presence of obstacles

Two possible solutions



Eight solutions of the PUMA 560



Another solution

$$\theta_4' = \theta_4 + 180^\circ,$$

$$\theta_5' = -\theta_5,$$

$$\theta_6' = \theta_6 + 180^\circ.$$

Number of solutions vs. nonzero a_i

A 6R manipulator | *Link length*

$a_1 = a_3 = a_5 = 0$	≤ 4
$a_3 = a_5 = 0$	≤ 8
$a_3 = 0$	≤ 16
$All \quad a_i \neq 0$	≤ 16

The more the link length parameters are nonzero, the bigger the maximum number of solutions!

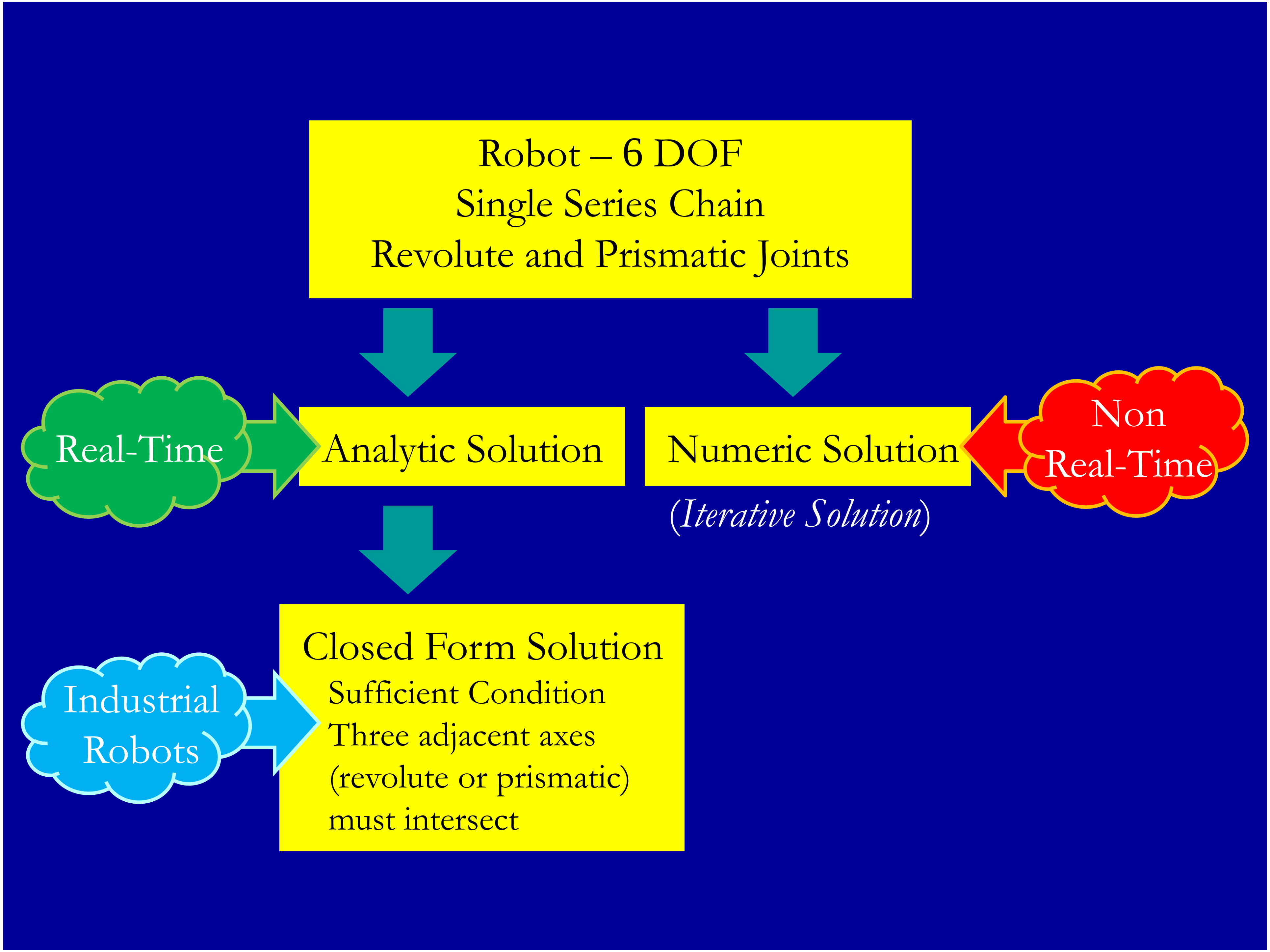
Method of solution

- Closed form solutions: based on analytic expressions or on the solution of a polynomial of degree 4 or less *Algebraic/geometric*
- Numerical solutions: all systems with revolute and prismatic joints having a total of 6 degrees of freedom in a single chain are solvable. *Much slower*

Closed-form solutions

- Analytical solution to system of equations
- Can be solved in a fixed number of operations (therefore, computationally fast/known speed)
- Results in all possible solutions to the manipulator kinematics
- Often difficult or impossible to find
- Most desirable for real-time control
- Most desirable overall

6R Manipulator: Three neighboring joint axes intersect at a point.


Robot – 6 DOF
Single Series Chain
Revolute and Prismatic Joints
Real-Time
Analytic Solution
Numeric Solution
Non
Real-Time
(Iterative Solution)
Industrial
Robots
Closed Form Solution
Sufficient Condition
Three adjacent axes
(revolute or prismatic)
must intersect

$$ {}^{0}_{N}T(\theta_1, \theta_2, \ldots, \theta_N) = {}^{0}_{N}T $$

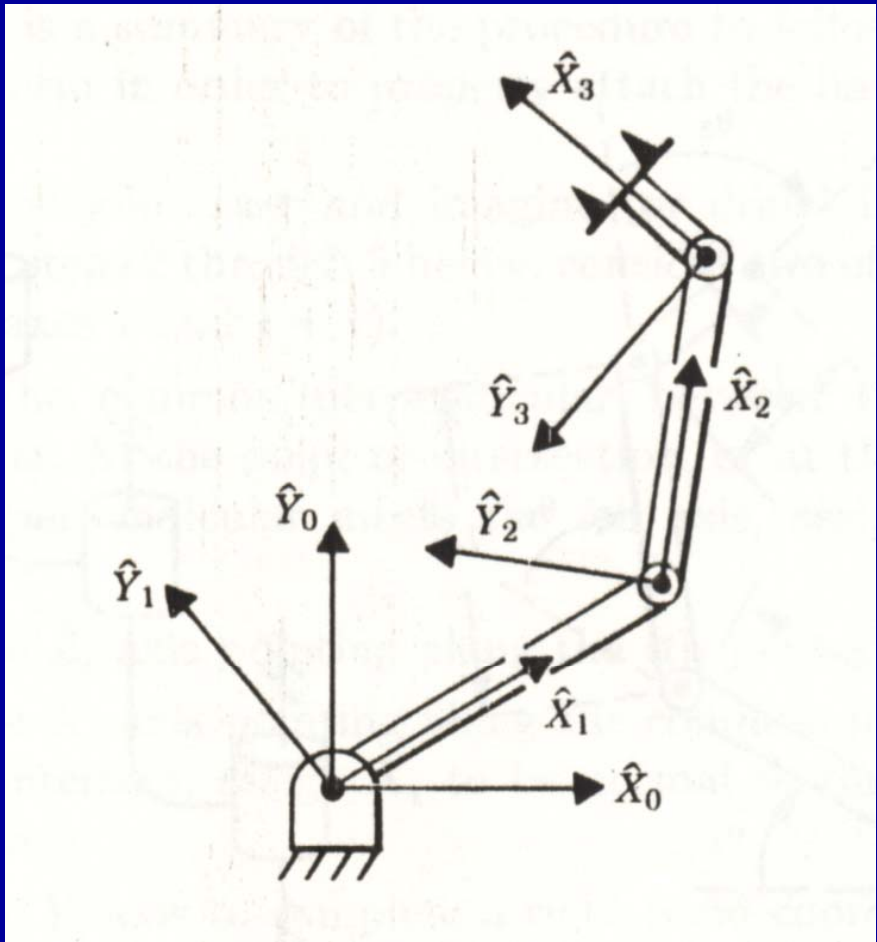
A given desired pose of the tool (numbers!)

A function of the joint variables (equation!): found by solving the forward kinematics

Numerical solutions

- Results in a numerical, iterative solution to system of equations, for example Newton/Raphson techniques
- Unknown number of operations to solve
- Only returns a single solution
- Accuracy is dictated by user
- Because of these reasons, this is much less desirable than a closed-form solution
- Can be applied to all robots.

Algebraic solution (Fig. 4.7)



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

$$
{}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_0 \\ s\theta_1 c\alpha_0 & c\theta_1 c\alpha_0 & -s\alpha_0 & -s\alpha_0 d_1 \\ s\theta_1 s\alpha_0 & c\theta_1 s\alpha_0 & c\alpha_0 & c\alpha_0 d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & L_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & L_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{B}_{W}T = {}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T
$$

$$
{}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1 c_1 + l_2 c_{12} \\ s_{123} & c_{123} & 0.0 & l_1 s_1 + l_2 s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

$$
{}^{B}_{W}T = \begin{bmatrix} c_{\phi} & -s_{\phi} & 0.0 & x \\ s_{\phi} & c_{\phi} & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

$$
c_{12} = c_1 c_2 - s_1 s_2
$$

$$
s_{12} = s_1 c_2 + c_1 s_2
$$

$$c_\phi = c_{123}$$

$$s_\phi = s_{123}$$

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

A set of 4 nonlinear equation that must be solved for $\theta_1, \theta_2, \theta_3$

$$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_2$$

$$\rightarrow c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\rightarrow s_2 = \pm\sqrt{1 - c_2^2} \quad \Rightarrow \theta_2 = A\tan 2(s_2, c_2).$$

Elbow-*up* or elbow-*down*

$$x = k_1 c_1 - k_2 s_1, \quad y = k_1 s_1 + k_2 c_1,$$

$$k_1 = l_1 + l_2 c_2, \quad k_2 = l_2 s_2.$$

$$r = \sqrt{k_1^2 + k_2^2}, \quad \gamma = A\tan 2(k_2, k_1),$$

$$k_1 = r\cos\gamma, \quad k_2 = r\sin\gamma.$$

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1 = \cos(\gamma + \theta_1),$$

$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1 = \sin(\gamma + \theta_1).$$

$$\gamma + \theta_1 = A\tan 2(\frac{y}{r}, \frac{x}{r}) = A\tan 2(y, x),$$

$$\boxed{\theta_1 = A\tan 2(y, x) - A\tan 2(k_2, k_1).}$$

$$\theta_1 + \theta_2 + \boxed{\theta_3} = A\tan 2(s_\phi, c_\phi) = \phi.$$

Some Inverse-Kinematic Formulas

The single equation

$$\sin\theta = a \qquad (1)$$

has two solutions, given by

$$\theta = \pm A\tan 2\left(\sqrt{1-a^2}, a\right).$$

Likewise, given

$$\cos\theta = b, \qquad (2)$$

there are two solutions: $\theta = A\tan 2\left(b, \pm\sqrt{1-b^2}\right).$

If both (1) and (2) are given, then there is a unique solution given by

$$\theta = A\tan 2(a, b).$$

The transcendental equation

$$a\cos\theta + b\sin\theta = 0$$

has two solutions

$$\theta = A\tan 2(a, -b)$$

and

$$\theta = A\tan 2(-a, b)$$

The equation

$$a\cos\theta + b\sin\theta = c$$

is also solved by

$$\theta = A\tan 2(b,a) \pm A\tan 2\left(\sqrt{a^2+b^2-c^2},c\right).$$

The set of equations

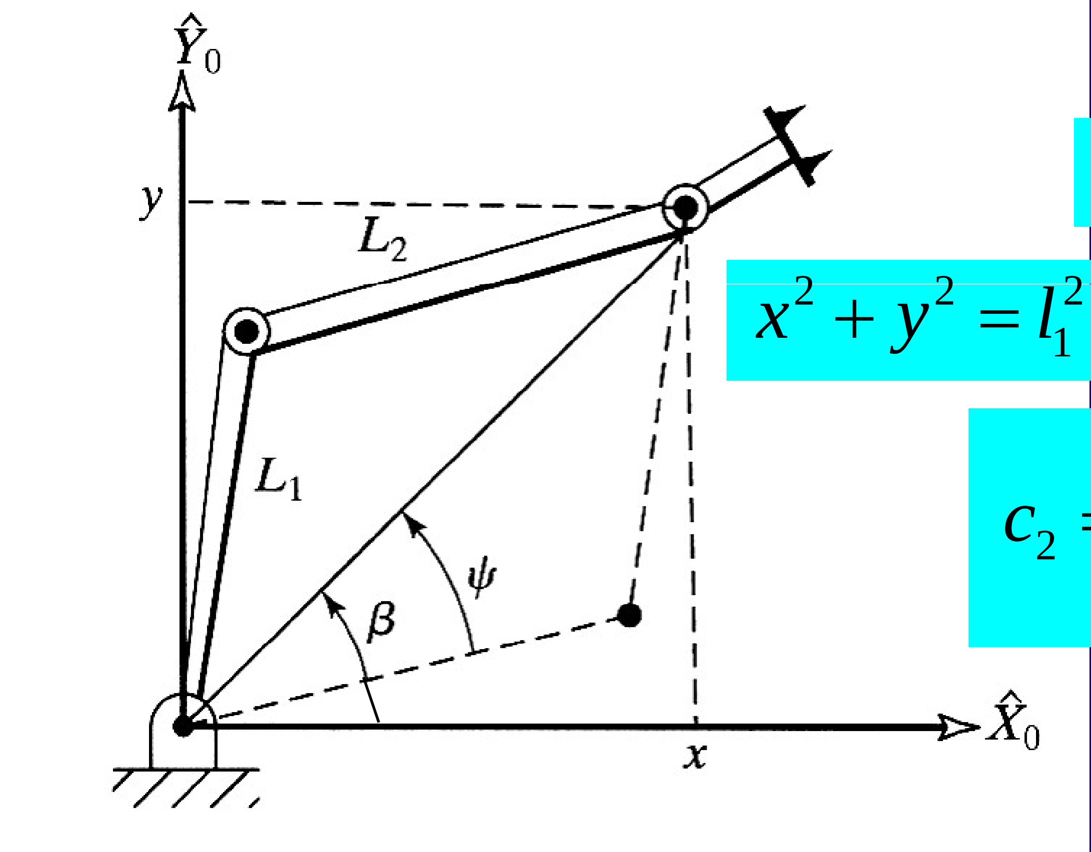
$$a\cos\theta - b\sin\theta = c,$$
$$a\sin\theta + b\cos\theta = d$$

also is solved by $\theta = A\tan 2(ad-bc, ac+bd).$

A Geometric Approach

- For most manipulators, many of the a_i, d_i are zero, the α_i are 0 or $\pm\pi/2,$ etc.
- In these cases, a geometric approach is the simplest and most natural.
- The general idea is to decompose the spatial geometry of the arm into several plane-geometry problems: solving simple trigonometry problems

Geometric Solution: 3-link Planar



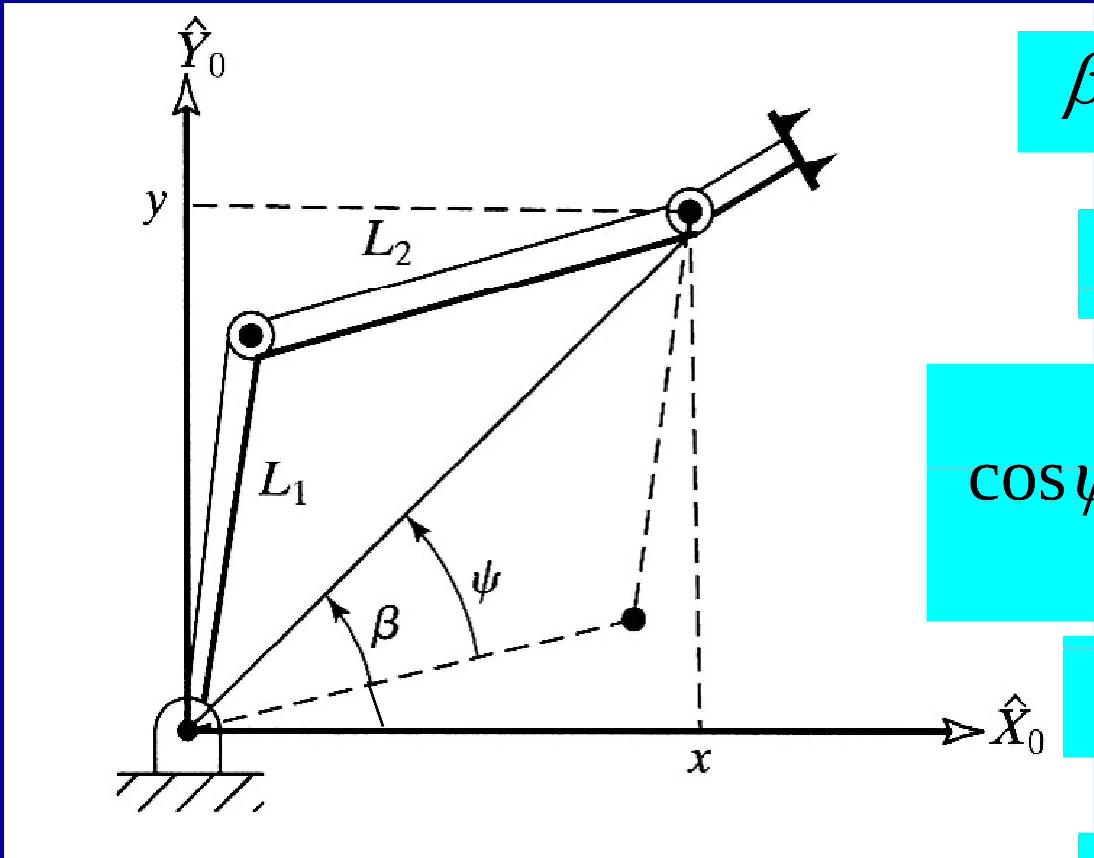
Law of cosines

$$x^2 + y^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos(\theta_2 - 180)$$

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

Pythagorean (or Pythagoras') theorem

Geometric Solution: 3-link Planar



$$\beta = A\tan 2(y, x)$$

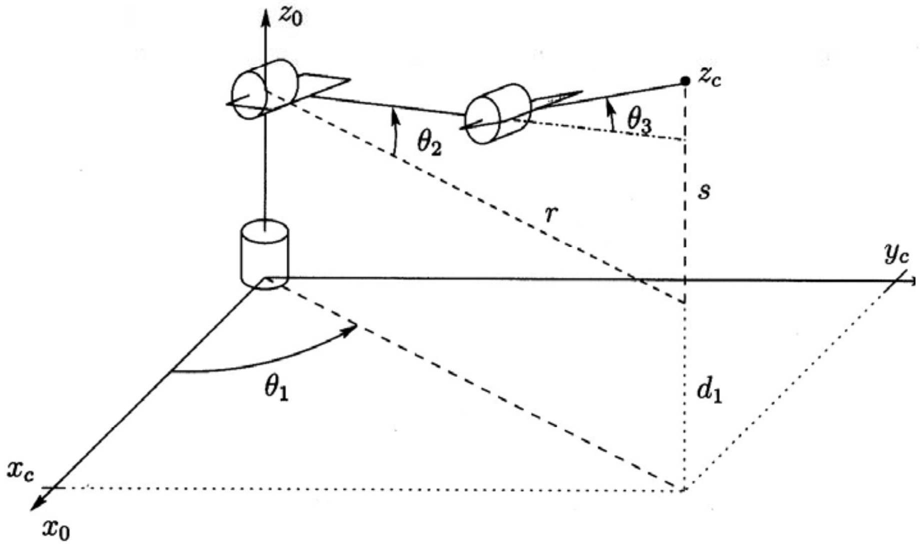
Law of cosines

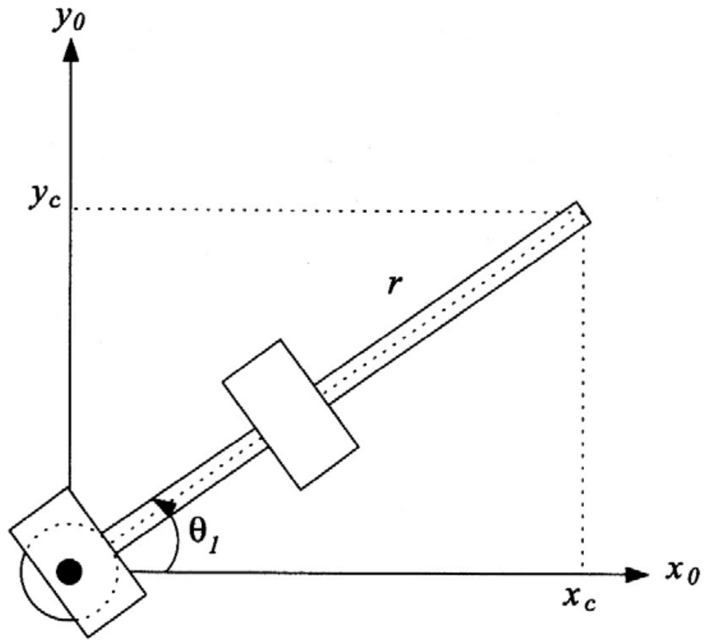
$$\cos\psi = \frac{x^2 + y^2 + l_1^2 - l_2^2}{2l_1\sqrt{x^2 + y^2}}$$

$$\theta_1 = \beta \pm \psi$$

$$\theta_1 + \theta_2 + \theta_3 = \phi$$

Elbow manipulator

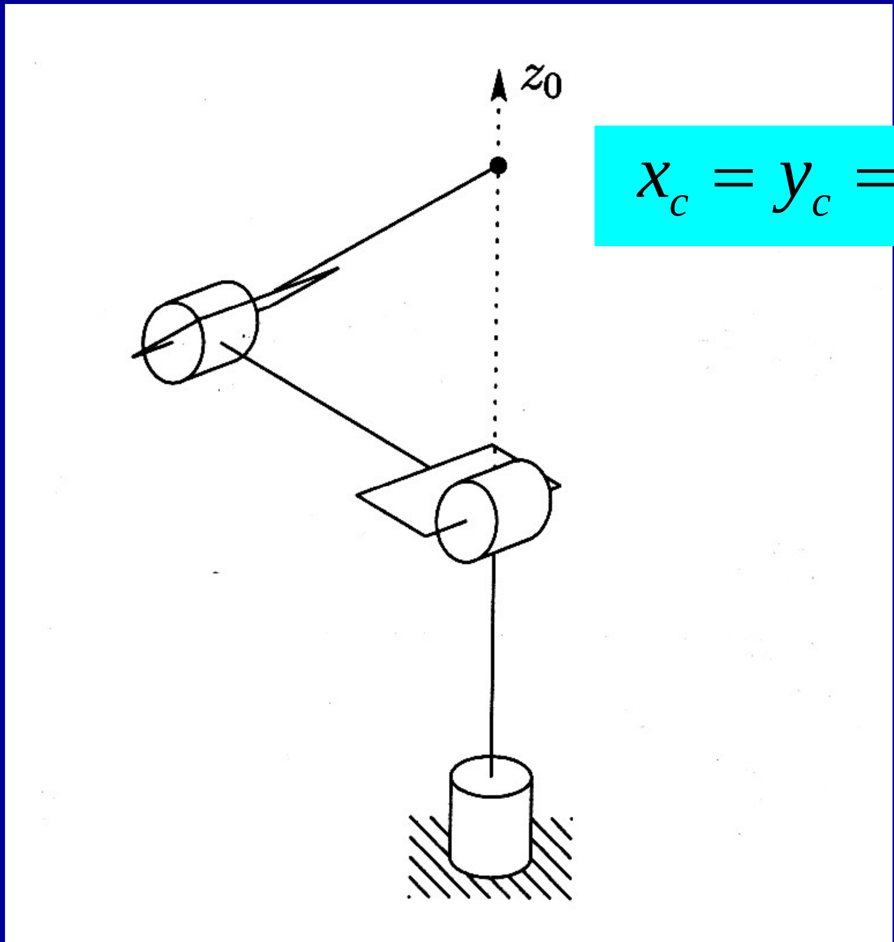




$$\theta_1 = A\tan 2(y_c, x_c)$$

$$\theta_1 = \pi + A\tan 2(y_c, x_c)$$

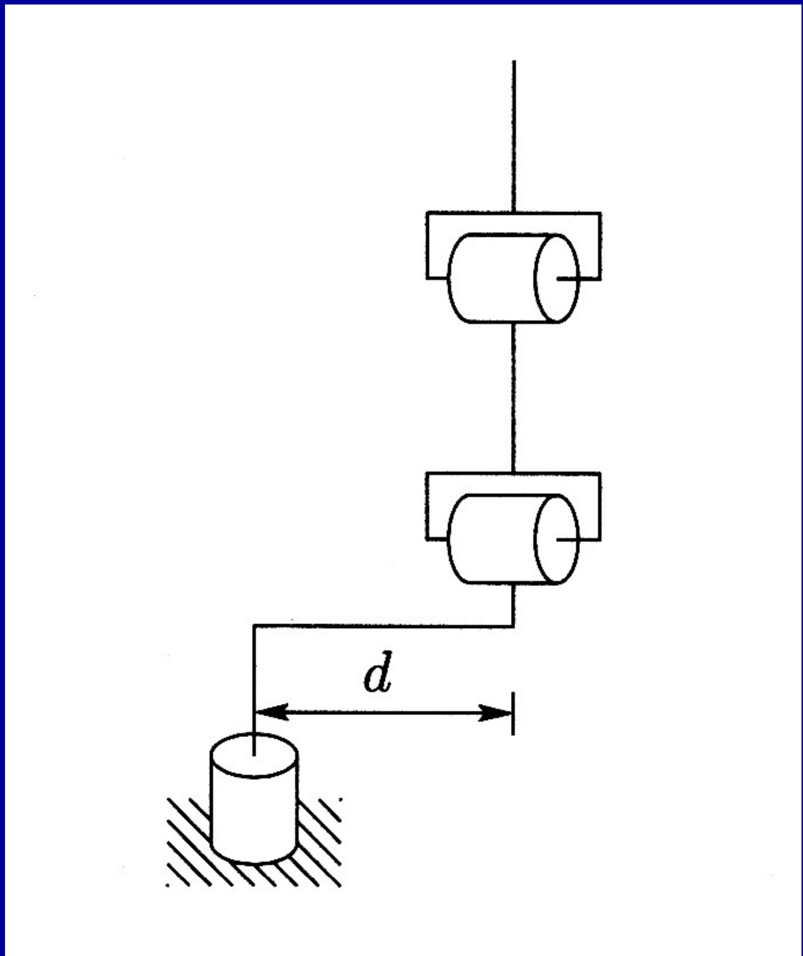
Elbow Manipulator: Singular configuration



$$x_c = y_c = 0$$

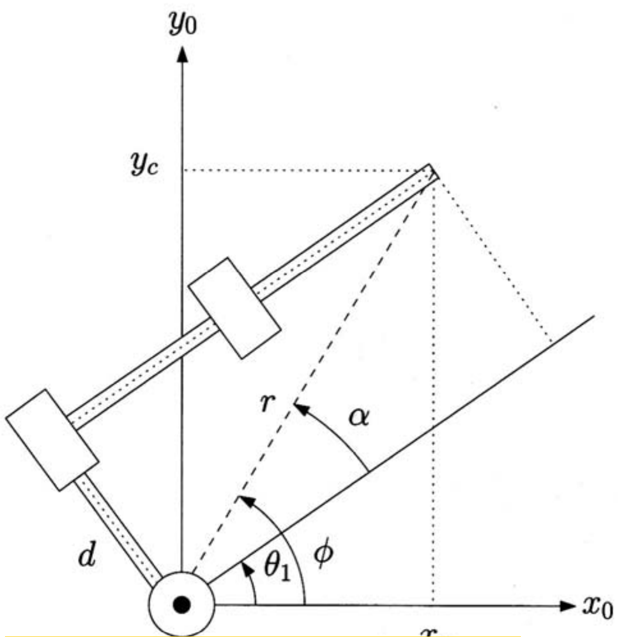
The wrist center P_c intersects z_0; Hence, any value of θ_1 leaves P_c fixed. There are thus infinitely many solutions for θ_1 when P_c intersects z_0.

Elbow manipulator: Shoulder offset

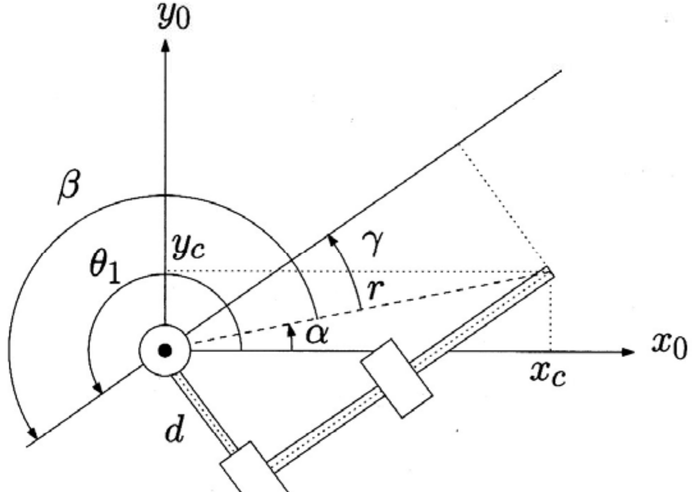


There will, in general, be only two solutions for θ_1.

These correspond to the so-called left arm and right arm configurations.



Left arm configuration



Right arm configuration

$$\theta_1 = \phi - \alpha$$

$$\phi = A\tan 2(y_c, x_c)$$

$$\alpha = A\tan 2\left(d, \sqrt{r^2 - d^2}\right)$$

$$= A\tan 2\left(d, \sqrt{x_c^2 + y_c^2 - d^2}\right)$$

$$\theta_1 = A\tan 2(y_c, x_c) + A\tan 2\left(-d, -\sqrt{r^2 - d^2}\right)$$

$$\theta_1 = \alpha + \beta$$

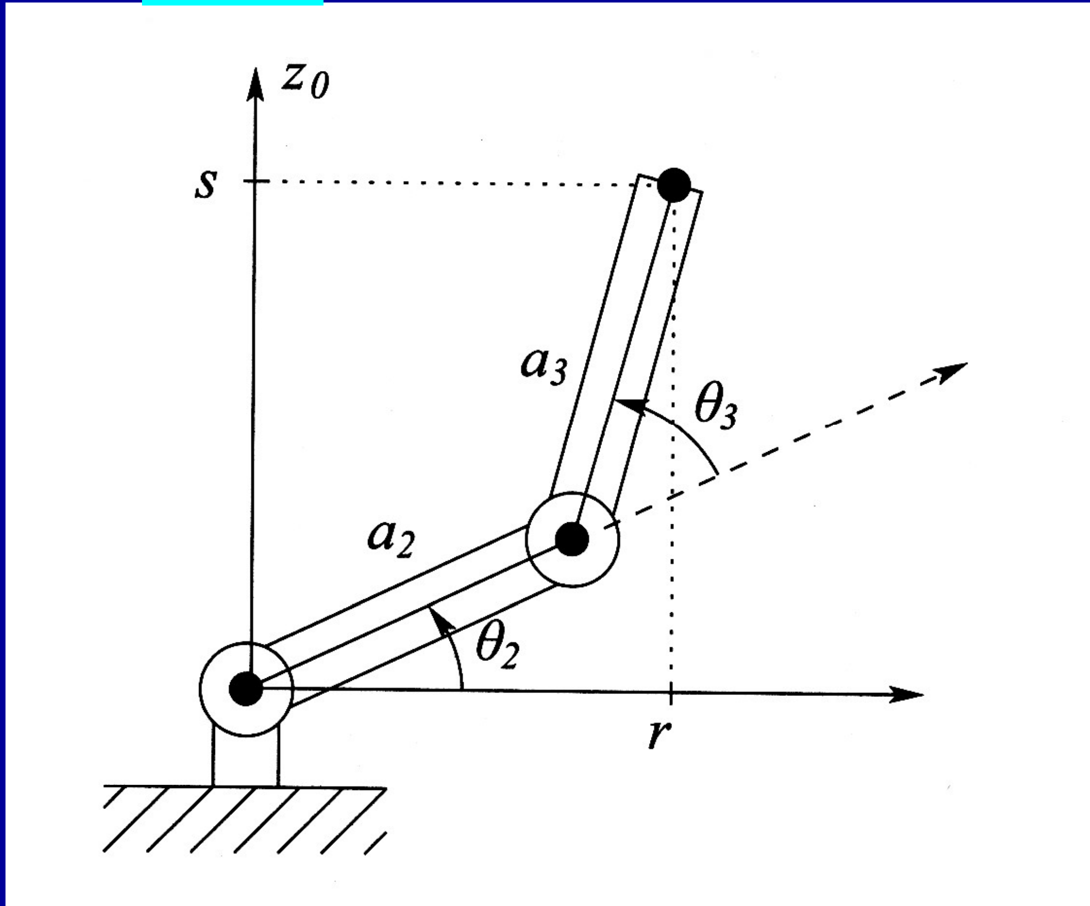
$$\alpha = A\tan 2(y_c, x_c)$$

$$\beta = \gamma + \pi$$

$$\gamma = A\tan 2\left(d, \sqrt{r^2 - d^2}\right)$$

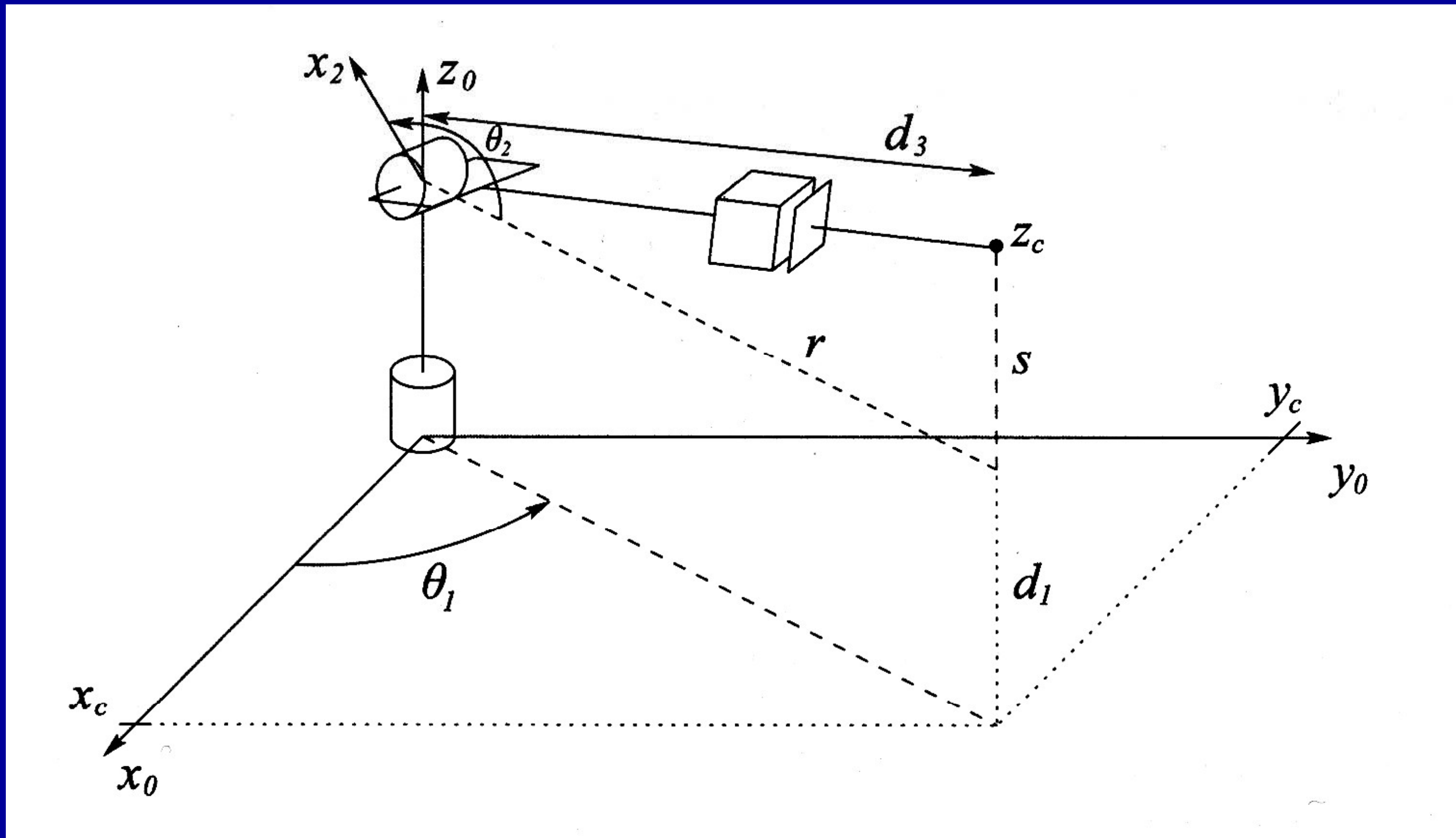
Homework #10 (1 pt.) – Due Jan. 13

Find the angles θ_2, θ_3 for the elbow manipulator given θ_1.



Homework #11 (1 pt.) – Due Jan. 13

Solve the inverse position kinematics for a 3 DOF spherical manipulator.



Algebraic solution by reduction to polynomial

Tangent of the half angle substitution

$$u = \tan\frac{\theta}{2},$$

$$\cos\theta = \frac{1-u^2}{1+u^2},$$

$$\sin\theta = \frac{2u}{1+u^2}.$$

Example 4.3

$$a\cos\theta + b\sin\theta = c$$

$$a(1-u^2) + 2bu = c(1+u^2)$$

$$(a+c)u^2 - 2bu + (c-a) = 0$$

$$u = \frac{b \pm \sqrt{b^2 - a^2 - c^2}}{a+c}$$

$$\theta = 2\tan^{-1}\left(\frac{b \pm \sqrt{b^2 - a^2 - c^2}}{a+c}\right).$$

Sine and Cosine of a Sum/Difference

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$\sin 2A=2\sin A\cos A$$
$$\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$$

Tangent of a Double Angle

$$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

$$\tan(2A) = \frac{\sin(2A)}{\cos(2A)} = \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

Divide both parts of the fraction by $\cos^2 A$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Sine and Cosine of a Half Angle

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$$

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = 2\cos^2(\theta/2) - 1$$

$$= 1 - 2\sin^2(\theta/2)$$

$$\sin^2(\theta/2) = \frac{1-\cos\theta}{2} \quad \rightarrow \sin(\theta/2) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\rightarrow \cos(\theta/2) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Tangent of a Half Angle

$$\tan(\theta/2) = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$\tan(\theta/2) = \sqrt{\frac{(1-\cos\theta)(1+\cos\theta)}{(1+\cos\theta)^2}} = \frac{\sqrt{1-\cos^2\theta}}{1+\cos\theta}$$

$$= \frac{\sqrt{\sin^2\theta}}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

$$\sin\theta = -\sin(-\theta) = -\cos(\theta + 90^\circ) = \cos(\theta - 90^\circ)$$

$$\cos\theta = \cos(-\theta) = \sin(\theta + 90^\circ) = -\sin(\theta - 90^\circ)$$

$$\tan(90^\circ + \theta) = -\cot\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\tan(\theta - 180^\circ) = \tan\theta$$

Pieper's solution when three axes intersect (*e.g.*, spherical wrists)

- A completely general robot with six degrees of freedom does not have a closed form solution.
- The technique involves decoupling the position and orientation problems. The position problem positions the wrist center, while the orientation problem completes the desired orientation.

When the last three axes intersect, the origins of link frames {4}, {5}, and {6} are all located at this point of intersection.

$$^{0}P_{4ORG} = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}P_{4ORG} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix} = {}^{0}_{1}T\,{}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

$${}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 1 \end{bmatrix} = {}^2_3T \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix} = \begin{bmatrix} a_3 c_3 + d_4 s\alpha_3 s_3 + a_2 \\ a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2 \\ a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2 \\ 1 \end{bmatrix}$$

$${}^0P_{4ORG} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix}, \quad \begin{aligned} g_1 &= c_2 f_1 - s_2 f_2 + a_1 \\ g_2 &= s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1 \\ g_3 &= s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1 \end{aligned}$$

$$\begin{aligned} r = x^2 + y^2 + z^2 &= g_1^2 + g_2^2 + g_3^2 \\ &= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 + 2a_1(c_2 f_1 - s_2 f_2) \end{aligned}$$

$$r = (k_1 c_2 + k_2 s_2) 2a_1 + k_3,$$
$$z = (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4,$$

Dependence on θ_1 *has been eliminated*
Dependence on θ_2 *takes a simple form*

g_3

$$k_1 = f_1,$$
$$k_2 = -f_2,$$
$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3,$$
$$k_4 = f_3 c\alpha_1 + d_2 c\alpha_1.$$

Three Cases:

$a_1 = 0 \rightarrow r = k_3$

$s\alpha_1 = 0 \rightarrow z = k_4$

Otherwise, eliminate s_2 and c_2

We can solve for $\theta_1, \theta_2, \theta_3. \Rightarrow$ can compute ${}^0_4R\big|_{\theta=0}$.

Given

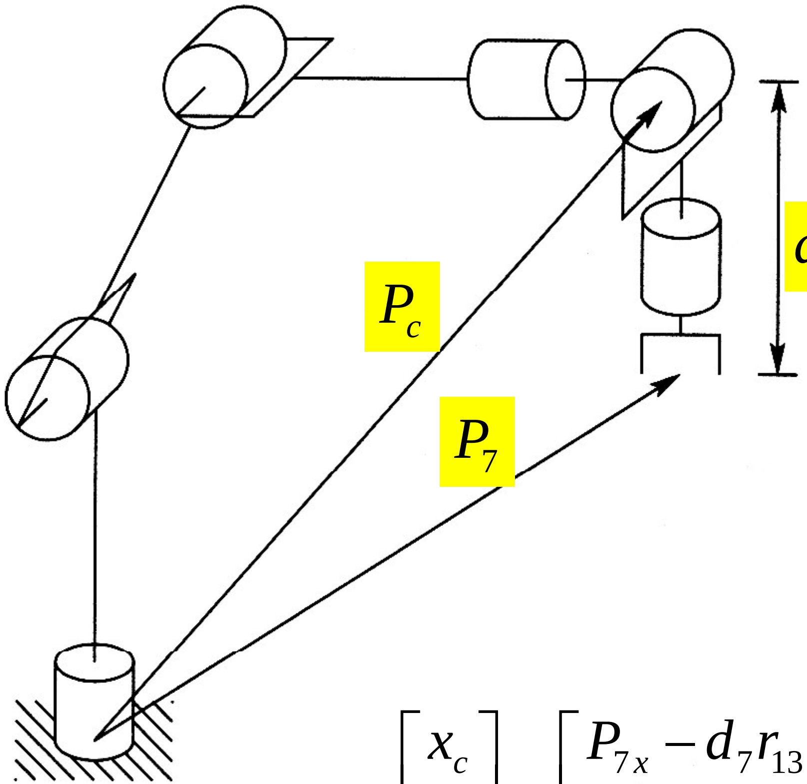
$${}^4_6R\big|_{\theta_4=0} = {}^0_4R^{-1}\big|_{\theta_4=0} \, {}^0_6R$$

Kinematic Decoupling

- Decouple the inverse kinematics problem into two simpler problems: inverse position kinematics, inverse orientation kinematics
- First finding the position of the intersection of the wrist axes, then finding the orientation of the wrist

Kinematic Decoupling

P_c *A function of only the first three joint variables*



$${}^0_6P(\theta_1,\ldots,\theta_6) = P_{des}$$

$${}^0_6R(\theta_1,\ldots,\theta_6) = R_{des}$$

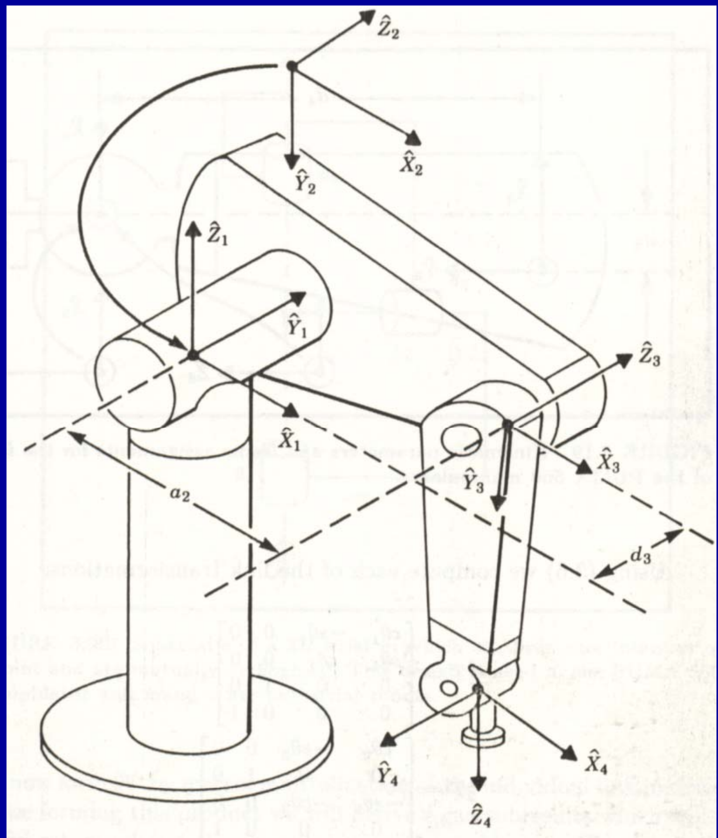
The direction of z_6 *w.r.t.* $\{0\}$

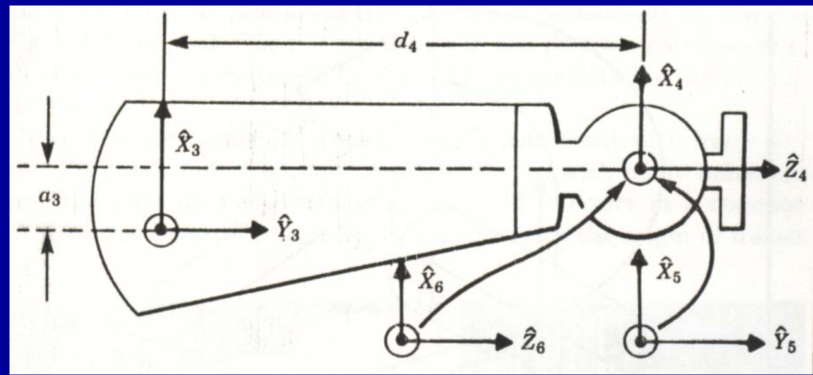
$$P_{des} = {}^0_cP + d_7 R_{des}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$${}^0_cP = P_{des} - d_7 R_{des}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}x_c\\y_c\\z_c\end{bmatrix} = \begin{bmatrix}P_{7x} - d_7 r_{13}\\P_{7y} - d_7 r_{23}\\P_{7z} - d_7 r_{33}\end{bmatrix}, \quad R_{des} = {}^0_3R\,{}^3_7R \rightarrow {}^3_7R = {}^0_3R^{-1} R_{des} = {}^0_3R^T R_{des}$$

Example: PUMA 560 (Refer to the textbook)





Repeatability and Accuracy

- Repeatability: how precisely a manipulator can return to a taught point?
- Accuracy: the precision with which a computed point can be attained.