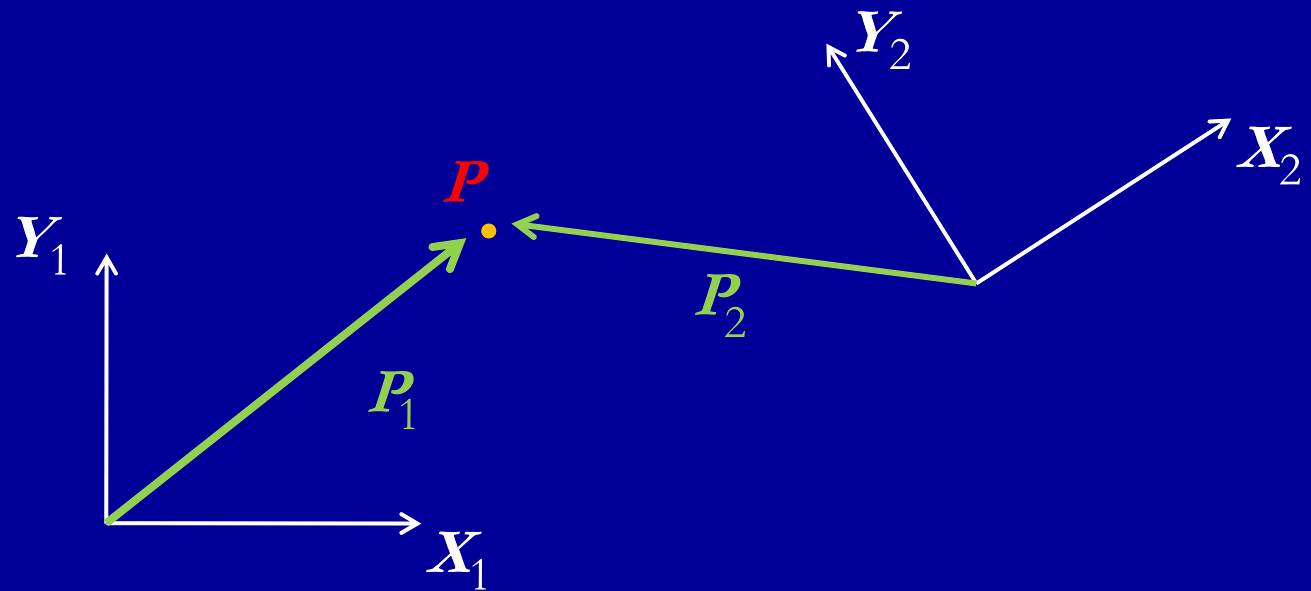
Spatial Descriptions and Transformations

December 14, 2006

Representing positions and orientations

- Define coordinate systems and develop conventions for representation
- A universe coordinate system

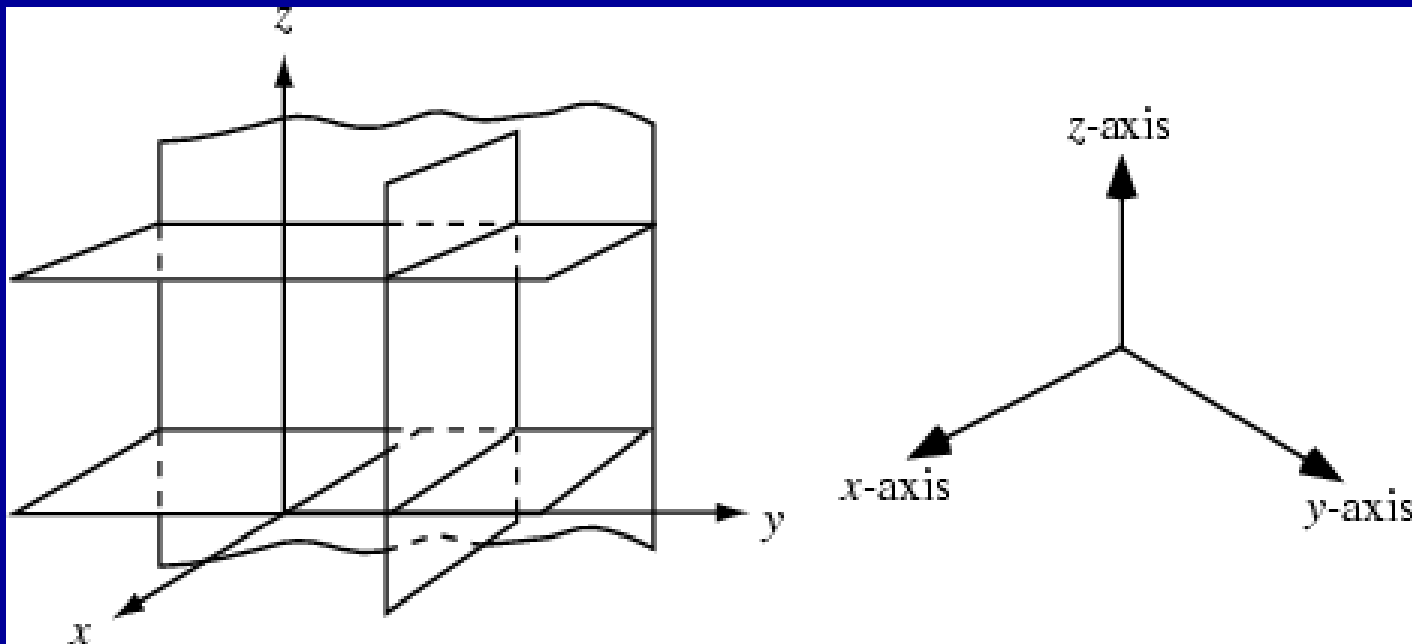


Coordinate System

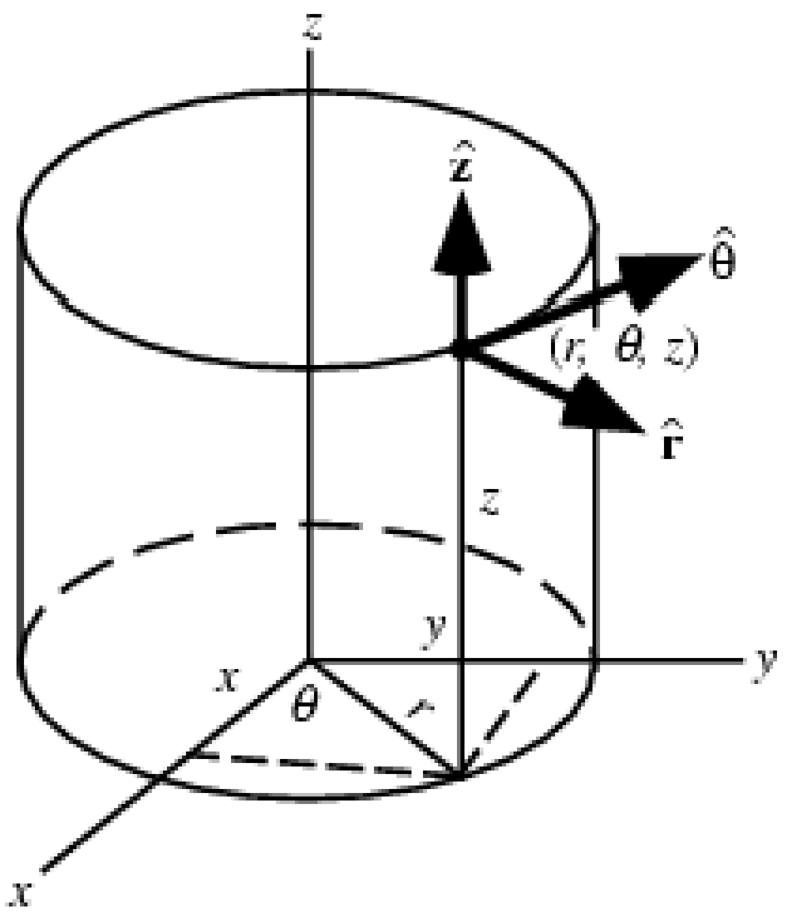
- A system for specifying points using coordinates measured in some specified way. Depending on the type of problem under consideration, coordinate systems possessing special properties may allow particularly simple solution.

Cartesian (*or* rectangular) **Coordinates**

- The simplest coordinate system consisting of coordinate axes oriented perpendicularly to each other.



Cylindrical Coordinates



$$r = \sqrt{x^2 + y^2}, \quad r \in [0, \infty)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \quad \theta \in [0, 2\pi)$$

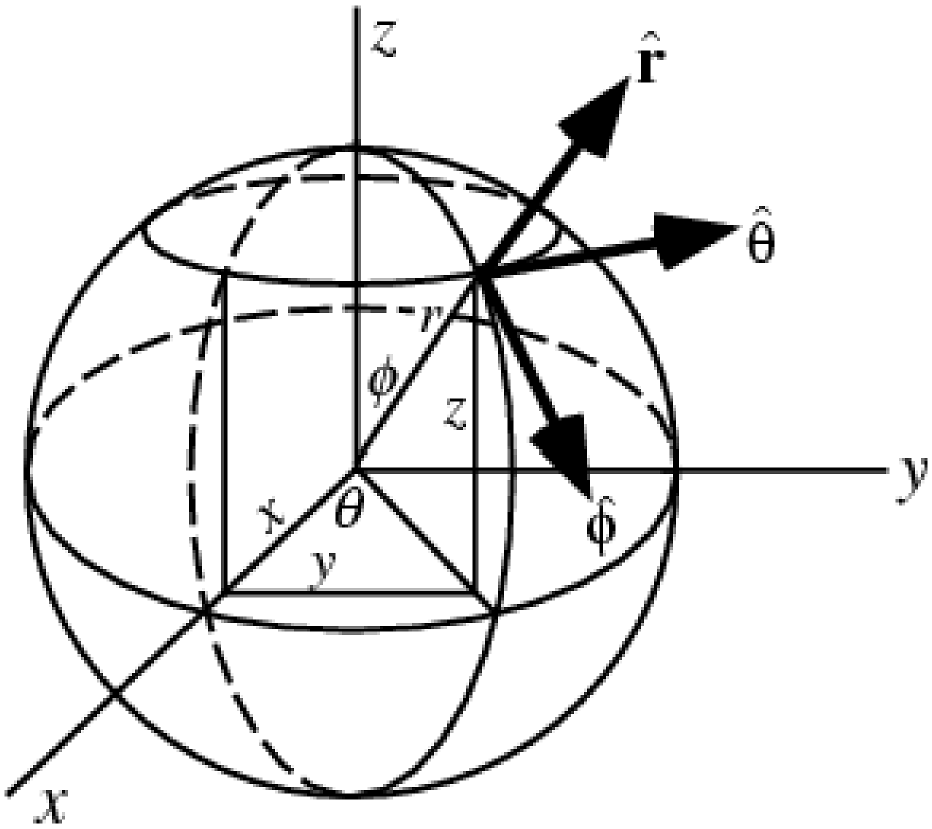
$$z = z, \quad z \in (-\infty, \infty)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Spherical Coordinates



$$r = \sqrt{x^2 + y^2 + z^2}, \quad r \in [0, \infty)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \quad \theta \in [0, 2\pi)$$

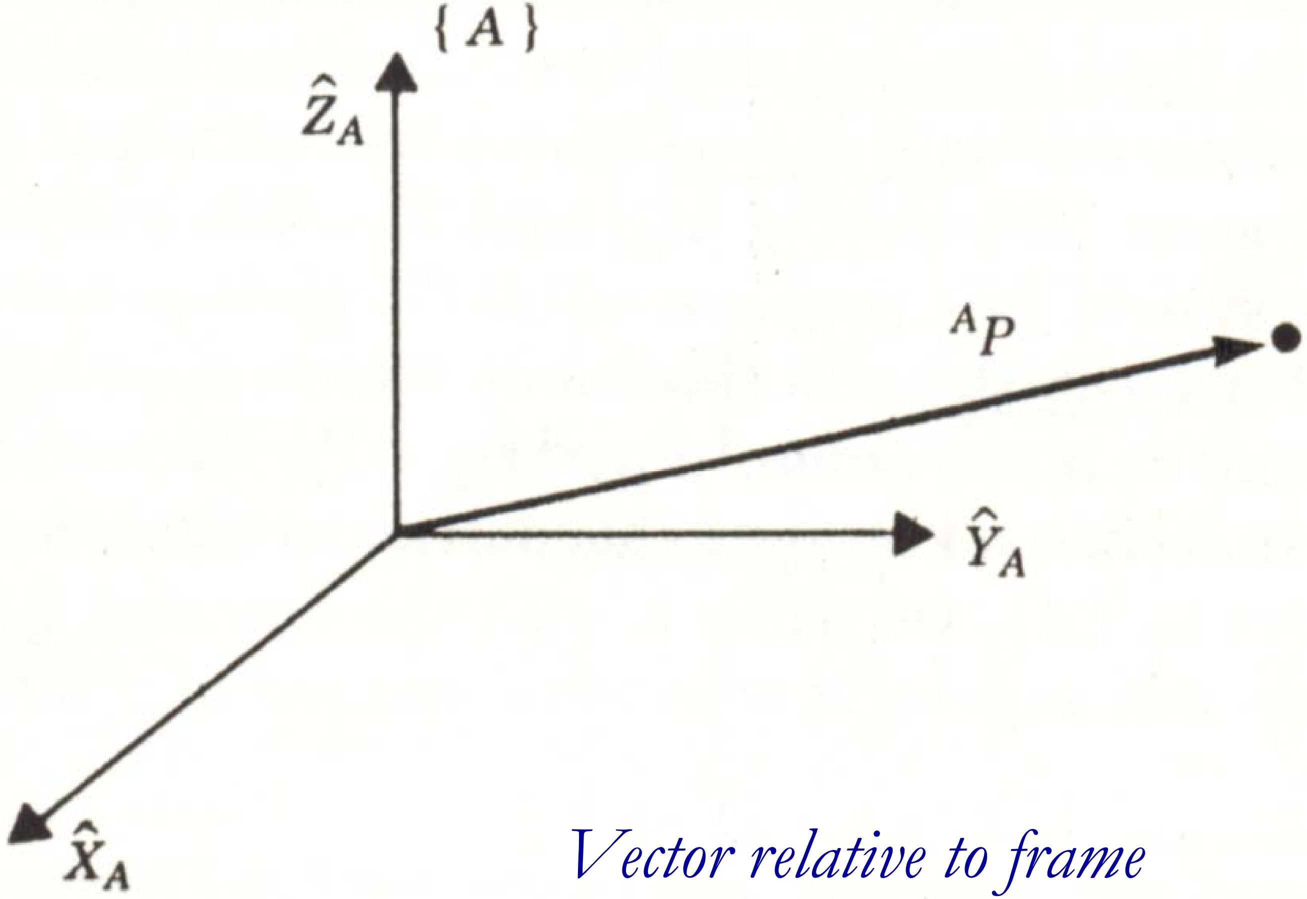
$$\phi = \cos^{-1}\left(\frac{z}{r}\right), \quad \phi \in [0, \pi)$$

$$x = r \sin\phi \cos\theta$$

$$y = r \sin\phi \sin\theta$$

$$z = r \cos\phi$$

Description of a position



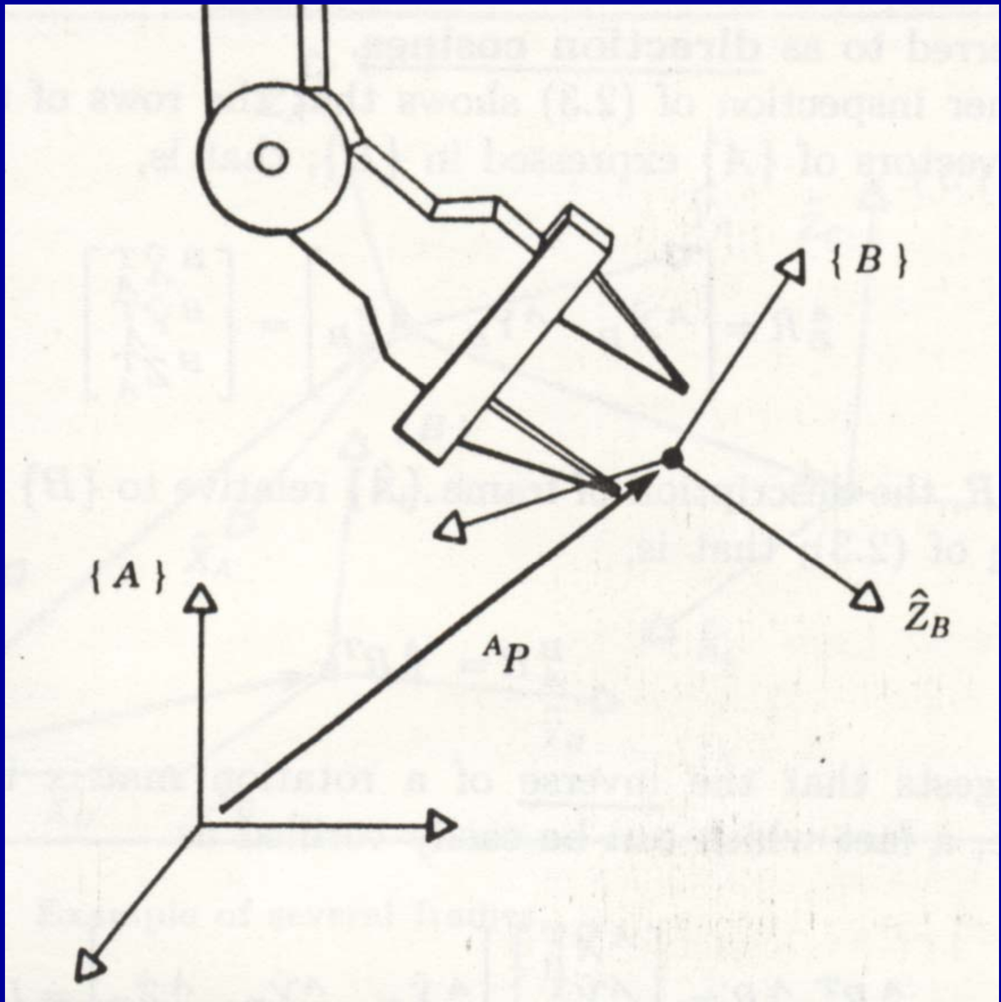
Vector relative to frame

Location

- A 3 x 1 position vector
- A leading superscript indicates the coordinate system

$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Description of an orientation



Positions of points are described with *vectors* and *orientations* of bodies are described with an *attached coordinate system*.

Description of an orientation

- In order to describe the orientation of a body, we will attach a coordinate system to the body and then give a description of this coordinate system relative to the reference system.
- One way to describe the body-attached coordinate system, $\{B\}$, is to write the unit vectors of its three principal axes in terms of the coordinate system $\{A\}$.

$${}^{A}\hat{X}_{B},\quad {}^{A}\hat{Y}_{B},\quad {}^{A}\hat{Z}_{B}$$

Rotation matrix

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. $$

The orientation of a body is represented with a *matrix*.

Note that the components of any vector are simply the *projections* of that vector onto the unit directions of its reference frame.

$$
{}^{A}_{B}R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}.
$$

Direction cosines

$$
{}^{A}_{B}R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}.
$$

$$
{}^{B}_{A}R = \begin{bmatrix} \hat{X}_A \cdot \hat{X}_B & \hat{Y}_A \cdot \hat{X}_B & \hat{Z}_A \cdot \hat{X}_B \\ \hat{X}_A \cdot \hat{Y}_B & \hat{Y}_A \cdot \hat{Y}_B & \hat{Z}_A \cdot \hat{Y}_B \\ \hat{X}_A \cdot \hat{Z}_B & \hat{Y}_A \cdot \hat{Z}_B & \hat{Z}_A \cdot \hat{Z}_B \end{bmatrix}.
$$

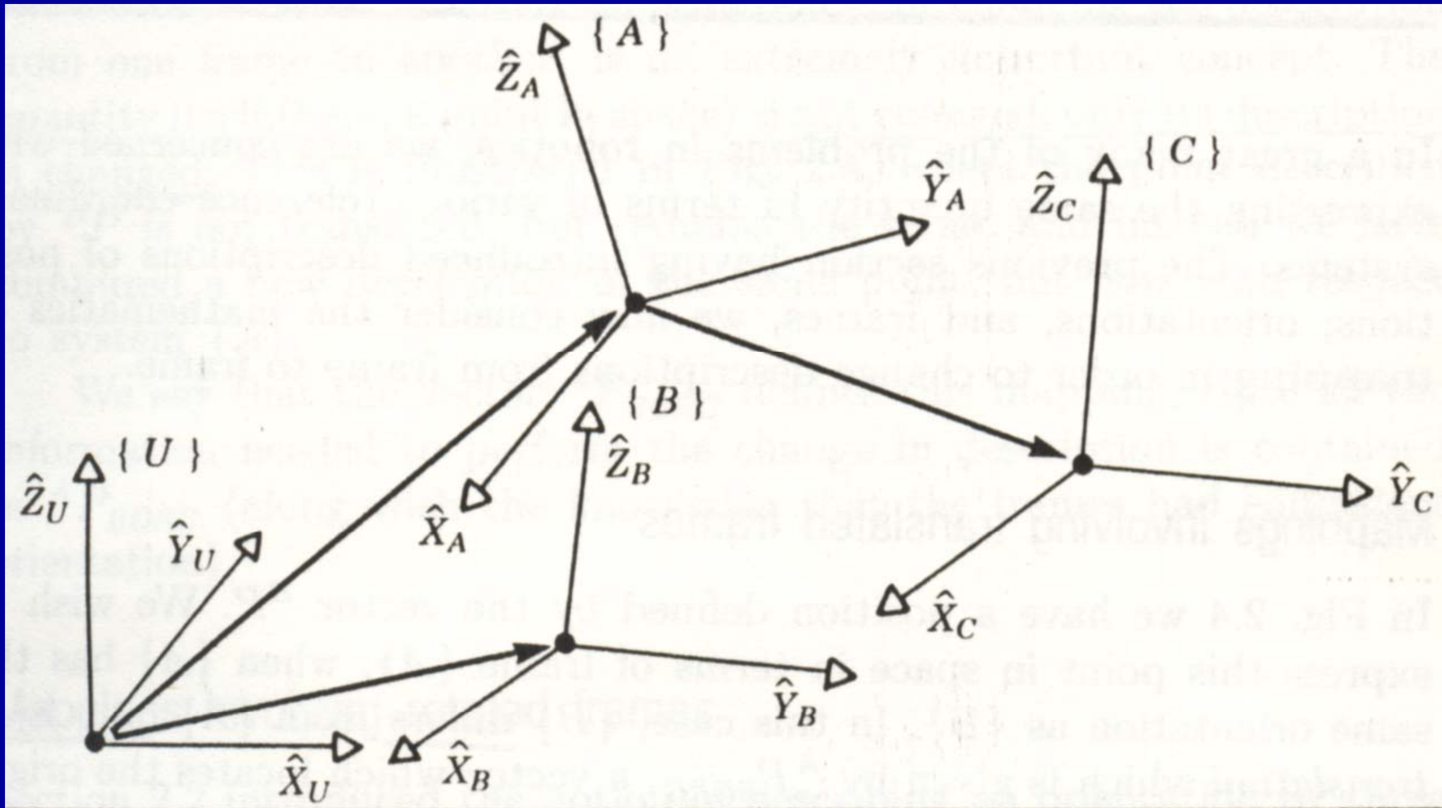
$${}^{B}_{A}R = {}^{A}_{B}R^{T}$$

$${}^{A}_{B}R^{T}\,{}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B}^{T} \\ {}^{A}\hat{Y}_{B}^{T} \\ {}^{A}\hat{Z}_{B}^{T} \end{bmatrix} \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = I_{3}$$

$${}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T}.$$

The inverse of a matrix with orthonormal columns is equal to its transpose.

Description of a frame: *graphical representation*



For convenience, the point whose position we will describe is chosen as the origin of the body-attached frame.

A frame: *a position and an orientation pair*

- A set of four vectors giving position and orientation information.
- A position vector and a rotation matrix
- Frame $\{B\}$ is described by

$$\{B\} = \{{}^{A}_{B}R, \quad {}^{A}P_{BORG}\}.$$

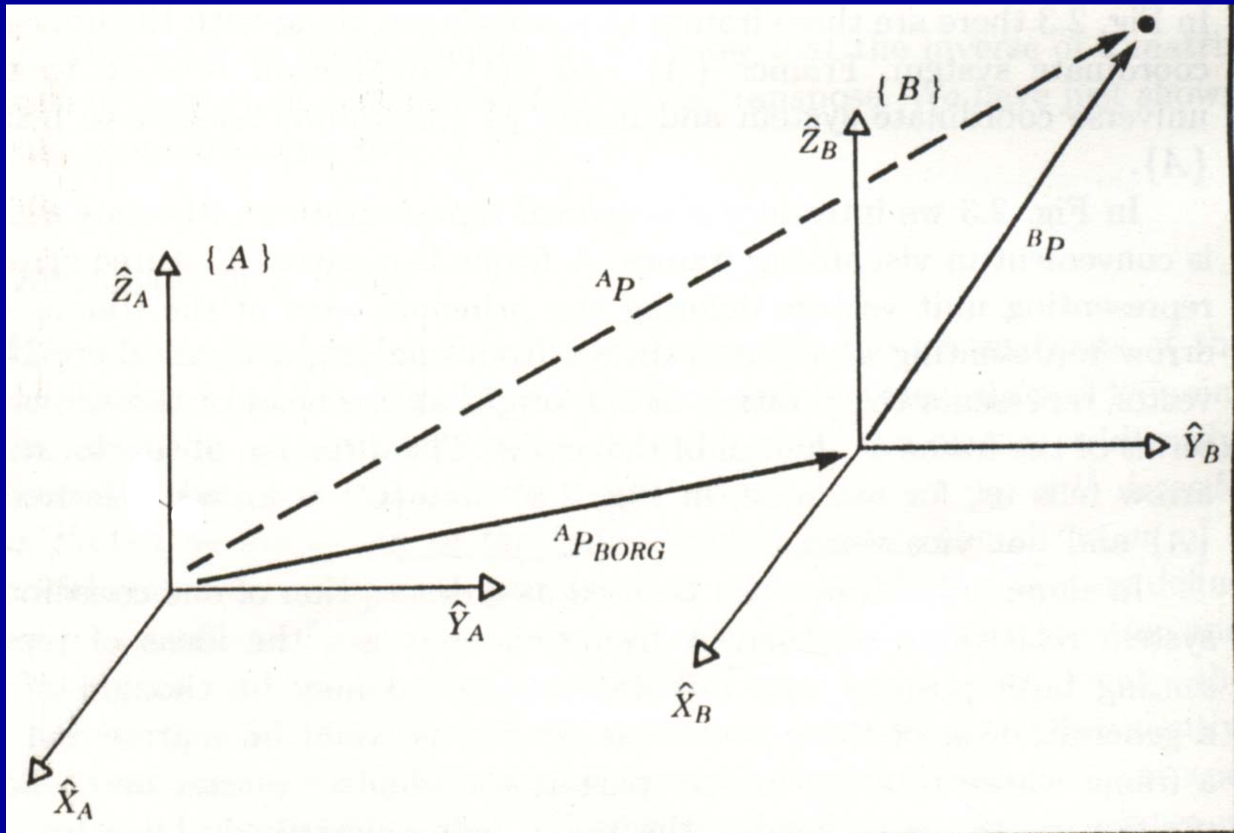
Mapping

- Changing the description from one frame to another
- The quantity itself is not changed; only its description is changed.

$$ {}^{A}_{B}T \textbf{\textit{ maps}} $$

$$ {}^{B}P \rightarrow {}^{A}P. $$

Mappings involving translated frames

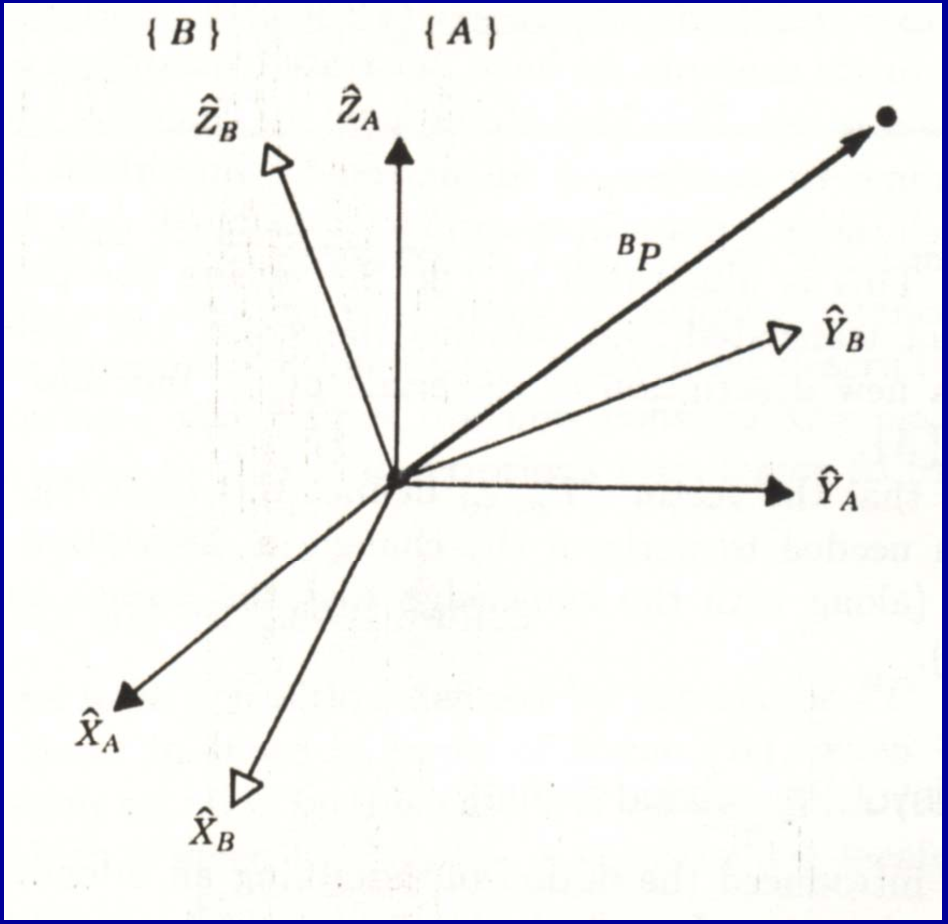


Translational mapping

$$ {}^{A}P = {}^{B}P + {}^{A}P_{BORG} $$

Both vectors are defined relative to frames of the same orientation.

Mappings involving rotated frames

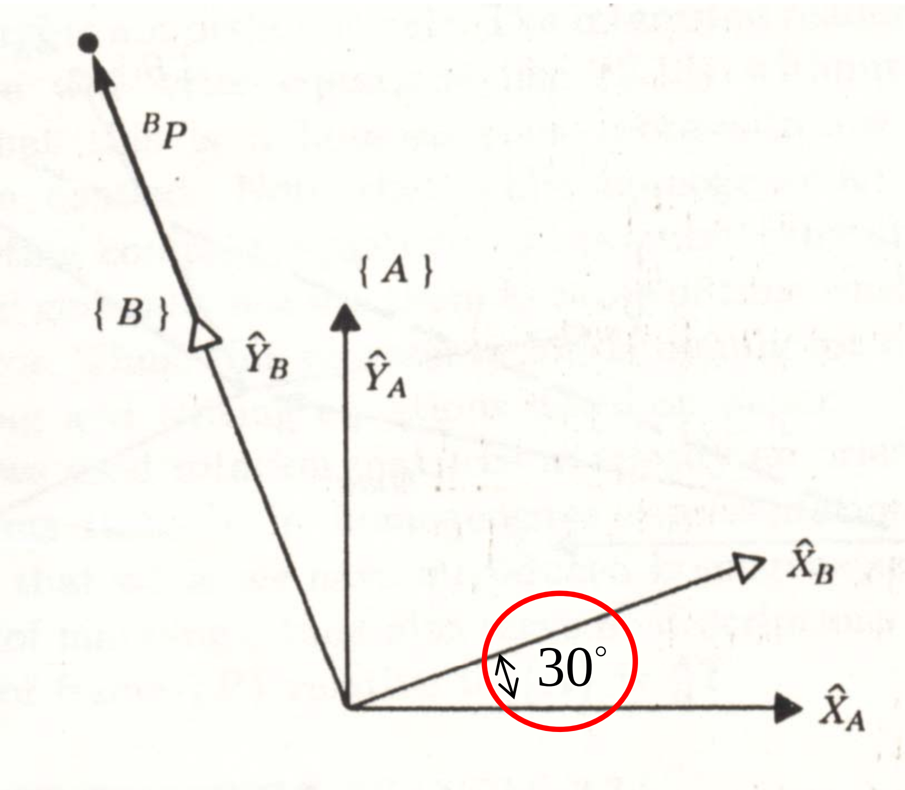


Rotating the description of a vector

$$ {}^{A}P = {}^{A}_{B}R \, {}^{B}P $$

The origins of the two frames are coincident.

Example 2.1

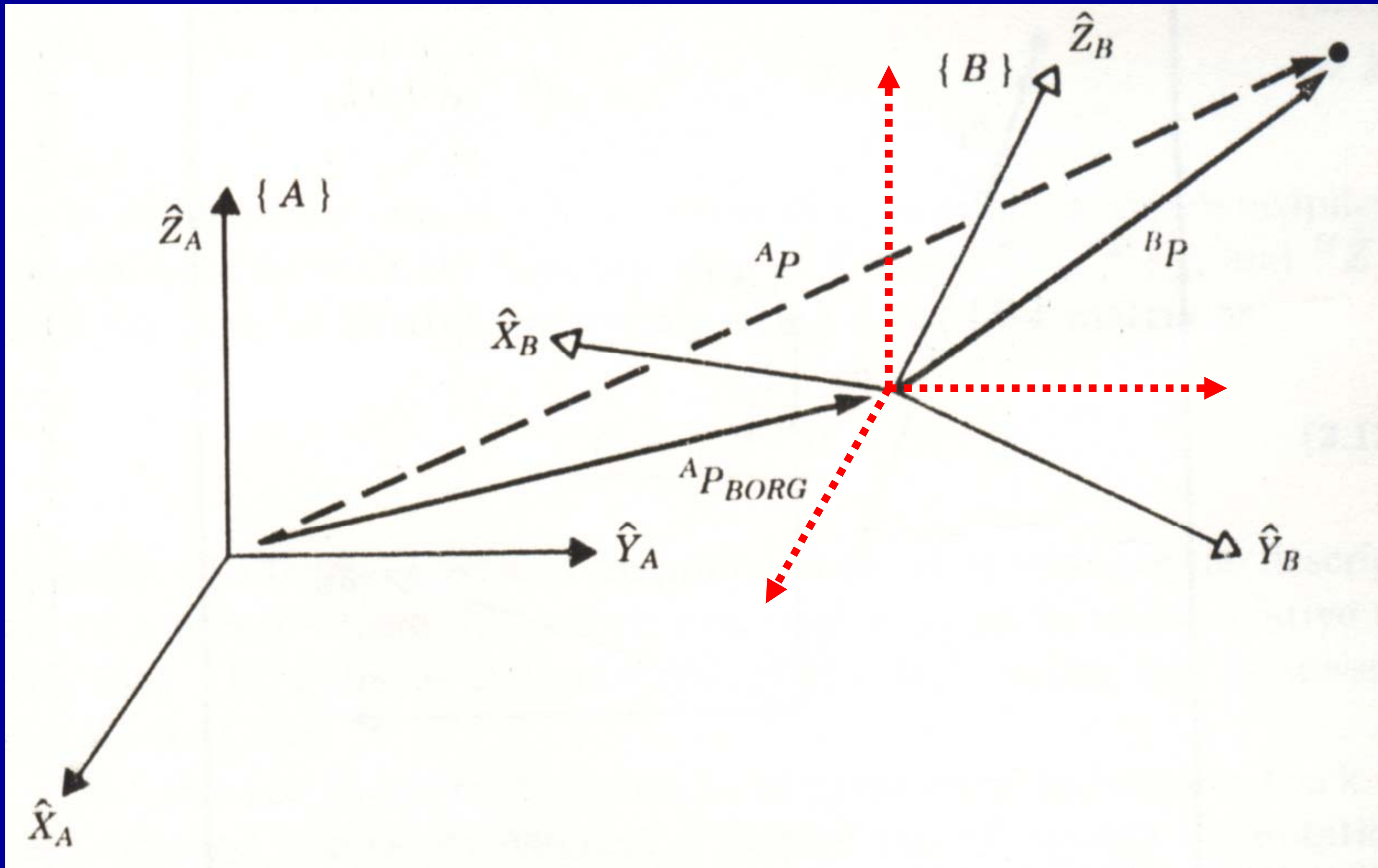


$$ {}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

$$ {}^{B}P = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix} $$

$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix} $$

Mappings involving general frames



General transform of a vector

$$ {}^{A}P = {}^{A}_{B}R \, {}^{B}P + {}^{A}P_{BORG} $$

$$ {}^{A}P = {}^{A}_{B}T \, {}^{B}P $$

an operator in matrix form

A Homogeneous Transform

Combines the operations of rotation and translation into a single matrix multiplication

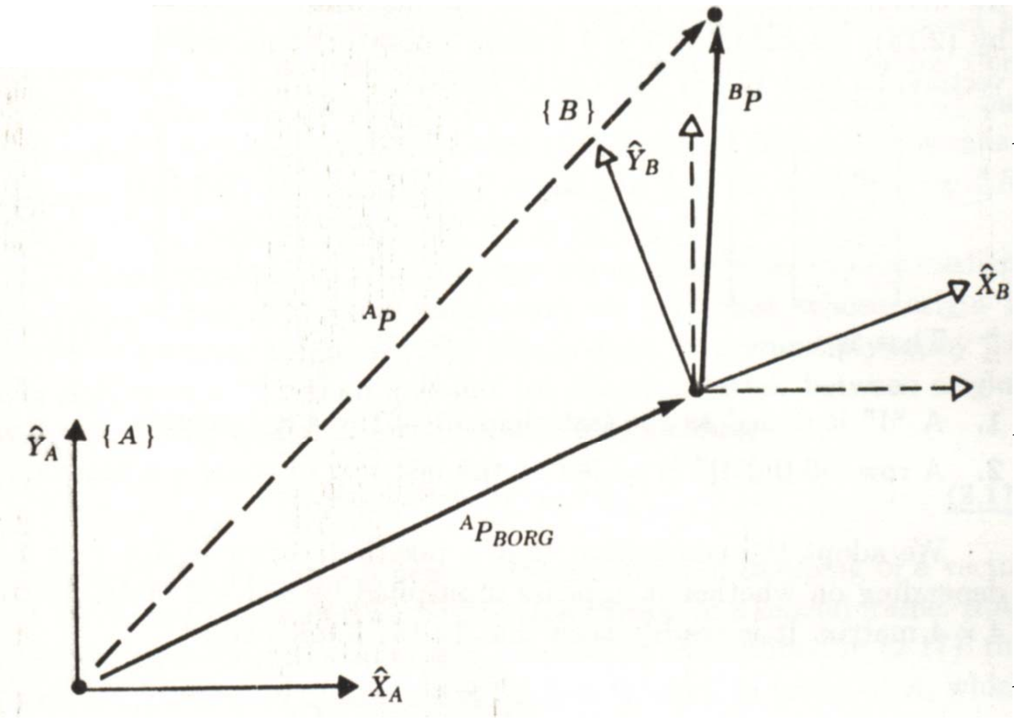
$$\begin{bmatrix} {}^{A}P \\ \cdots \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^{A}_{B}R & & \vdots & {}^{A}P_{BORG} \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ \cdots \\ 1 \end{bmatrix}$$

$${}^{A}_{B}T$$

A Homogeneous Transformation

- Simultaneously represent the position and orientation of one coordinate frame relative to another.
- Can be used to perform coordinate transformations.

Example 2.2



$$
{}^A_B T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^B P = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix}
$$

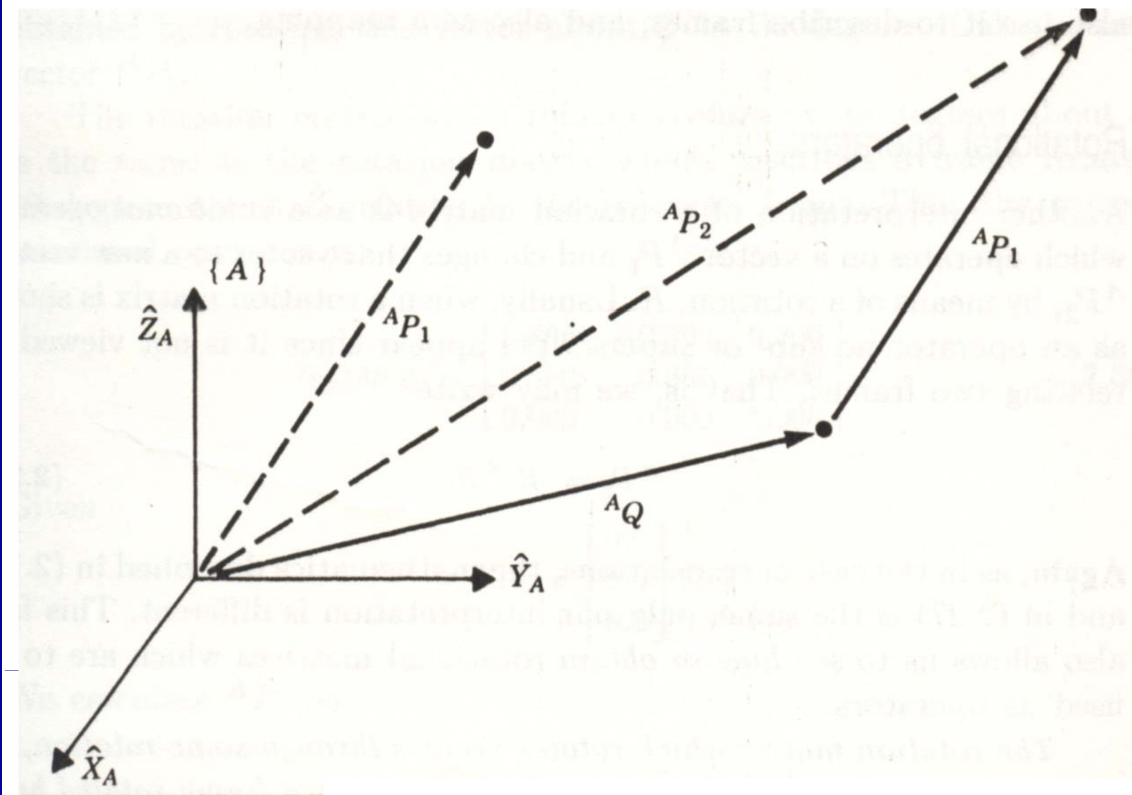
$$
{}^A P = {}^A_B T \, {}^B P = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}
$$

Translating

- A translation moves a point in space a finite distance along a given vector direction.
- Only one coordinate system need be involved.

T operates on ${}^{A}P_2$ to create ${}^{A}P_1$.

Translational operators



$$^AP_2 = {}^AP_1 + {}^AQ$$

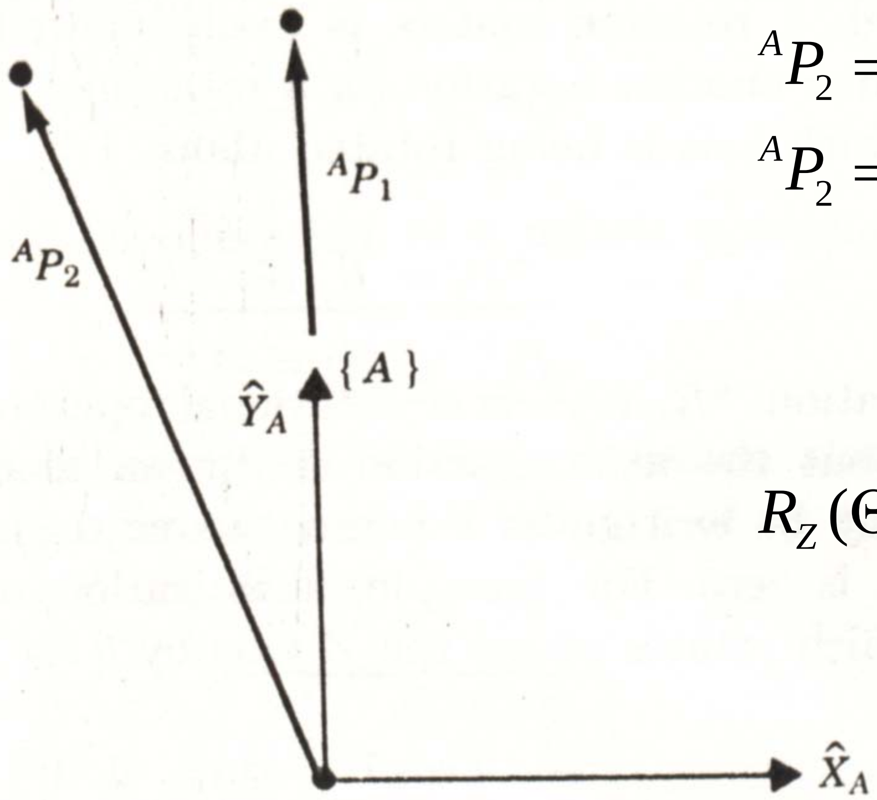
$$^AP_2 = D_Q(q)\,{}^AP_1$$

$$D_Q(q) = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$q = \sqrt{q_x^2 + q_y^2 + q_z^2}$$

Translating a point in space is accomplished with the same mathematics as mapping the point to a second frame.

Rotational Operators: Example 2.3



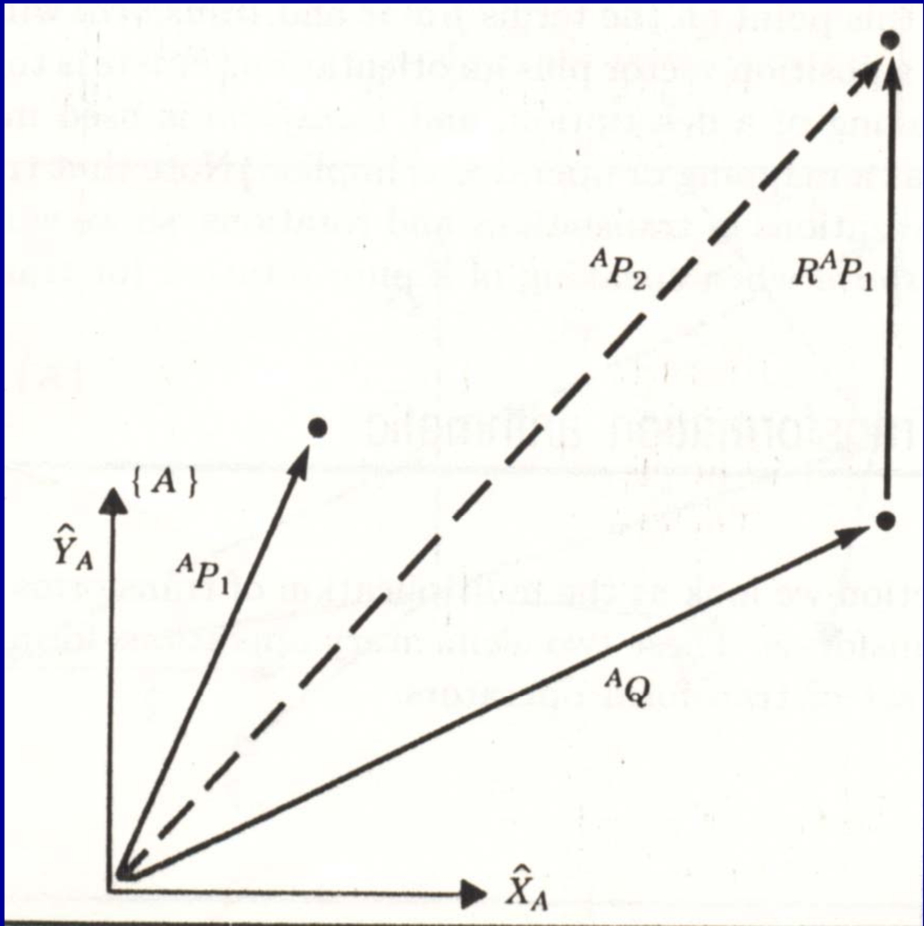
$$^AP_2 = R\,^AP_1$$

$$^AP_2 = R_K(\theta)\,^AP_1$$

$$R_Z(\Theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A homogeneous transform

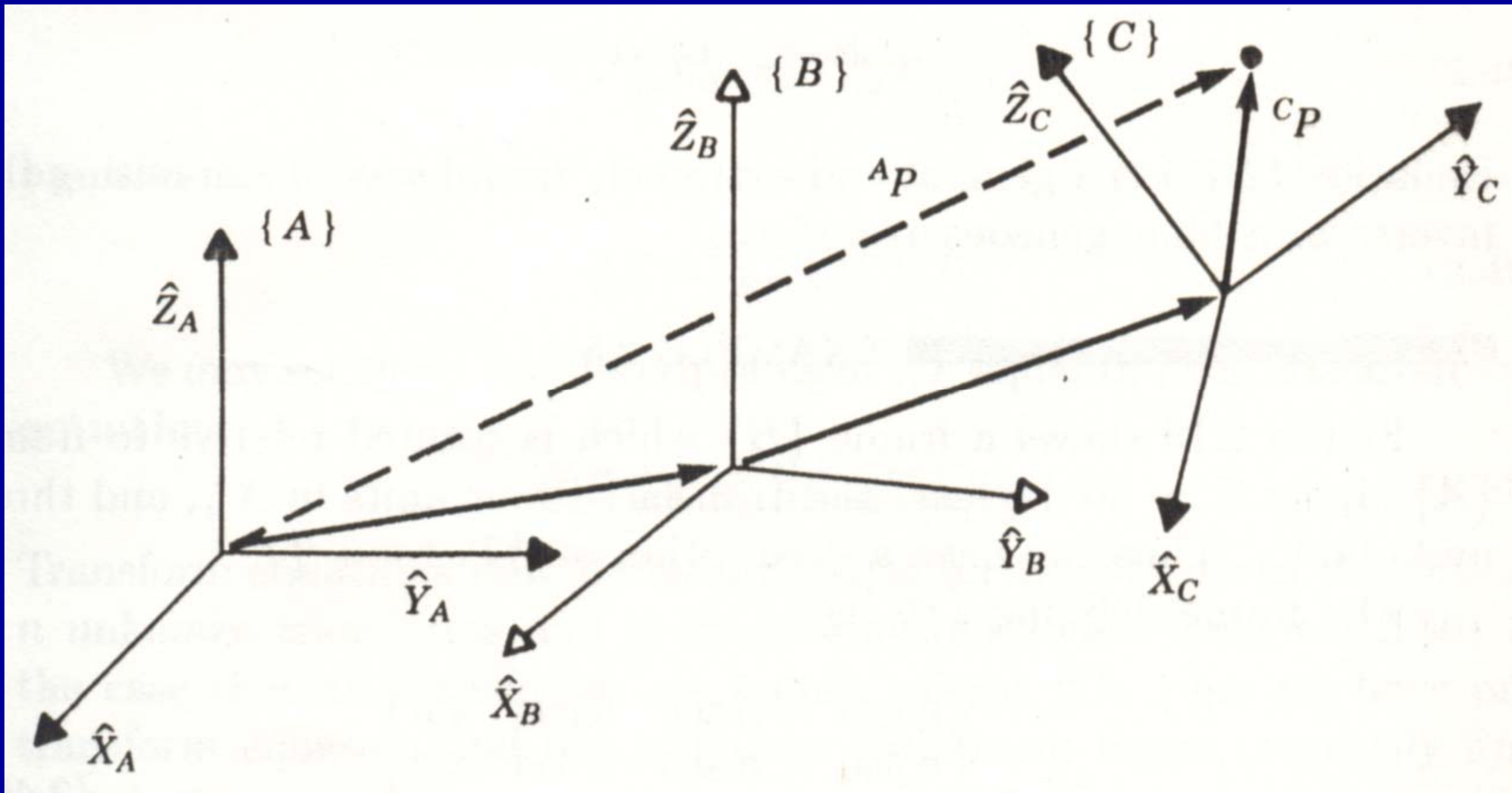
Transformation operators: Example 2.4



The transform that rotates by *R* and translates by *Q* is the same as the transform that describes a frame rotated by *R* and translated by *Q* relative to the reference frame.

$$^AP_2 = T\,^AP_1$$

Compound transformations



$$ {}^{B}P = {}^{B}_{C}T \, {}^{C}P, $$

$$ {}^{A}P = {}^{A}_{B}T \, {}^{B}P. $$

$$ {}^{A}P = {}^{A}_{B}T \, {}^{B}_{C}T \, {}^{C}P, $$

$$ {}^{A}_{C}T = {}^{A}_{B}T \, {}^{B}_{C}T. $$

$$
{}^{A}_{C}T = \begin{bmatrix} & {}^{A}_{B}R\,{}^{B}_{C}R & & \vdots & {}^{A}_{B}R\,{}^{B}P_{CORG} + {}^{A}P_{BORG} \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}
$$

Inverting a transform

$$
{}^{B}_{A}T = \begin{bmatrix} & {}^{B}_{A}R & & \vdots & {}^{B}P_{AORG} \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}
$$

$$
{}^{B}_{A}T = {}^{A}_{B}T^{-1}
$$

$${}^{B}_{A}R = {}^{A}_{B}R^{T}$$

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$

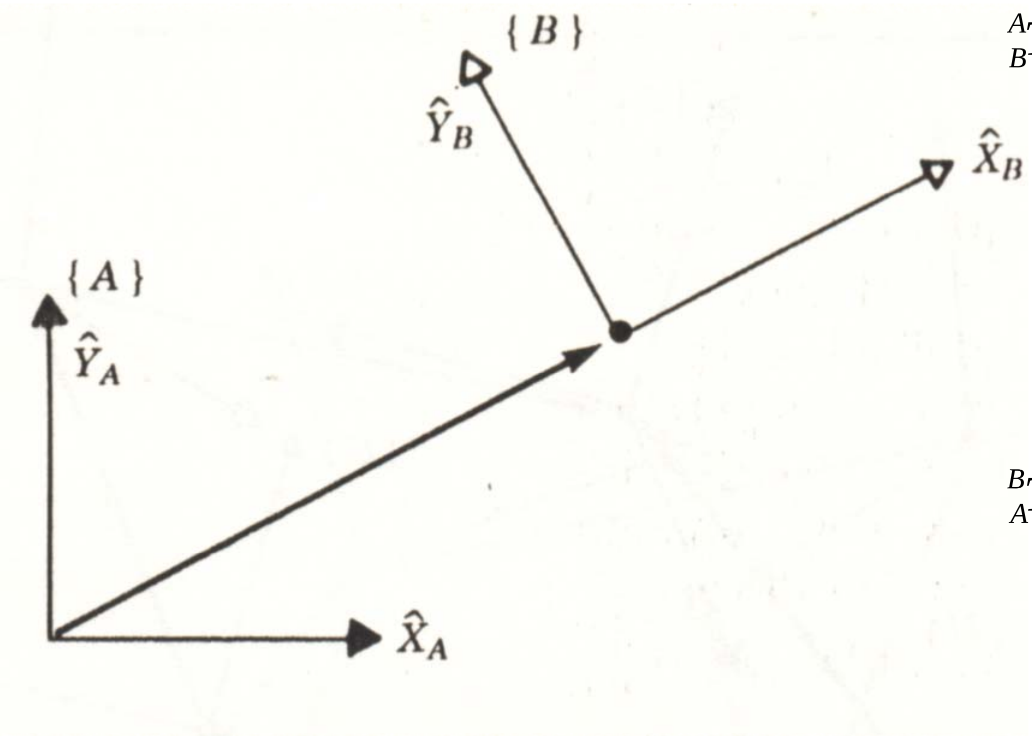
$$\mathbf{0}$$

$${}^{B}({}^{A}P_{BORG}) = {}^{B}_{A}R\,{}^{A}P_{BORG} + {}^{B}P_{AORG}$$

$${}^{B}P_{AORG} = -{}^{B}_{A}R\,{}^{A}P_{BORG} = -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG}$$

$$
{}_{A}^{B}T = \begin{bmatrix} & {}_{B}^{A}R^{T} & & \vdots & -{}_{B}^{A}R^{T}\,{}^{A}P_{BORG} \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}
$$

Example 2.5

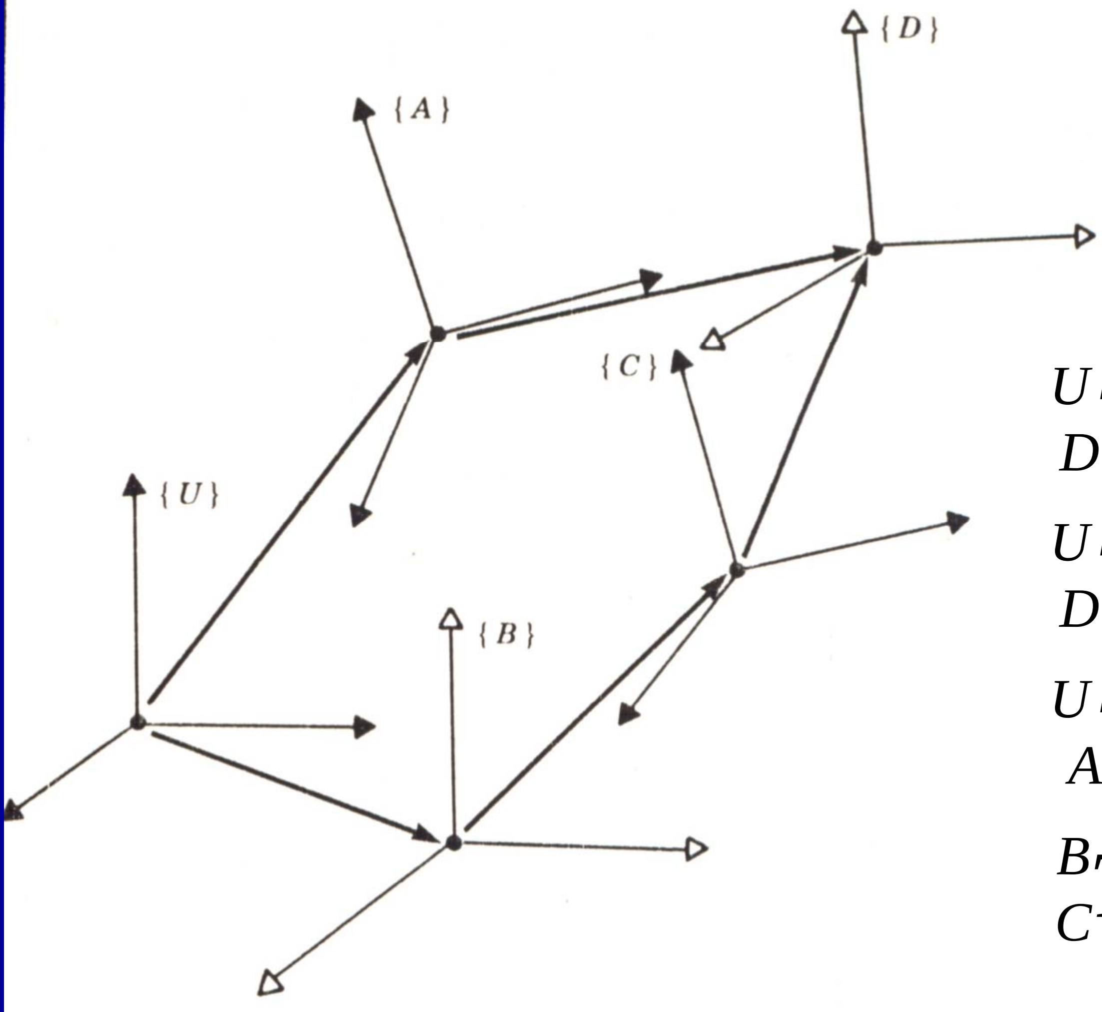


$$ {}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 4.0 \\ 0.500 & 0.866 & 0.000 & 3.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}_{A}T = \begin{bmatrix} 0.866 & 0.500 & 0.000 & -4.964 \\ -0.500 & 0.866 & 0.000 & -0.598 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Check by MATLAB.

Transform equations



$$ {}^{U}_{D}T = {}^{U}_{A}T\,{}^{A}_{D}T, $$

$$ {}^{U}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T. $$

$$ {}^{U}_{A}T\,{}^{A}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T. $$

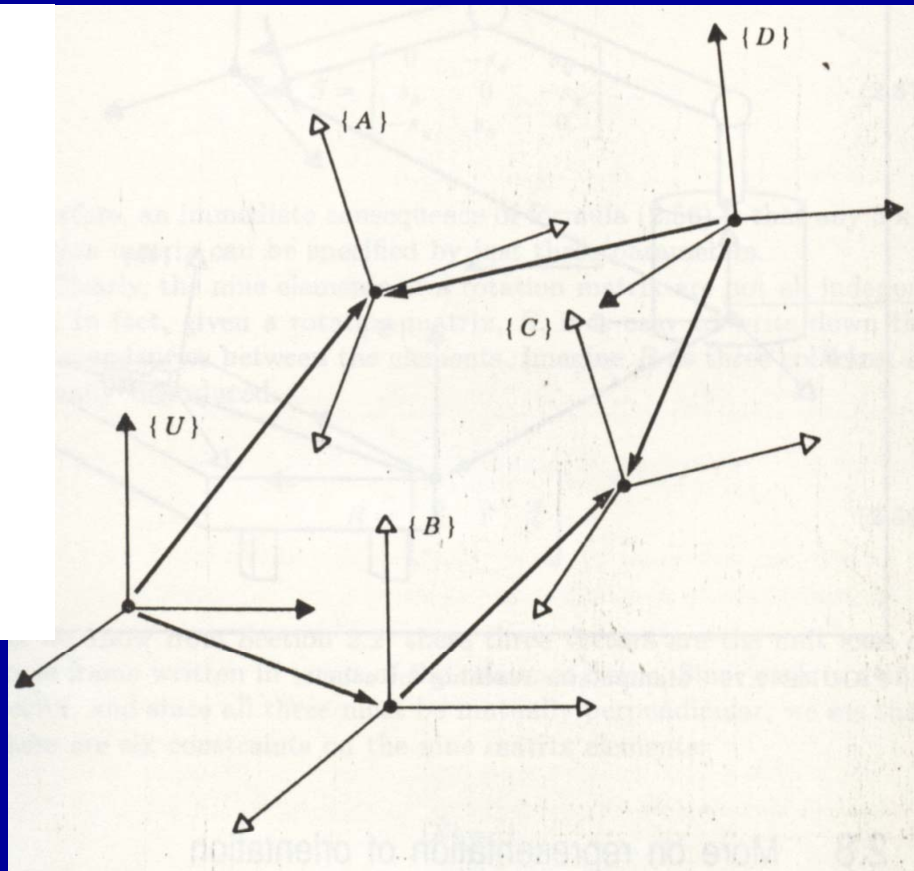
$$ {}^{B}_{C}T = {}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T\,{}^{C}_{D}T^{-1}. $$

Example of a transform equation

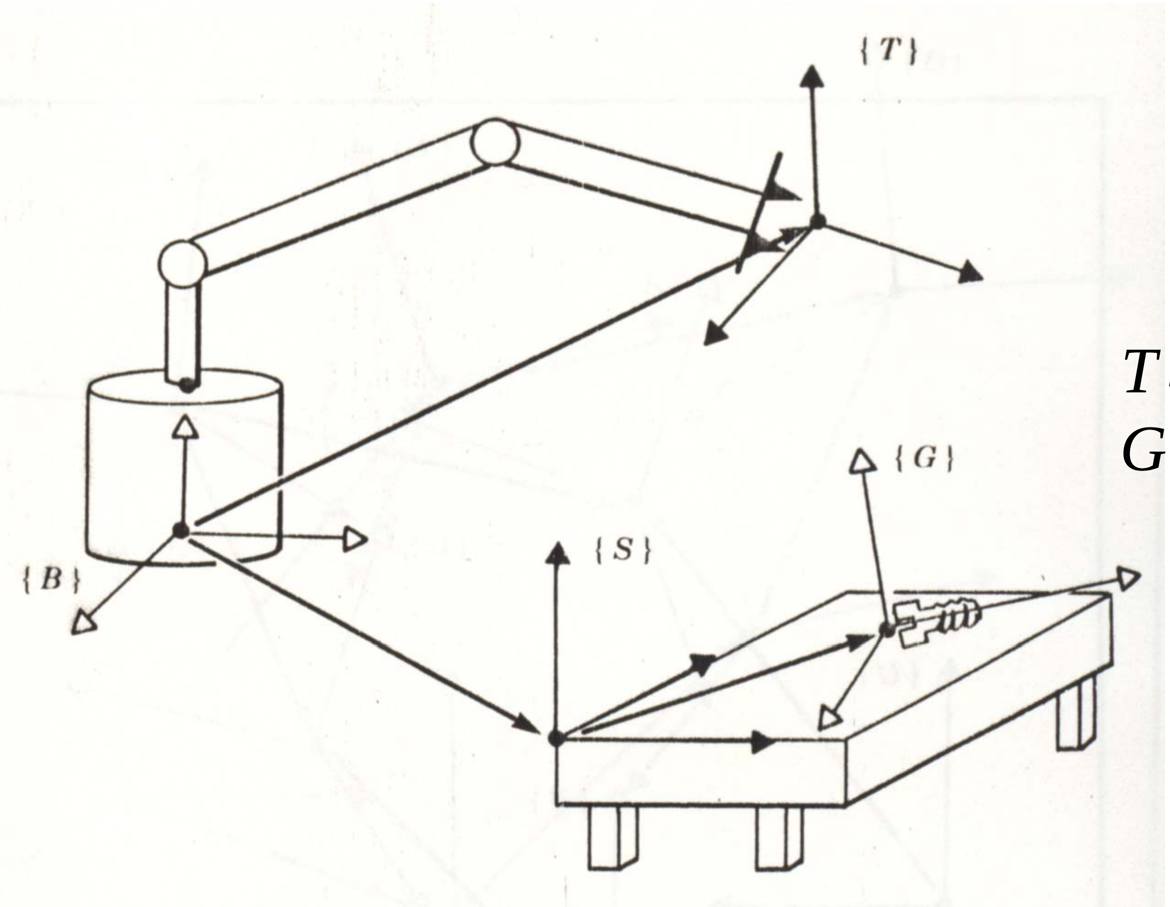
$$^{U}_{C}T = {}^{U}_{A}T\,{}^{D}_{A}T^{-1}\,{}^{D}_{C}T,$$

$$^{U}_{C}T = {}^{U}_{B}T\,{}^{B}_{C}T.$$

$$^{U}_{A}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{D}_{C}T^{-1}\,{}^{D}_{A}T.$$



Example 2.6: Manipulator reaching for a bolt ${}^{T}_{G}T$



$${}^{T}_{G}T = {}^{B}_{T}T^{-1}\,{}^{B}_{S}T\,{}^{S}_{G}T.$$

Proper orthonormal matrices

- Rotation matrices are special in that all columns are mutually orthogonal and have unit magnitude.
- The determinanat of a rotation matrix is always equal to +1.

Special Orthogonal group of order n

For any $R \in SO(n)$ the following properties hold.

$$R^T = R^{-1} \in SO(n)$$

The columns (and therefore the rows) of R are mutually orthogonal.

Each column (and therefore each row) of R is a unit vector.

$\det R = 1$.

For any proper orthonormal matrix R, there exists a skew-symmetric matrix such that

$$R = (I_3 - S)^{-1}(I_3 + S),$$

$$S = \begin{bmatrix} 0 & -s_x & s_y \\ s_x & 0 & -s_z \\ -s_y & s_z & 0 \end{bmatrix}$$

Any 3 x 3 rotation matrix can be specified by just three parameters.

Proper orthonormal matrices

Six constraints on the nine matrix elements:

$$\left|\hat{X}\right| = 1, \left|\hat{Y}\right| = 1, \left|\hat{Z}\right| = 1,$$

$$\hat{X} \cdot \hat{Y} = 0, \hat{X} \cdot \hat{Z} = 0, \hat{Y} \cdot \hat{Z} = 0.$$

Each is a unit vector, and all three must be mutually perpendicular.

Example 2.7

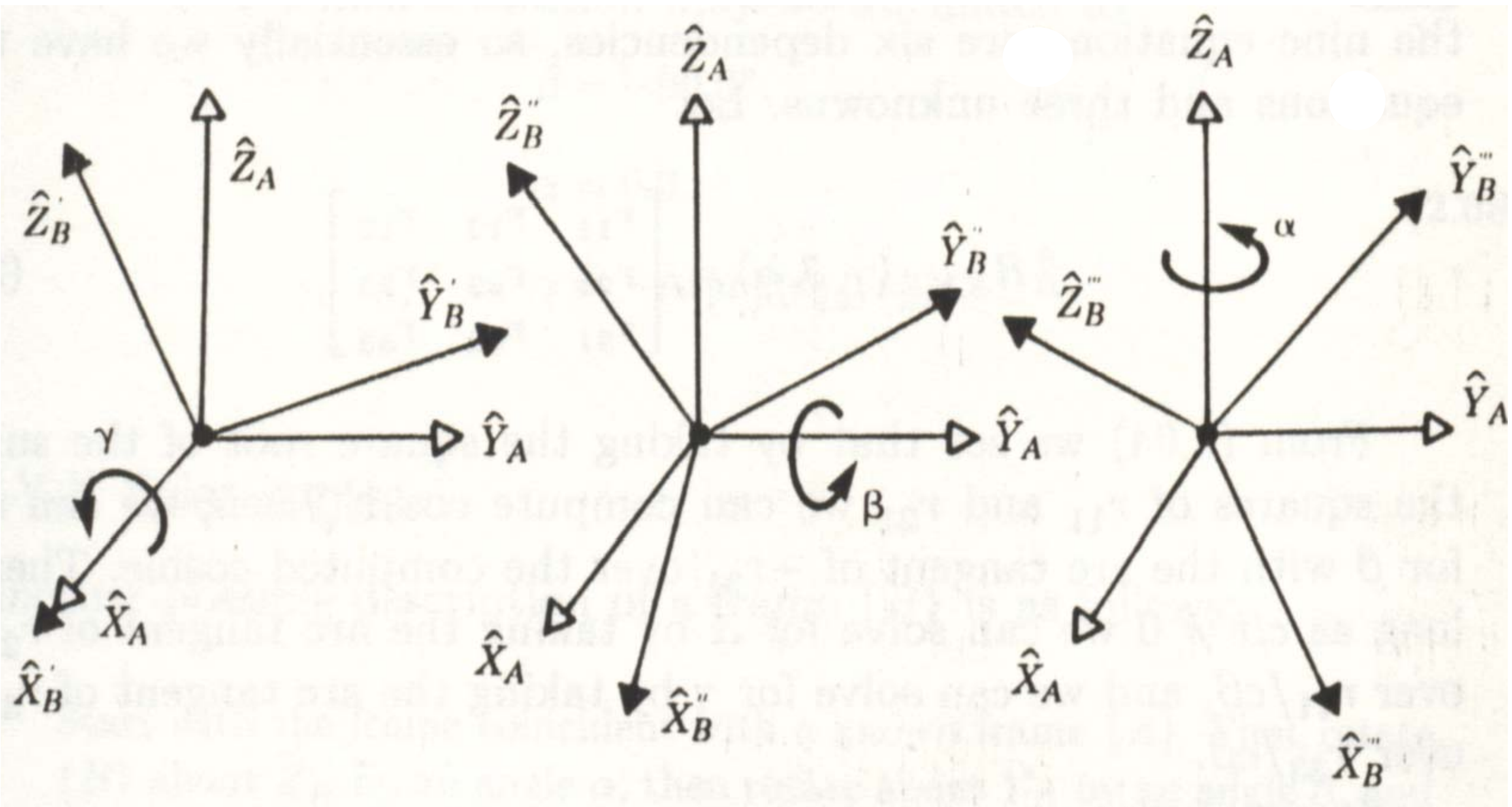
$$R_z(30) = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}$$

$$R_x(30) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.866 & -0.500 \\ 0.000 & 0.500 & 0.866 \end{bmatrix}$$

$$R_z(30)R_x(30) \neq R_x(30)R_z(30)$$

Rotations don't generally commute.

X-Y-Z fixed angles (roll-pitch-yaw)



$R_X(\gamma)$ $R_Y(\beta)$ $R_Z(\alpha)$

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}. $$

Inverse problem:

$${}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

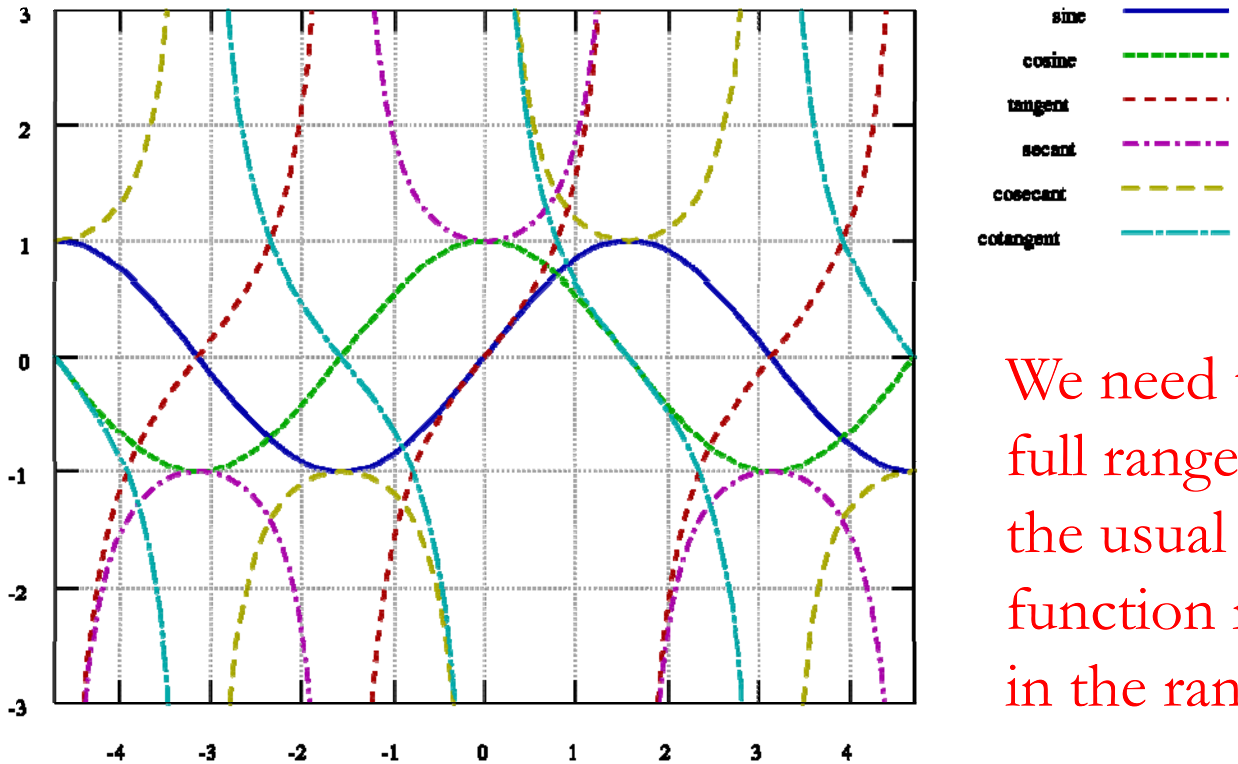
$$\beta = \boxed{A\tan 2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right),$$

$$\alpha = A\tan 2(r_{21}/c\beta, r_{11}/c\beta),$$

$$\gamma = A\tan 2(r_{32}/c\beta, r_{33}/c\beta).$$

A two argument arc tangent function

Trigonometric Functions



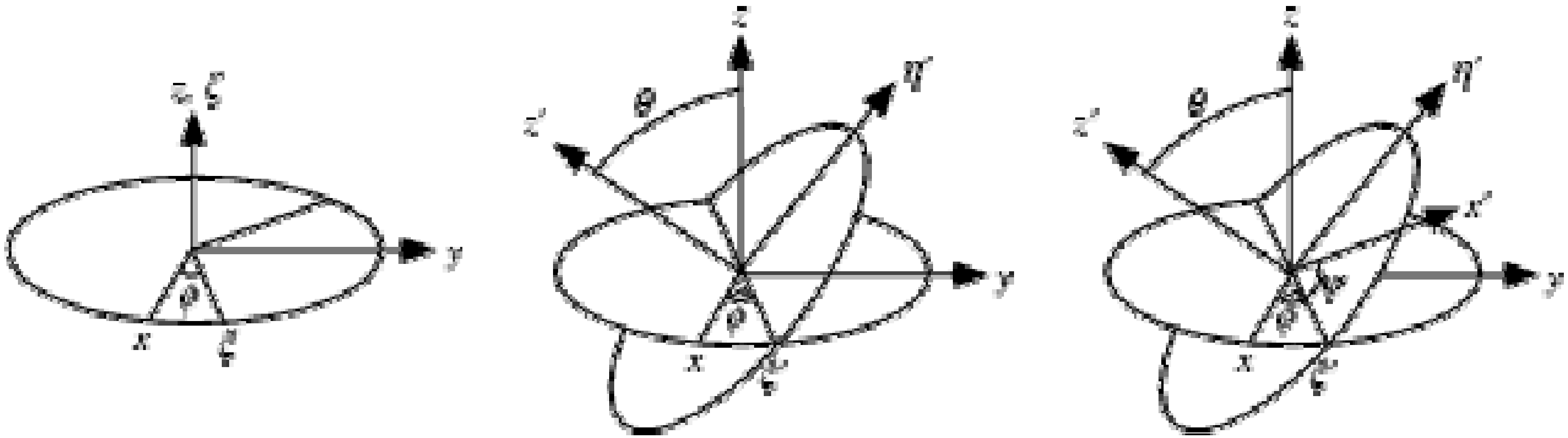
We need to express the full range of angles, but the usual inverse tangent function returns an angle in the range $(-\pi/2, \pi/2)$.

$$A\tan 2(-1,1) = -\frac{\pi}{4}, \quad A\tan 2(1,-1) = +\frac{3\pi}{4}$$

Euler's Rotation Theorem

- An arbitrary rotation may be described by only three parameters (*angles*).

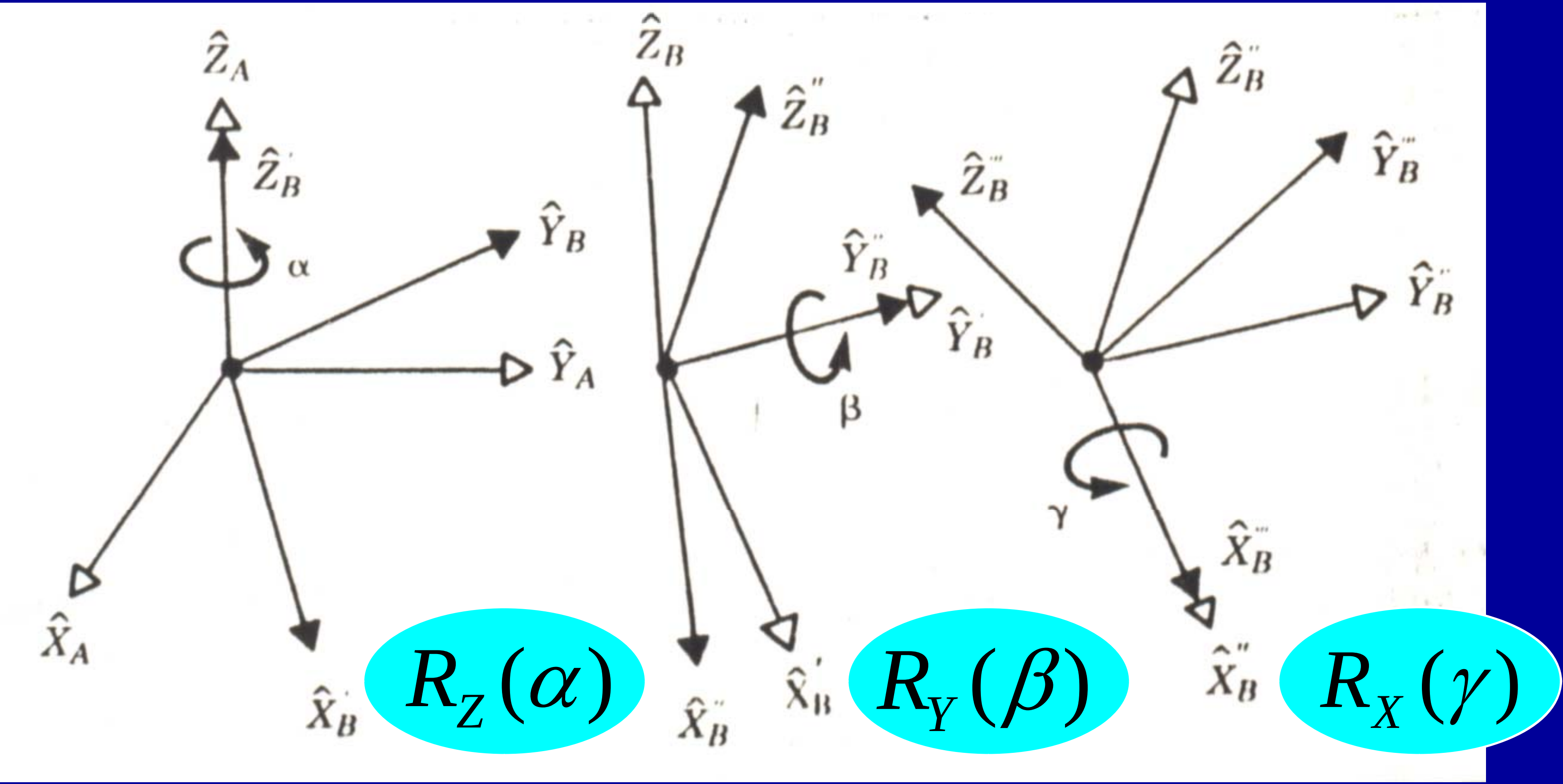
Euler angles



If the rotations are written in terms of rotation matrices *B*, *C*, and *D*, then a general rotation *A* can be written as *A* = *BCD*.
The three angles giving the three rotation matrices are called Euler angles. There are several conventions for Euler angles, depending on the axes about which the rotations are carried out. [Appendix B]

Z-Y-X Euler angles

Each rotation is performed about an axis of the moving system {B} rather than one of the fixed reference {A}.



$$ {}^{A}_{B}R = {}^{A}_{B'}R \, {}^{B'}_{B''}R \, {}^{B''}_{B}R $$

$$ {}^{A}_{B}R_{Z'Y'X'} = R_Z(\alpha) R_Y(\beta) R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}. $$

Three rotations taken about fixed axes yield the same final orientation as the same three rotations taken in opposite order about the axes of moving frame.

Homework #5 : Z-Y-Z Euler Angles (1 pt.) – Due Dec. 21

- Derive the rotational matrix
- Extract Z-Y-Z Euler angles

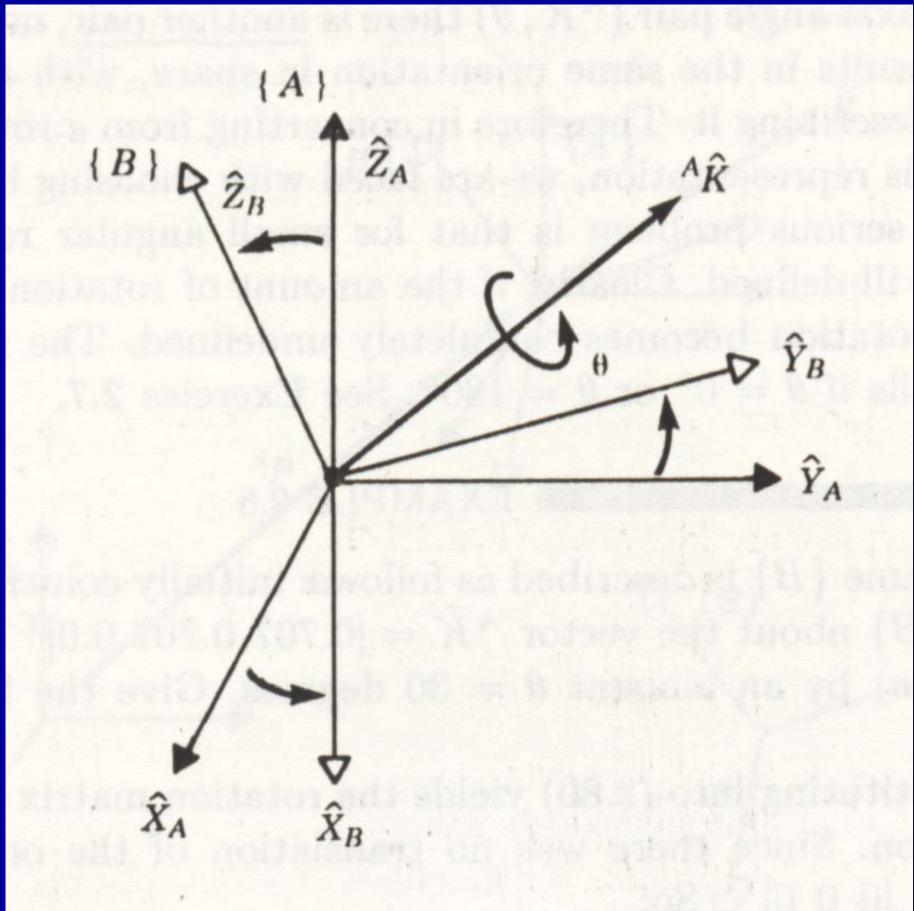
Rules for Composition of Rotational Transformations:
Relative to the *current frame*: *postmultiply*
Relative to *fixed frame*: *premultiply*

Suppose R is defined by the following sequence of basic rotations in order specified:

1. A rotation of θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = R_x(\delta)R_z(\alpha)R_x(\theta)R_z(\phi)R_y(\beta)$$

Equivalent angle-axis representation



If the axis is a *general* direction, any orientation may be obtained through proper axis and orientation selection.

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix},$$

$$v\theta = 1 - \cos\theta$$

$${}^A_B R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

$$\theta = A\cos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}.$$

Homework #6 (1 pt.)

– Due Jan. 13

Verify this equation.

Change of Coordinates

An $n \times n$ matrix A represents a linear transformation from R^n to R^n in the sense that it takes a vector x to a new vector y according to

$$y = Ax$$

The vector y is called the image of x under the transformation A. If the vectors x and y are represented in terms of the standard unit vectors

$$e_1 = [1,0,\ldots,0]^T,\ldots,e_n = [0,0,\ldots,1]^T$$

then the column vectors of A represent the images of the basis vectors $e_1,\ldots,e_n$.

Change of Coordinates (cont'd)

Often it is desired to represent vectors with respect to a second coordinate frame with different basis vectors $f_1, \ldots, f_n.$ In this case the matrix representing the same linear transformation as *A*, but relative to this new basis, is given by

$$A' = T^{-1}AT$$

where *T* is a nonsingular matrix with column vectors $f_1, \ldots, f_n.$ The transformation $T^{-1}AT$ is called a similarity transformation of the matrix *A*.

The use of similarity transformations allows us to express the same rotation easily with respect to different frames.

Relative to frame {A}

Frames {A} and {B} are related by

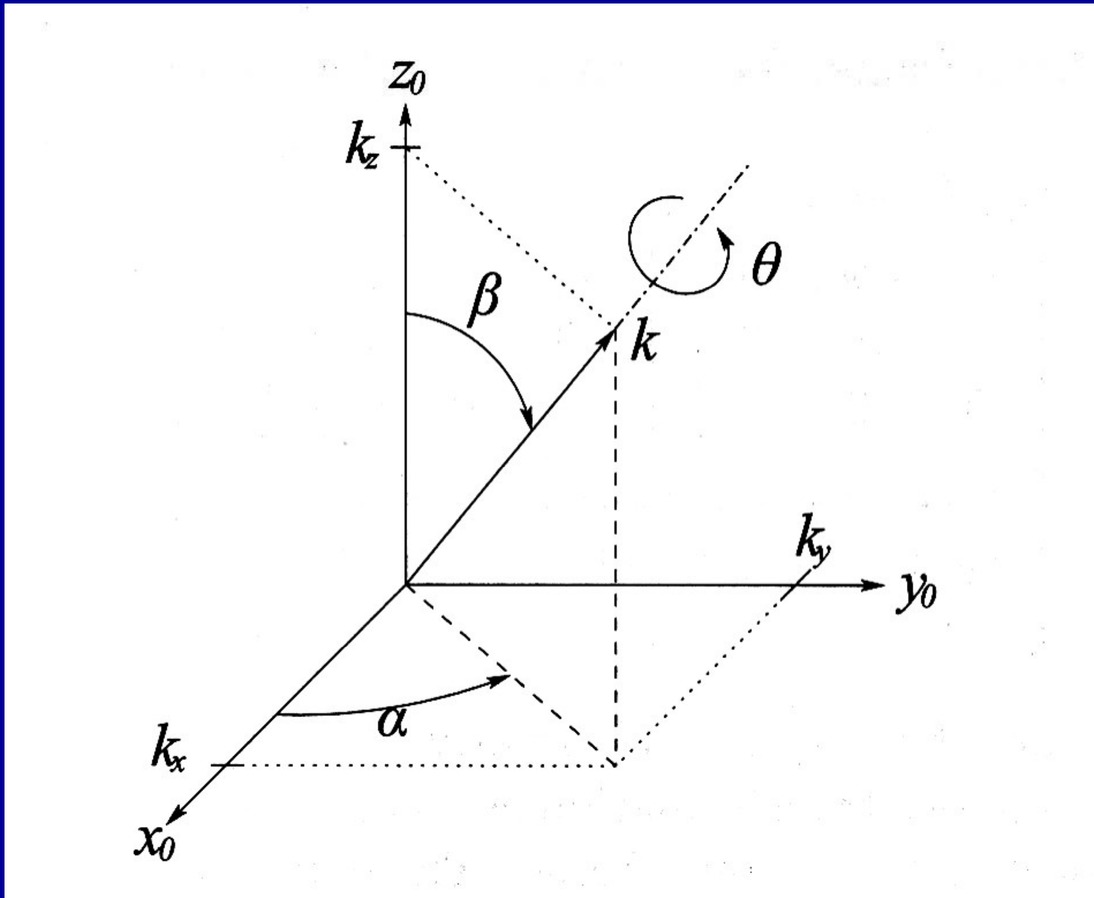
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T = {}^A_B R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Relative to frames {B}, we have

$$R'_z(\theta) = T^{-1} R_z(\theta) T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

A rotation about $\hat{Z}_A$ but expressed relative to {B}

Rotation about an arbitrary axis



The rotational transformation $R = R_z(\alpha)R_y(\beta)$ will bring the world z-axis into alignment with the vector K. Therefore, a rotation about the axis K can be computed using a similarity transformation as

$$R_K(\theta) = RR_z(\theta)R^{-1}$$
$$= R_z(\alpha)R_y(\beta)R_z(\theta)R_y(-\beta)R_z(-\alpha)$$

$$\sin\alpha = \frac{k_y}{\sqrt{k_x^2 + k_y^2}} \quad \cos\alpha = \frac{k_x}{\sqrt{k_x^2 + k_y^2}}$$

$$\sin\beta = \sqrt{k_x^2 + k_y^2} \quad \cos\beta = k_z$$

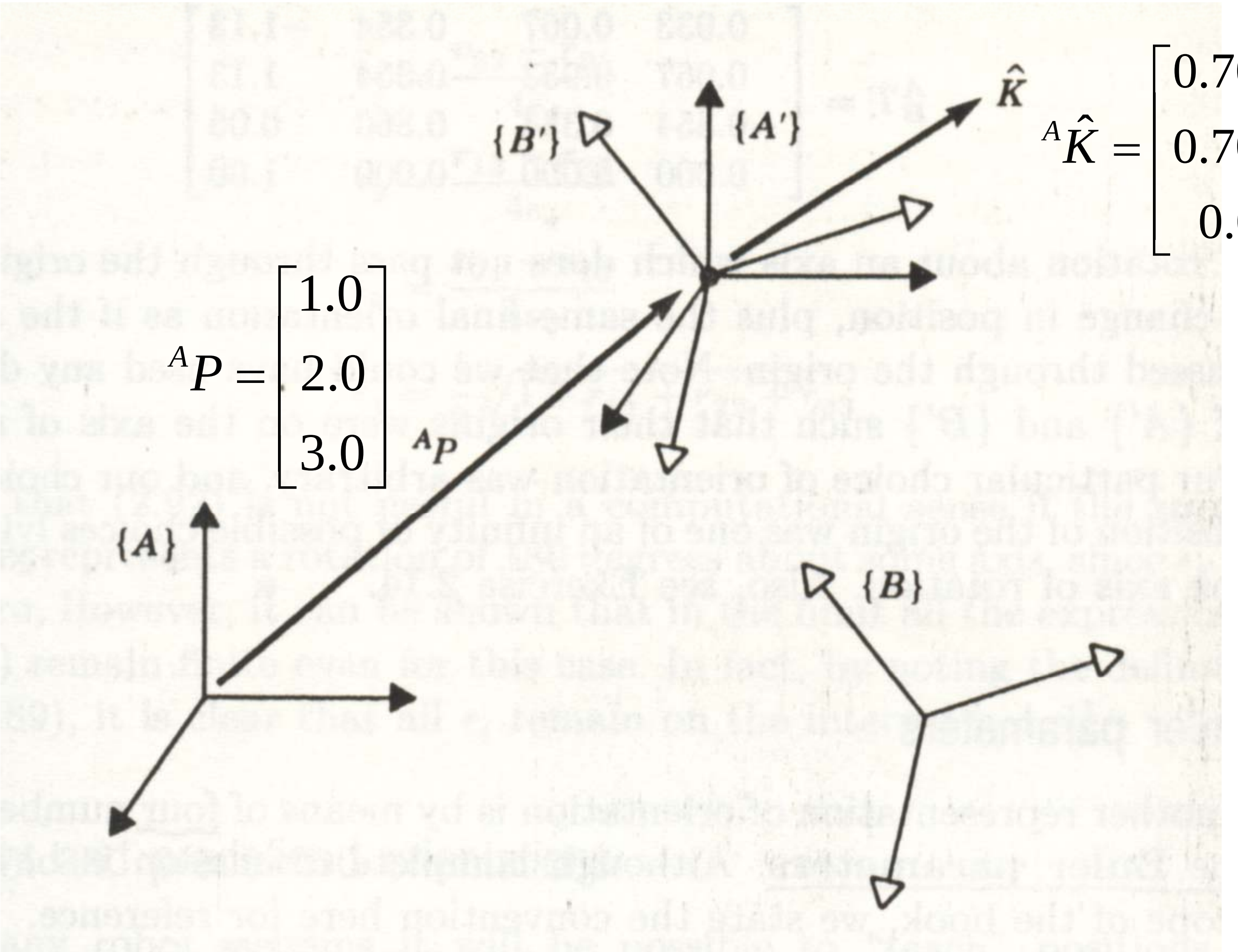
Suppose R is generated by a rotation of 90° about z_0 followed by a rotation of 30° about y_0 followed by a rotation of 60° about x_0. Then

$$R = R_x(60)R_y(30)R_z(90) = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ$$

$$\hat{K} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}} - \frac{1}{2} \\ \frac{1}{2\sqrt{3}} + \frac{1}{2} \end{bmatrix}$$

Example 2.9



$$^{A}\hat{K} = \begin{bmatrix} 0.707 \\ 0.707 \\ 0.0 \end{bmatrix}, \quad \theta = 30^\circ$$

$$^{A}P = \begin{bmatrix} 1.0 \\ 2.0 \\ 3.0 \end{bmatrix}$$

$$
{}^{A}_{A'}T = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 0.0 & 2.0 \\ 0.0 & 0.0 & 1.0 & 3.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}, \quad {}^{B'}_{B}T = \begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 \\ 0.0 & 1.0 & 0.0 & -2.0 \\ 0.0 & 0.0 & 1.0 & -3.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
$$

$$
{}^{A'}_{B'}T = \begin{bmatrix} 0.933 & 0.067 & 0.354 & 0.0 \\ 0.067 & 0.933 & -0.354 & 0.0 \\ -0.354 & 0.354 & 0.866 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
$$

$$
{}^{A}_{B}T = {}^{A}_{A'}T \, {}^{A'}_{B'}T \, {}^{B'}_{B}T = \begin{bmatrix} 0.933 & 0.067 & 0.354 & -1.13 \\ 0.067 & 0.933 & -0.354 & 1.13 \\ -0.354 & 0.354 & 0.866 & 0.05 \\ 0.000 & 0.000 & 0.000 & 1.00 \end{bmatrix}.
$$

A rotation about an axis that does not pass through the origin causes a change in position, plus the same final orientation as if the axis had passed through the origin.

Euler parameters

- The four parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3,$ and ε_4 describing a finite rotation θ about an arbitrary axis $\hat{K} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}^T$
- Defined by

$$\varepsilon_1 = k_x \sin\frac{\theta}{2}, \varepsilon_2 = k_y \sin\frac{\theta}{2}, \varepsilon_3 = k_z \sin\frac{\theta}{2}$$

$$\varepsilon_4 = \cos\frac{\theta}{2}$$

Euler parameters (cont'd)

- A quaternion in scalar-vector representation

$$(\varepsilon, \varepsilon_4) = \varepsilon_1 i + \varepsilon_2 j + \varepsilon_3 k + \varepsilon_4.$$

$$\varepsilon \cdot \varepsilon + \varepsilon_4^2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1.$$

An arbitrary rotation may be described by only three parameters, a relationship must exist between these four quantities.

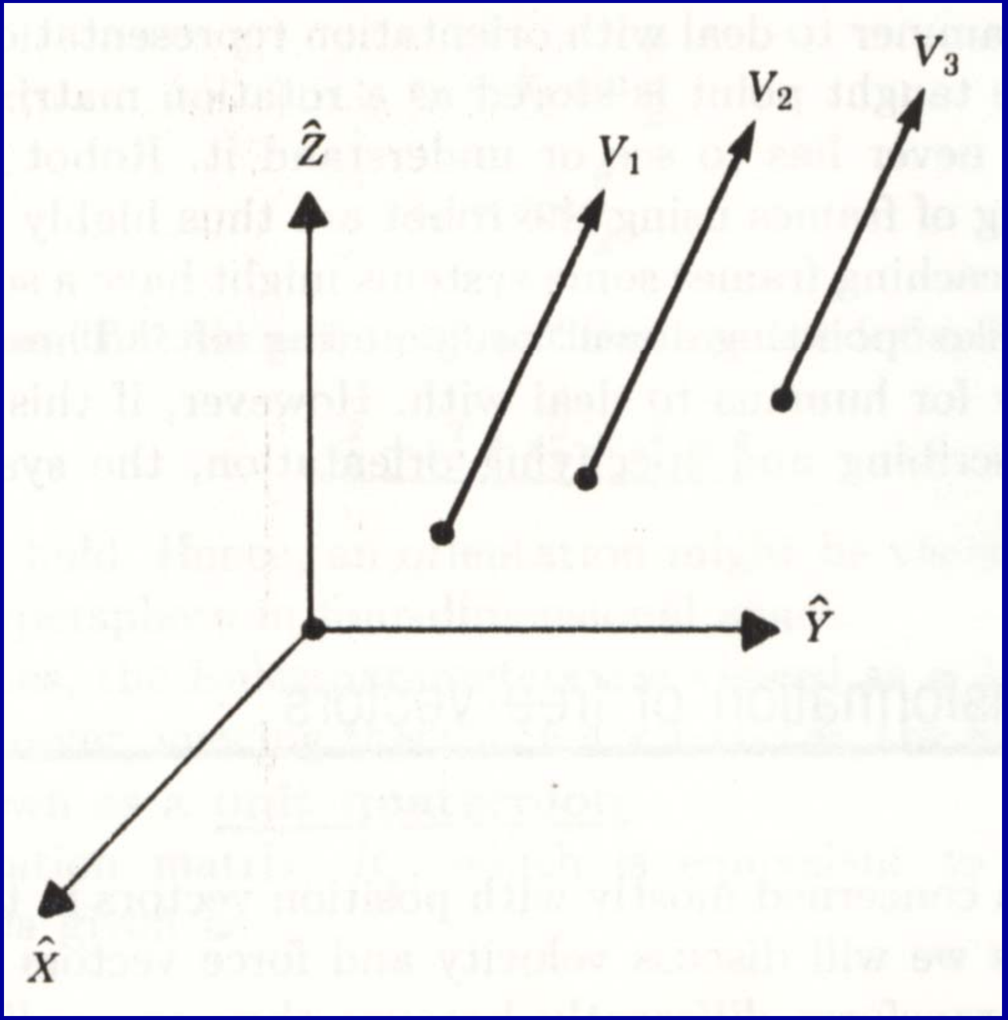
Line vector

- Refers to a vector that is dependent on its line of action, along with direction and magnitude, for causing its effects.
- A force vector

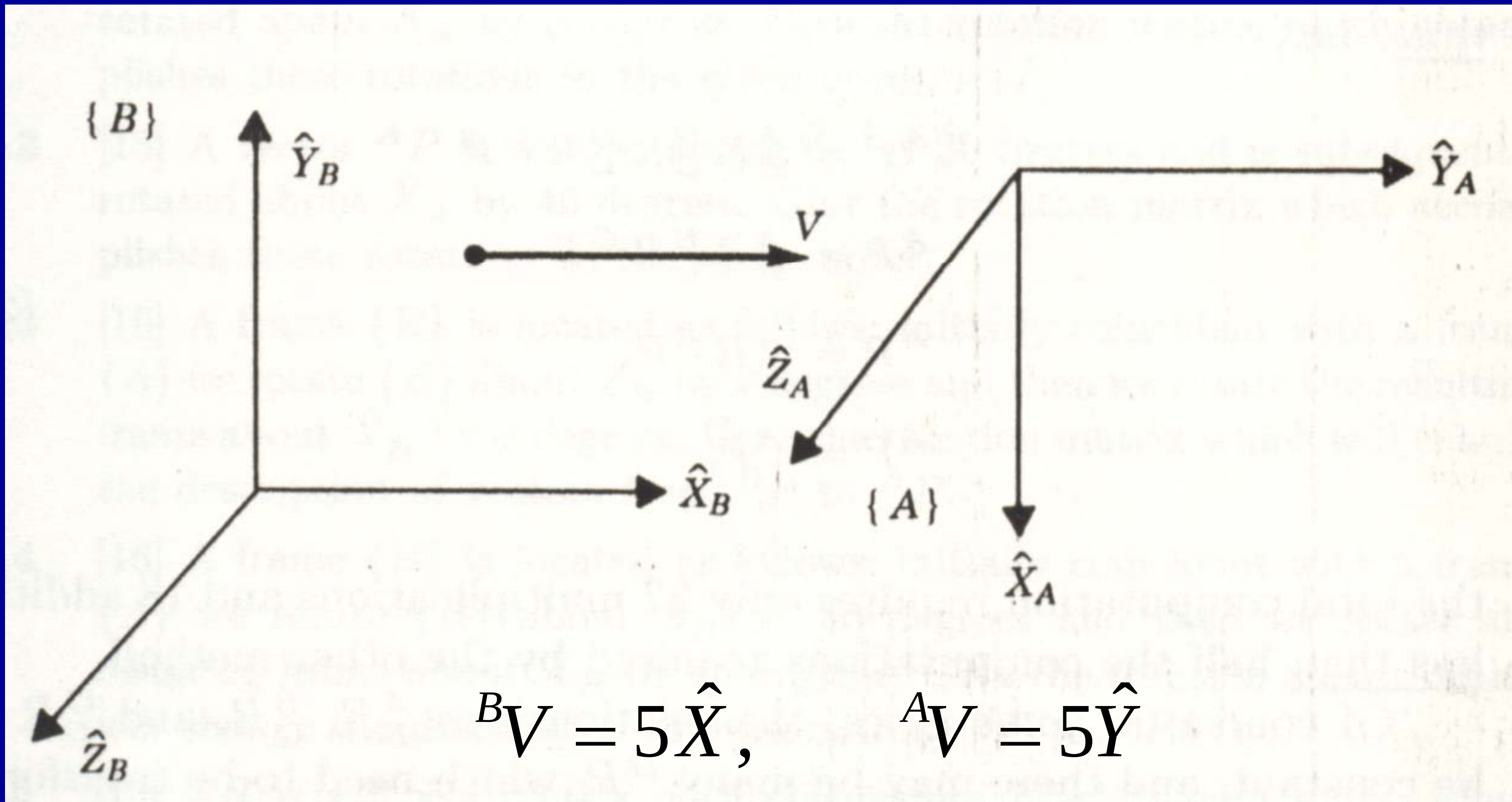
Free vector

- Refers to a vector that may be positioned anywhere in space without loss or change of meaning, provided that magnitude and direction are preserved.
- A pure moment vector, the velocity of a point

Equal velocity vectors



Transforming velocities



$$^{B}V = 5\hat{X}, \qquad ^{A}V = 5\hat{Y}$$

Rigid motions

A rigid motion is an ordered pair of (d, R).

$$d \in R^3$$

a pure translation

$$R \in SO(3)$$

a pure rotation

$$SE(3) = R^3 \times SO(3)$$

Special Euclidean Group

Quiz #2 – Due Today

- All the problems are from the course text. Unless otherwise indicated, each problem is worth 0.5 points.
- 2.27, 2.28, 2.29, 2.30, 2.31, 2.33

Homework #7 – Dec. 21

- 2.4, 2.12, 2.13, 2.37