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Tutorial

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# Geometry and Screw Theory for Robotics

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# 1

# Motion of a Rigid Body

*This Chapter shows the fundamental differences between the motion properties of a point mass and of a rigid body. The rigid body does* not *live in a Euclidean space. Hence, this Chapter explains which relevant concepts from non-Euclidean (i.e., differential) geometry are needed to describe the kinematics and dynamics of a rigid body.*

## 1.1 Introduction

The motion of a particle mass is easy to describe because its configuration can be associated to a point of the three-dimensional Euclidean space. After having chosen coordinates, each point can be associated to a triple of real numbers in $\mathbb{R}^3$; but the most important thing is that the algebraic and topological properties of $\mathbb{R}^3$ correspond to real physical properties of the motion of the particle: forces can be added; velocity vectors too; magnitudes of vectors correspond to magnitudes of forces and velocities; orthogonality of a force and velocity vector gives zero power; the velocity and acceleration vectors are the time derivatives of the point's position vector; Newton's Laws link a three-dimensional force vector to a three-dimensional acceleration vector, through the scalar quantity "mass," for point masses, as well as spherically-symmetric rigid bodies such as planets and canon balls.

In contrast to the simplicity of the point mass motion properties, the motion and the dynamics of a rigid body are much more complex. A rigid body is composed of an infinite number of point masses, which are constrained not to move with respect to each other. It turns out that the dimension of the space necessary to describe the configuration of a rigid body is six: three dimensions for orientation, and three for translation. And the force–acceleration relation is now a full six-by-six matrix, and not a scalar anymore. Moreover, the acceleration involved in this dynamic relation is *not* just the second-order time derivative of the position/orientation vector of the rigid body.

Even the short overview above should make it clear that it is wrong to treat the six position/orientation coordinates of a rigid body in the same way as one treats the three position coordinates of a point: the *geometrical properties* of rigid bodies are fundamentally different from the geometrical properties of point masses. For example, if one continuously increases one of these six numbers (i.e., one that corresponds to orientation representation), the rigid body arrives at the *same* configuration after every rotation over 360 degrees; this "curvature" property does not occur when one indefinitely increases any of the three coordinates of a point configuration.

Locally (i.e., in the neighborhood of a specific configuration) it *is* possible to describe a configuration using six real numbers, but this description is not an *intrinsic property* of the motion. (An intuitive definition of an “intrinsic property” is: any property that does not change if one changes the coordinate representation; the total mass of a point or body is such an intrinsic property, at least in non-relativistic dynamics.)

A lot of powerful tools are available which allow to describe the motion of rigid bodies in a geometrical and global way. These methods are related to the geometry of *lines* and *screws* and to the differential geometric concept of a *Lie group*. This text makes ample use of these (and other) mathematical concepts, because their mathematical properties correspond perfectly to the (ideal) physical properties of moving rigid bodies. The quality of engineering results depends critically on the quality and faithfulness of the mathematical models on which the engineering is based: especially in robotics, engineers have made (and are still making) fundamental errors in their reasoning, for the sole reason that that reasoning was based on mathematical properties of the coordinates in their model for which no corresponding physical properties exist.

In a nutshell: it’s not because one can use $n$ numbers as coordinates on a given space, that the objects in that space have exactly the same properties as the $n$-tuples in $\mathbb{R}^n$!

## 1.2 The Euclidean Space

The space of rigid body motion (position and orientation; translational and angular velocity; and translational and angular acceleration) has different properties than the well-known three-dimensional Euclidean space we live in. A Euclidean space is a continuous set of points together with an extra structure which is able to describe *orthogonality* and to *measure length*. This extra structure is the *scalar product*. We indicate an Euclidean space of dimension $n$ with $\mathcal{E}(n)$. The Euclidean space $\mathcal{E}(n)$ is therefore the set of points of the space. We indicate the three-dimensional space we live in with $\mathcal{E}(3)$ or simply $\mathcal{E}$. Sometimes we omit the information about the dimension and just indicate the Euclidean space as $\mathcal{E}$. It is important to realize that since a point can be fixed or moving with respect to an observer, for a given Euclidean space we need to specify an observer which does not move within it. We suppose that there exists an inertial observer we consider as reference in which the points belonging to the Euclidean space are *not* moving. It is possible to see in (Stramigioli 2001, Stramigioli & Bruyninckx 2001) that the hypothesis we use considering a reference Euclidean space hides important consequences.

### 1.2.1 Points versus vectors

The points of $\mathcal{E}$ are *not* vectors but points: they do not have a versus and a direction and as such it makes *no* sense to, for example, sum them! We can associate a vector to them only once we have chosen a reference point: if we choose such a reference point $o \in \mathcal{E}$, we can associate to every point $p \in \mathcal{E}$ a vector going from $o$ to $p$ which we indicate with $(p - o)$. It is not here the place to go into details, but this association is possible since the space is Euclidean: for curved spaces this is *not* possible. For example, what would be the meaning of the “sum” of two points on the surface of the earth?

However, once we have vectors, we can sum them; for example, it makes sense to sum the velocities or accelerations that a moving point mass gets under the influence of separate forces acting on the point. And for points, the velocity and acceleration are indeed vectors, because they represent the “timed” distance between consecutive

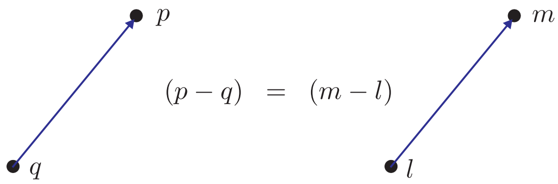



Figure 1.1: The distance between points are represented by vectors.

positions of the same point. These vectors are called *free vectors*, because their physical meaning does not change if one moves them parallel to themselves to another place in the space. Mathematically speaking, we use as a vector space the *quotient space* of all different vectors, using parallelism and equal length as equivalence relations.

For example, as shown in Figure 1.1, we consider the vector $(p-q)$ equivalent to the vector $(m-l)$ where $p, q, l, m \in \mathcal{E}$. The set of these free vectors are denoted with $\mathcal{E}_*(n)$ or $\mathcal{E}_*$ if it is not necessary to show the dimension $n$.

### 1.2.2 The scalar product

We can now introduce the *scalar product* which characterizes a Euclidean space. It is a map which gives a real number once we give two vectors belonging to $\mathcal{E}_*$ as input:

$$\langle , \rangle : \mathcal{E}_* \times \mathcal{E}_* \to \mathbb{R} \quad ; \quad (v, w) \mapsto \langle v, w \rangle.$$

The scalar product satisfies the following properties:

1. $\langle v, w \rangle = \langle w, v \rangle$ (symmetry).
2. $\langle v, v \rangle \geq 0$ (positiveness).
3. $\langle \alpha v_1 + \beta v_2, w \rangle = \alpha \langle v_1, w \rangle + \beta \langle v_2, w \rangle$ (linearity).

### 1.2.3 Measuring the distance between points

We can define the distance $d(p, q)$ between the points $p$ and $q$ as the length of $(p-q) \in \mathcal{E}_*$. This distance defines a *norm*:

$$|| \cdot || : \mathcal{E}_* \to \mathbb{R} \quad ; \quad v \mapsto \sqrt{\langle v, v \rangle}.$$

### 1.2.4 Orthogonality

Orthogonality of vectors $v, w \neq 0$ is also defined. We say that two vectors with non-zero length are orthogonal if their scalar product is zero:

$$v \perp w \Leftrightarrow \langle v, w \rangle = 0.$$

The cosine of an angle is also a consequence of the scalar product. It can be defined using only the scalar product:

$$\cos v \angle w := \frac{\langle v, w \rangle}{||v|| \cdot ||w||}.$$

### 1.2.5 Vector Product

It is also possible to define a *vector product* (after a choice of orientation (Stramigioli 2000)!):

$$\wedge : \mathcal{E}_*(3) \times \mathcal{E}_*(3) \to \mathcal{E}_*(3); (v, w) \mapsto v \wedge w.$$

The vector product is skew-symmetric, i.e., $v \wedge w = -(w \wedge v)$. The following "tilde-relation" is a one-to-one relation between a *point* and a *mapping of points*:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Leftrightarrow \tilde{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}. \tag{1.1}$$

Using this notation, If we choose a coordinate system $\Psi_o$ and express $v, w \in \mathcal{E}_*(3)$ in $\Psi_o$ as $v^o, w^o \in \mathbb{R}^3$, we can write:

$$v^o \wedge w^o = \tilde{v}^o w^o,$$

where $\tilde{v}^o$ is now the matrix corresponding to the above-mentioned mapping. We can therefore express the vector product of two vectors as the product of a skew-symmetric matrix and a vector. The following equalities hold:

$$v^o \wedge w^o = \tilde{v}^o w^o = -w^o \wedge v^o = -\tilde{w}^o v^o = \tilde{w}^T v^o.$$

Issues about the vector product will be further explained in Remark. 4, p.14.

### 1.2.6 Coordinate Systems

If we want to work with a Euclidean space, we need to define coordinate systems in order to be able to represent points and vectors by means of numbers to work with. Formally, we define a coordinate system for an $n$-dimensional Euclidean space $\mathcal{E}(n)$ as an $(n+1)$-tuple composed of a point $o \in \mathcal{E}(n)$ and $n$ **linear independent** vectors belonging to $\mathcal{E}_*(n)$:

$$\Psi_o := (o, e_1, e_2, \ldots, e_n) \in \mathcal{E}(n) \times \underbrace{\mathcal{E}_*(n) \times \mathcal{E}_*(n)}_{n \text{ times}}.$$

The symbol capital 'Psi' is used to indicate a coordinate system.

A special kind of coordinate systems is called *orthonormal*. Orthonormal coordinate systems are defined as those coordinates systems for which

$$||e_i|| = 1 \quad \forall i \quad \text{(unit vectors)},$$

and

$$\langle e_i, e_j \rangle = 0 \quad \forall i \neq j \quad \text{(orthogonality)}.$$

An orthonormal coordinate system is a system for which its defining vectors are orthogonal with respect to each other, and have all unity length. As an example consider Figure 1.2 for the case of $\mathcal{E}(2)$. Part a) represents the set $\Psi_o$ representing a coordinate system and composed of $o, e_1, e_2$; b) shows it in the way we are used to use it shifting the vectors to $o$; c) shows the parallelogram rule to see how to calculate the coordinates

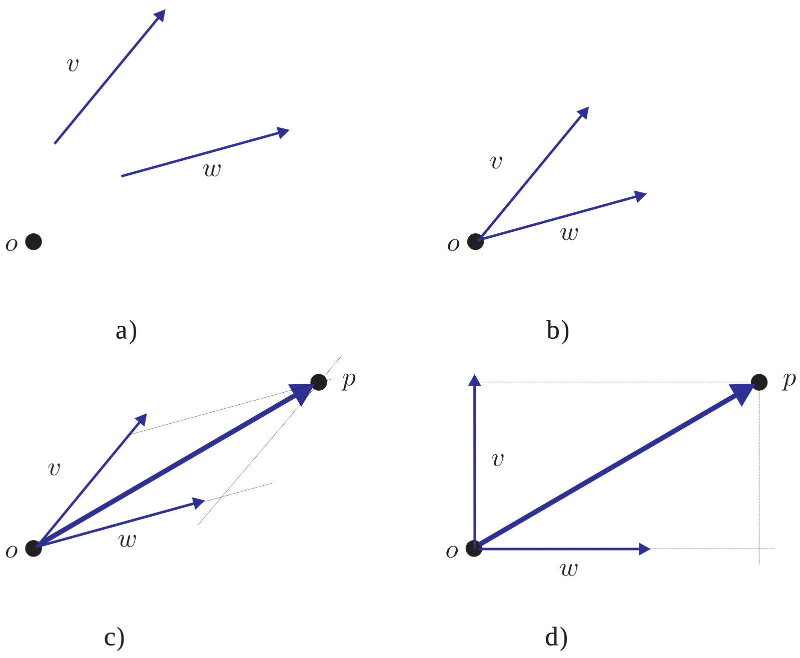



Figure 1.2: Example of the use of coordinates

for a point $p$; and d) shows an orthonormal frame if we suppose that the length of $e_1$ and $e_2$ is equal to unity.

We say that an orthonormal coordinate system $(o, \hat{x}, \hat{y}, \hat{z})$ for $\mathcal{E}(3)$ is *right handed* if, and only if,

$$\hat{x} \wedge \hat{y} = \hat{z}.$$

We implicitly use only right-handed coordinate systems. Given a coordinate system, we can associate to each point or vector a set of real numbers representing its coordinates in the given frame. The coordinates $x_i$ of a point $p \in \mathcal{E}$ are:

$$x_i = \langle (p - o), e_i \rangle \in \mathbb{R}, \quad \forall i.$$

The coordinates of a vector $v \in \mathcal{E}_*$ are:

$$x_i = \langle v, e_i \rangle \in \mathbb{R} \quad \forall i.$$

In 3D we work with $\mathcal{E}(3)$ and we indicate sometimes for an orthonormal coordinate system $\hat{x} := e_1$, $\hat{y} := e_2$ and $\hat{z} := e_3$. From now on we will implicitly always use orthogonal, positively-oriented coordinate systems.

### Coordinate mapping

For any Cartesian coordinate system $\Psi_i = (o_i, \hat{x}_i, \hat{y}_i, \hat{z}_i)$, we can define a coordinate mapping $\psi_i$ which associates to each point a set of numbers in the following way:

$$\psi_i : \mathcal{E}(3) \to \mathbb{R}^3 \quad ; \quad p \mapsto \begin{pmatrix} \langle (p - o_i), \hat{x}_i \rangle \\ \langle (p - o_i), \hat{y}_i \rangle \\ \langle (p - o_i), \hat{z}_i \rangle \end{pmatrix}.$$

We indicate with $\psi_i$ the coordinate map associated to the coordinate frame $\Psi_i = (o_i, \hat{x}_i, \hat{y}_i, \hat{z}_i)$.

**Change of coordinates**

If we take two coordinate systems, we can consider the mapping representing the change of coordinates from one system to the other. A change of coordinates from $p^1$ (coordinates with respect to $\Psi_1$) to $p^2$ (coordinates with respect to $\Psi_2$) can be expressed as:

$$p^2 = (\psi_2 \; o \; \psi_1^{-1})(p^1).$$

Or, in mapping form:

$$\mathbb{R}^3 \xrightarrow{\psi_1^{-1}} \mathcal{E}(3) \xrightarrow{\psi_2} \mathbb{R}^3 \quad ; \quad p^1 \mapsto p \mapsto p^2.$$

**Projective coordinates**

A lot of concepts of importance in kinematics and dynamics are better explained in projective terms. It is not here the place to treat projective geometry in details, but we will just say a few things in order to use it as a tool. For more mathematical details the reader can consult Appendix A. First of all, the set $\mathbb{PR}^n$ denotes an $n$-dimensional projective real space. Each of the elements of $\mathbb{PR}^n$ is an $(n+1)$-tuple of real numbers. Two of these $(n+1)$ vectors of real numbers $a$ and $b$ in $\mathbb{PR}^n$ are related by the projective equivalence relation if $a$ is a multiple of $b$:

$$a \sim b \Leftrightarrow \exists \lambda \neq 0 : a = \lambda b.$$

It is possible to study three-dimensional Euclidean space using projective coordinates $\mathbb{PR}^3$. A representative of a finite point $p \in \mathcal{E}$ can then be:

$$P^1 := \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Any point $\lambda P^1$ with $\lambda \neq 0$ is also an equivalent projective representation of the same point. For reasons which are explained in Appendix A, a point whose numerical representation has the last component equal to zero, represents a point at infinity. Using projective geometry, it is therefore possible to consider points at infinity in a consistent way.

## 1.3 Configurations of a rigid body

A lot of geometrical and analytical methods have been developed over the last three centuries to describe the (changes of) configuration of rigid bodies. The first differential geometrical interpretation seems to be due to Study (Selig 1996). In this course we will treat the finite twists representation and the Lie group representation of the so-called *Special Euclidean group* $SE(3)$. Another possible analytical representation are *bi-quaternions*. Details about this topics can be found in (Selig 1996, Stramigioli 2001, Karger & Novak 1978, Abraham & Marsden 1994). The previously mentioned methods are all global, geometrical, well-defined methods. This cannot be said of descriptions like *Euler angles* which are only valid locally, and are not as powerful as the methods described in the following sections.

### 1.3.1 A Lie Group approach: $SE(3)$

Given a certain body in space, we can describe the motion of this body from a certain position to a new one. This set of motions are characterized by the fact that they are what are called in mathematics *orientation preserving isometries*. *Isometries* means that the motion does not change the distance between points and *orientation preserving* implies that the motion does not map a right-handed coordinate frame onto a left-handed frame.

#### Coordinates changes versus motions

It is a general fact that if we consider a global map of points for an $n$-dimensional Euclidean space $\mathcal{E}$ which is bijective (one-to-one and onto), we can always associate to it an equivalent change of coordinates in the following sense. Suppose to have a bijective mapping $h$:

$$h : \mathcal{E} \to \mathcal{E} \quad s.t. \quad h(p_1) = h(p_2) \Rightarrow p_1 = p_2. \tag{1.2}$$

Once we have chosen a global[1] coordinate function

$$\psi : \mathcal{E} \to \mathbb{R}^n, \tag{1.3}$$

we can consider the numerical representation of $h$ in the coordinates $\psi$ by defining:

$$h_n : \mathbb{R}^n \to \mathbb{R}^n \quad ; \quad x \to (\psi \ o \ h \ o \ \psi^{-1})(x). \tag{1.4}$$

If we then define a new coordinate map

$$\psi' : \mathcal{E} \to \mathbb{R}^n \quad ; \quad p \mapsto (\psi \ o \ h)(p), \tag{1.5}$$

the numerical representation $h_n$ of $h$ is clearly equivalent to a change of coordinate from $\Psi$ to $\Psi'$. If, for hypothesis $h$ is a positive isometry, the converse is clearly only true if the coordinate functions are isometries from $\mathcal{E}$ to $\mathbb{R}^n$. Coordinate functions satisfying this hypothesis are the set of right-handed frames or the set of left-handed frames, but not a combination of them. For this reason, we will first study coordinate changes and then define motions based on the previous arguments.

#### Rotations

We start by considering changes of coordinates in 2D. Consider two coordinate systems, with their origin in common, but rotated over an angle $\theta$ with respect to one another, as shown in Figure 1.3. What is the relation between the coordinates of any point $p \in \mathcal{E}(2)$ in $\Psi_1$ and $\Psi_2$? The definitions above yield the following for the coordinates of the point $p$ in the first frame:

$$p^1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \psi_1(p) = \begin{pmatrix} \langle (p-o), \hat{x}_1 \rangle \\ \langle (p-o), \hat{y}_1 \rangle \end{pmatrix},$$

or, equivalently,

$$(p - o) = x_1 \hat{x}_1 + y_1 \hat{y}_1, \tag{1.6}$$

[1] A global coordinate function exists since the space $\mathcal{E}$ is Euclidean.

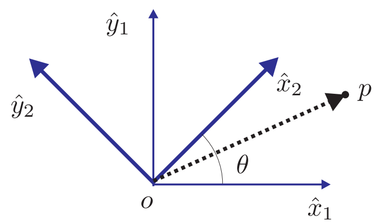



Figure 1.3: Change of coordinates under rotation.

with $x_1, y_1 \in \mathbb{R}$ (scalar coordinates) and $\hat{x}_1, \hat{y}_1 \in \mathcal{E}_*(2)$ (unit position vectors in 2D).

Similarly, the coordinates $p^2 \in \mathbb{R}^2$ of the point $p$ with respect to the second frame are:

$$p^2 := \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \langle (p-o), \hat{x}_2 \rangle \\ \langle (p-o), \hat{y}_2 \rangle \end{pmatrix},$$

The expression for $(p-o)$ in $\Psi_1$, Eq. (1.6), gives:

$$p^2 = \begin{pmatrix} \langle x_1\hat{x}_1 + y_1\hat{y}_1, \hat{x}_2 \rangle \\ \langle x_1\hat{x}_1 + y_1\hat{y}_1, \hat{y}_2 \rangle \end{pmatrix} = \begin{pmatrix} x_1\langle \hat{x}_1, \hat{x}_2 \rangle + y_1\langle \hat{y}_1, \hat{x}_2 \rangle \\ x_1\langle \hat{x}_1, \hat{y}_2 \rangle + y_1\langle \hat{y}_1, \hat{y}_2 \rangle \end{pmatrix}.$$

In matrix form, this gives:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \langle \hat{x}_1, \hat{x}_2 \rangle & \langle \hat{y}_1, \hat{x}_2 \rangle \\ \langle \hat{x}_1, \hat{y}_2 \rangle & \langle \hat{y}_1, \hat{y}_2 \rangle \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix},$$

or

$$p^2 = R_1^2\, p^1,$$

where $R_1^2$ is the *rotation matrix*, defined as:

$$R_1^2 := \begin{pmatrix} \langle \hat{x}_1, \hat{x}_2 \rangle & \langle \hat{y}_1, \hat{x}_2 \rangle \\ \langle \hat{x}_1, \hat{y}_2 \rangle & \langle \hat{y}_1, \hat{y}_2 \rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

**Remark 1 (Orthonormal matrices)** *The matrix $R_1^2$ representing the change of coordinates between two coordinate systems rotated with respect to one another, has the following properties:*

- $\det(R_1^2)$*=1;*
- $R_2^1 = (R_1^2)^{-1} = (R_1^2)^T$*;*
- *the columns and rows vectors of $R_1^2$ have length 1 and are orthogonal.*

These *mathematical* properties correspond to *physical* properties of rotations: a rotation doesn't change the *volume* of a set of three independent vectors; the inverse of a rotation consists of reversing the projection axes; and, the projections of a set of three orthogonal unit vectors give rise to an *orthogonal* matrix.

In 3D, if the coordinates we use are all right handed, we have a similar result as in 2D for a general rotation around the origin:

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \langle \hat{x}_1, \hat{x}_2 \rangle & \langle \hat{y}_1, \hat{x}_2 \rangle & \langle \hat{z}_1, \hat{x}_2 \rangle \\ \langle \hat{x}_1, \hat{y}_2 \rangle & \langle \hat{y}_1, \hat{y}_2 \rangle & \langle \hat{z}_1, \hat{y}_2 \rangle \\ \langle \hat{x}_1, \hat{z}_2 \rangle & \langle \hat{y}_1, \hat{z}_2 \rangle & \langle \hat{z}_1, \hat{z}_2 \rangle \end{pmatrix}}_{R_1^2} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}. \tag{1.7}$$

$R_1^2$ is therefore almost identical to the two-dimensional case, but now we are dealing with a $3 \times 3$ matrix.

**Remark 2** *It is important to note that:*

- *The columns of $R_1^2$ are the unit vectors along the axes of $\Psi_1$, expressed in the coordinate system $\Psi_2$.*
- *The rows of $R_1^2$ are the the unit vectors along the axes of $\Psi_2$, expressed in the coordinate system $\Psi_1$.*

A direct consequence of this is that the *inverse* of $R_1^2$, i.e., $R_2^1$, is equal to the *transpose* of $R_1^2$. This property holds for any rotation matrix $R$:

$$R^{-1} = R^T$$

The following identities are useful properties when rotating any vectors $v, w \in \mathbb{R}^3$:

$$R(v \wedge w) = (Rv) \wedge (Rw), \tag{1.8}$$

and

$$R\tilde{w}R^T = \widetilde{(Rw)}, \tag{1.9}$$

where the "tilde" operator is defined in Eq. (1.1).

As an example of a rotation in $\mathcal{E}(3)$, consider a change of coordinates due to a rotation of $\theta$ around $\hat{y}_1$. The matrix representing this change of coordinates looks like:

$$R_1^2 = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}.$$

It is possible to see that the set of matrices satisfying $R^{-1} = R^T$ is a three-dimensional matrix Lie group, which is called the *orthonormal* group and indicated with $O(3)$. First, it fulfills all properties of a *group*:

- Associativity: $R_1, R_2, R_3 \in O(3) \Rightarrow (R_1R_2)R_3 = R_1(R_2R_3)$.
- Identity: $I \in O(3)$.
- Inverse: $R \in O(3) \Rightarrow R^{-1} \in O(3), RR^{-1} = I$.

Moreover, it is a *Lie* group, because the composition operation is *continuous*. Since $R^TR = I,\ \forall R \in O(3)$, and since the determinant of a product is the product of the determinants, the determinant of any matrix in $O(3)$ must be $\pm 1$. This shows that $O(3)$ is composed of two disjoint components, one whose matrices have determinant equal to $-1$, and one consisting of the matrices with determinant equal to $+1$. The former is not a group by itself, because the product of two such matrices has determinant $+1$. The latter is again a *Lie group*, called the *special orthonormal group*, and denoted by $SO(3)$:

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \quad ; \quad R^{-1} = R^T, \det R = 1\}.$$

**Remark 3** *Note that the previous rotations are all around a specific point in space, namely the common origin of the coordinate systems. This implies that $SO(3)$ can only be used to describe rotations around a specific point and not any rotation! This is not explicitly mentioned in most of the robotics literature.*

The Lie *group* $SO(3)$ is the appropriate mathematical model of the *orientations* of a rigid body. Also the *angular velocities* of rigid bodies must get a good mathematical model. This is given by the so-called *Lie algebra* of $SO(3)$, which is denoted with $\mathfrak{so}(3)$. A Lie algebra describes the *local* ("first order") properties of its parent Lie group (irrespective of what the Lie group represent), while maintaining the geometric properties.

From the theory on left and right invariant vector fields, Eq. (B.5) and Eq. (B.6), we can see that elements of the algebra should look like $T_L = R^{-1}\dot{R}$, or like $T_R = \dot{R}R^{-1}$, for any curve $R(t) \in SO(3)$. Because $R^{-1} = R^T$, differentiating the equation $R^T R = I$ yields:

$$\dot{R}^T R + R^T \dot{R} = 0 \Rightarrow R^T \dot{R} = -(R^T \dot{R})^T,$$

which means that the matrix $T_L$ is *skew-symmetric*. Similarly, $T_R$ is also skew-symmetric, so that $\mathfrak{so}(3)$ is the vector space of skew-symmetric matrices. Note that due to the structure of the matrix commutator of the algebra, the commutation of skew-symmetric matrices is still skew-symmetric.

There is a bijective relation between $3 \times 3$ skew-symmetric matrices and 3 vectors as it was shown in Eq. (1.1).

It is straightforward to see that the algebra matrix commutator of $\mathfrak{so}(3)$ corresponds to the vector product of the corresponding three vectors:

$$[\tilde{\omega}_x, \tilde{\omega}_y] = \widetilde{\omega_x \wedge \omega_y} \qquad \forall \omega_x, \omega_y \in \mathbb{R}^3.$$

The elements of $\mathfrak{so}(3)$ correspond to the angular velocity of the frames whose relative change of coordinates is represented by the matrix $R_1^2 \in SO(3)$. We will analyse in more detail the general case, i.e., angular and translational velocities. A lot of coordinate representations of rotations exists, besides the above-mentioned skew-symmetric matrix representation of $SO(3)$; for example, the *unit quaternions*, which are isomorphic to another Lie group, namely $SU(2)$ (the *Special Unitary group*. They are both *double covers* of $SO(3)$ (i.e., each rotation in $SO(3)$ corresponds to two elements in the double cover group). In contrast to $SO(3)$, they are both topologically speaking *simply connected*. $SO(3)$ is instead *not contractible* and it is a typical example of a non simply connected manifold "without holes." This means that if we define a continuous closed curve in $SO(3)$ (a ring), there is not always a way to continuously make this ring smaller and smaller in order to let it become a point.

**Remark 4 (Vector product)** *It is a misconception to interpret the vector product in three space as an extra structure on $\mathcal{E}_*$ (the scalar product being the other structure). This is NOT the case, because the vector product is a consequence of the fact that for $\mathcal{E}$ the Lie group of rotations $SO(3)$ can be intrinsically defined. As we have just seen, the Lie algebra $\mathfrak{so}(3)$ has a commutator defined on it and therefore, after a choice of orientation of space, the intrinsic bijection between $\mathfrak{so}(3)$ and $\mathcal{E}_*$ carries the commutation operation defined in $\mathfrak{so}(3)$ over to the vector product in $\mathcal{E}_*$.*

*It is possible to find two bijections in the following way. A vector of $\mathcal{E}_*$ is characterized by a direction $d$, an orientation $v$ and a module $m$. An element of $\mathfrak{so}(3)$ represents an angular velocity. It is possible to see that any angular motion leaves a line $d'$ invariant. This is the first step in this bijection: the vector of $\mathcal{E}_*$ which we associate to a vector in $\mathfrak{so}(3)$ will have as direction the line left invariant by this rotation ($d = d'$). We still need a module and an orientation. It is possible to show that there exists a positive definite metric on $\mathfrak{so}(3)$, called the* Killing form *(Selig 1996), which defines the magnitude of a rotation in a unique, coordinate-independent way. We can therefore choose as $m$ the module of this angular velocity vector calculated using the metric on $\mathfrak{so}(3)$. The only choice we are left*

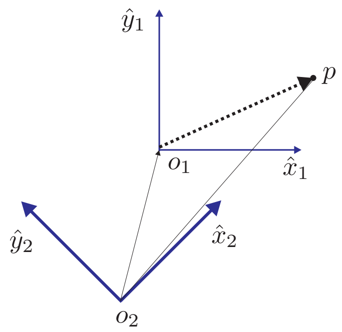



Figure 1.4: A general change of coordinates.

*with is that of the orientation. We clearly have two possible choices to orient the line. If we look at the rotation motion around its axis as a clockwise motion, we can orient the line as going away from us or as coming toward us. In the first case we say that we choose a right-handed orientation and in the second case a left-handed orientation. This is clearly related to the orientation chosen for the Euclidean space because if we consider a point* $x$ *moving due to the described rotation, we can consider a vector* $x(t) - o$ *and a vector* $x(t + dt) - o$ *where* $o$ *is the orthogonal projection*[2] *of* $x$ *on the axis of rotation. The direction of the vector* $(x(t) - o) \wedge (x(t + dt) - o)$ *is the associated direction of rotation.*

*Notice that if we look at this motion through a mirror, the orientation is changed because, under the same rotational motion, a line oriented toward the mirror will be seen through the mirror as a line oriented in the opposite direction. This is called a* reflection.

### Translations

Translations are much simpler than rotations. If we have two coordinate systems displaced with respect to each other, the change of coordinates is simple:

$$p^i = (\Psi_i \; o \; \Psi_j^{-1})(p^j) = p^j + p^i_j, \tag{1.10}$$

where $p^i_j$ represents the position of the origin of $\Psi_j$ with respect to the origin of $\Psi_i$, and expressed as a numerical vector in one of the two frames (either one, because they are not rotated with respect to each other). Obviously this set of translations is also a Lie group, due to the well-known additivity of translation vectors.

### General changes of coordinates

It is now possible to consider a general transformation including any rotation and/or translation. Figure 1.4 shows the change of coordinates from a 2D frame $\Psi_1$ to another frame $\Psi_2$. $p^2 \in \mathbb{R}^3$ contains the coordinates of the vector $(p - o_2) \in \mathcal{E}_*$ expressed in the frame $\Psi_2$. Taking the coordinates of the following vector equality

$$(p - o_2) = (p - o_1) + (o_1 - o_2),$$

yields

$$p^2 = R^2_1 p^1 + p^2_1,$$

[2] We can consider an orthogonal projection since we are in a Euclidean space

where $p_1^2$ are the coordinates of the vector $(o_1 - o_2)$ expressed in the frame $\Psi_2$. In matrix form, this change of coordinates becomes:

$$\begin{pmatrix} p^2 \\ 1 \end{pmatrix} = \begin{pmatrix} R_1^2 & p_1^2 \\ 0_3^T & 1 \end{pmatrix} \begin{pmatrix} p^1 \\ 1 \end{pmatrix}.$$

The transformation matrix in this equation is often called the *homogeneous (transformation) matrix* $H_1^2 \in \mathbb{R}^{4\times4}$:

$$H_1^2 := \begin{pmatrix} R_1^2 & p_1^2 \\ 0_3^T & 1 \end{pmatrix},$$

because it represents a coordinate transformation between two frames by a simple matrix multiplication. This matrix can also be interpreted as a projective coordinate change since we can consider the original three-dimensional vector as an element belonging to $\mathbb{PR}^3$.

We can recognize a couple of sub-cases, such as a pure translation:

$$H_1^2 := \begin{pmatrix} I & p_1^2 \\ 0_3^T & 1 \end{pmatrix},$$

and a pure rotation around the origin:

$$H_1^2 := \begin{pmatrix} R_1^2 & 0_3 \\ 0_3^T & 1 \end{pmatrix}.$$

The homogeneous transformation matrix can represent rotations around a point different from the origin: subsequent coordinate changes compose according to the so-called *chain rule*

$$H_1^4 = H_1^2 H_2^3 H_3^4, \tag{1.11}$$

for any intermediate frames $\Psi_2$ and $\Psi_3$. So, a rotation around any given point $p$ can now be expressed as a translation followed by a rotation and an inverse translation:

$$\begin{pmatrix} I & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & -p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & p - Rp \\ 0 & 1 \end{pmatrix}. \tag{1.12}$$

The rightmost transformation on the left-hand site transports the coordinates of the rigid body that rotates about $p$ from their expression with respect to the origin to their expression with respect to the point $p$; the middle transformation does the rotation about $p$; and the leftmost transforms all coordinates back to the origin again. It's clear that only the point $p$ is mapped on itself for all the possible $R$, hence the composite transformation is a rotation about $p$.

The inverse of an homogeneous transformation matrix is simple:

$$H_2^1 = (H_1^2)^{-1} = \begin{pmatrix} (R_1^2)^T & -(R_1^2)^T p_1^2 \\ 0_3^T & 1 \end{pmatrix} = \begin{pmatrix} R_2^1 & -R_2^1 p_1^2 \\ 0_3^T & 1 \end{pmatrix}.$$

The set of matrices

$$SE(3) := \left\{ \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \; s.t. \; R \in SO(3), p \in \mathbb{R}^3 \right\}$$

is called the *Special Euclidean* matrix group, which is of great importance in robotics. $SE(3)$ is a *Lie group*, as proven by the same arguments as for $SO(3)$ above. Hence,

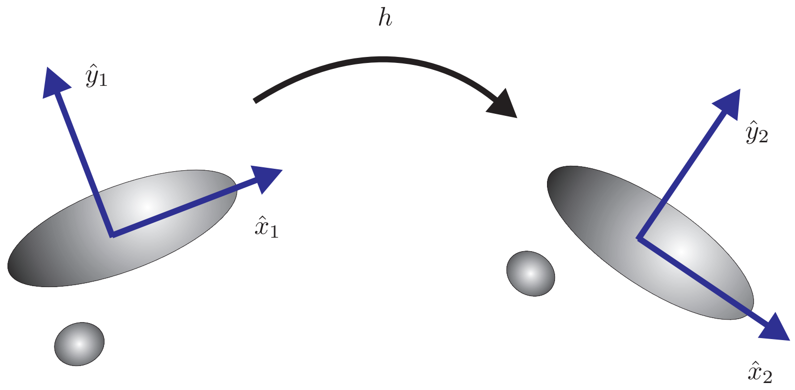



Figure 1.5: A representation of a motion.

velocities $\dot{H}_i^j$ map from the tangent space $T_{H_i^j}SE(3)$ to $\mathfrak{se}(3)$ either with left or right translations (see Appendix B). The elements of $\mathfrak{se}(3)$ are of the following form:

$$\mathfrak{se}(3) = \left\{ \begin{pmatrix} \Omega & v \\ 0 & 0 \end{pmatrix} \quad ; \quad \Omega \in \mathfrak{so}(3), v \in \mathbb{R}^3 \right\}. \tag{1.13}$$

This is proven from the definition by considering $H(t) \in SE(3)$ as a parameterized curve, and seeing (Appendix B) that elements of the form $\dot{H}H^{-1}$ and $H^{-1}\dot{H}$ belong to $\mathfrak{se}(3)$, and have the form given in Eq. (1.13). The Lie algebra $\mathfrak{se}(3)$ has its commutator:

Furthermore, the algebra commutator of $\mathfrak{se}(3)$ is such that since

$$\tilde{T}_i := \begin{pmatrix} \tilde{\omega}_i & v_i \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(3) \Rightarrow [\tilde{T}_1, \tilde{T}_2] = \begin{pmatrix} [\tilde{\omega}_1, \tilde{\omega}_2] & \tilde{\omega}_1 v_2 - \tilde{\omega}_2 v_1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(3).$$

### Motions versus changes of coordinates

The relationship between a *change of coordinates* and a *motion* follows from Figure 1.5:

> the moved points in the moved coordinate system $\Psi_2$ have the same coordinates as the original points in the original coordinate system $\Psi_1$.

In other words:

$$\psi_2 \, o \, h \, o \, \psi_1^{-1} = I,$$

where $I$ is the $4 \times 4$ identity matrix if $\psi_i$ are projective coordinates. From

$$\psi_2 \, o \, h \, o \, \psi_1^{-1} = \underbrace{\psi_2 \, o \, \psi_1^{-1}}_{H_1^2} \, o \, \psi_1 \, o \, h \, o \, \psi_1^{-1} = I,$$

it follows that

$$\psi_1 \, o \, h \, o \, \psi_1^{-1} = H_2^1,$$

which implies that:

> The representation of the motion from $\Psi_1$ to $\Psi_2$ in the coordinate $\Psi_1$ ($\psi_1 \, o \, h \, o \, \psi_1^{-1}$) is the *inverse* of the coordinate change from $\Psi_1$ to $\Psi_2$ ($\psi_2 \, o \, \psi_1^{-1}$).

The same motion expressed in $\Psi_2$ is:

$$\psi_2 \, o \, h \, o \, \psi_2^{-1} = \psi_2 \, o \, \underbrace{\psi_1^{-1} \, o \, \psi_1}_{I} \, o \, h \, o \, \underbrace{\psi_1^{-1} \, o \, \psi_1}_{I} \, o \, \psi_2^{-1} = H_1^2 H_2^1 H_2^1 = H_2^1.$$

To summarize:

- A general change of Cartesian coordinates in $\mathcal{E}(3)$ from $\Psi_i$ to $\Psi_j$ can be expressed with a matrix of the form:

$$H_i^j = \begin{pmatrix} R_i^j & p_i^j \\ O_3^T & 1 \end{pmatrix} \in SE(3),$$

  where $R_i^j \in SO(3)$ and $p_i^j \in \mathbb{R}^3$.

- For a rigid motion

$$h : \mathcal{E} \to \mathcal{E}; \; p \mapsto q,$$

  which maps $\Psi_i$ to $\Psi_j$, we have that

$$q^i = H_j^i p^i, \quad \text{and} \quad q^j = H_j^i p^j,$$

### 1.3.2 The finite twist description

This description is the geometrical one used in Screw Theory (Ball 1900), and it is based on *Chasles' Theorem*. Chasles' Theorem (which will be proved later on using Lie groups) says that any motion of a rigid body can be achieved as a rotation around a geometrical line $l$ together with a pure translation along $l$. The line $l$ is called the *(screw) axis* of the motion. For a pure translation, this axis is at infinity.

It can furthermore be proven that a pure non-trivial rotation and a pure non-trivial translation are commutative if and only if they are around and along the same geometrical line. Therefore, it is not important if we consider first the rotation and then the translation, or vice versa.

#### The screw axis

The axis of the motion is a line. The set of finite lines in space can be parameterized by a minimum of *four* scalars; for example by the so-called *Denavit-Hartenberg line coordinates*. Many other *non-minimum* coordinate sets are available for lines, the most common of which are the so-called *Plücker coordinates*. These also allow directly to describe lines at infinity, which describe pure translations. We will use these coordinates in what follows.

We can define a line as the linear locus passing through two points $p_1$ and $p_2$. Once we have chosen a coordinate frame $\Psi_i$, we can consider their projective coordinates:

$$P_1^i = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ s_1 \end{pmatrix} \quad \text{and} \quad P_2^i = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ s_2 \end{pmatrix}.$$

We can consider the following six-dimensional real vector which contains the *Plücker ray coordinates* for the line joining $p_1$ and $p_2$ with respect to frame $\Psi_i$:

$$l^i := \left( \begin{vmatrix} x_1 & x_2 \\ s_1 & s_2 \end{vmatrix}, \begin{vmatrix} y_1 & y_2 \\ s_1 & s_2 \end{vmatrix}, \begin{vmatrix} z_1 & z_2 \\ s_1 & s_2 \end{vmatrix}, \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}, \begin{vmatrix} z_1 & z_2 \\ x_1 & x_2 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right)^T, \tag{1.14}$$

defined using 2-by-2 determinants of the coordinates of the points. The question arises immediately wether $l^i$ is a meaningful numerical representation of the line joining $p_1$ and $p_2$ in the coordinate $\Psi_i$ ? The answer is yes if we consider it as an element belonging to $\mathbb{PR}^5$ instead of as an element belonging to $\mathbb{R}^6$ (as we will explain shortly), and the advantage of this representation is that it automatically allows to represent lines at infinity without any extra definitions! The previous coordinates are well posed and useful for the following reasons:

- If instead of taking $P_1^i$ and $P_2^i$ we take two other equivalent projective coordinates $\alpha P_1^i$ and $\beta P_2^i$ we obtain the same projective coordinates for the line, namely $\alpha\beta l^i \sim l^i$.
- If instead of taking $P_1^i$ we take another point on the line, namely $P_1^i + \alpha(P_1^i - P_2^i)$, the properties of determinants assure that the line representation does not change.
- If both points are at infinity (i.e., $s_1 = s_2 = 0$), $l^i$ is still properly defined and characterized by its first three components being equal to 0.

It is easily possible to see that Eq. (1.14) can be written as:

$$l^i = \begin{pmatrix} s_2 p_1 - s_1 p_2 \\ p_1 \wedge p_2 \end{pmatrix} = \begin{pmatrix} s_2 p_1 - s_1 p_2 \\ p_1 \wedge (p_2 - p_1) \end{pmatrix}. \tag{1.15}$$

From this last expression, it is possible to consider two cases: the first one when the defining points are finite ($s_1, s_2 \neq 0$) and the second when the points are at infinity ($s_1, s_2 = 0$) and therefore describe a line at infinity. In the first case, we have as coordinates of a finite line a vector of the form:

$$\begin{pmatrix} \omega \\ r \wedge \omega \end{pmatrix}, \tag{1.16}$$

where $\omega$ is a vector pointing from $p_1$ to $p_2$ and therefore spanning the line, and $r$ can be any point on the line since

$$r = p_1 + \alpha(p_2 - p_1) \Rightarrow r \wedge \omega = p_1 \wedge \omega.$$

Furthermore, since the representation of the line is projective, for any $\alpha \neq 0$,

$$\alpha \begin{pmatrix} \omega \\ r \wedge \omega \end{pmatrix}$$

represents the same line. This finite line can be associated to a rotation axis. If furthermore, besides the rotation axis we define a direction of the line to consider the positive rotation around the axis and a magnitude representing the rotation angle, we can completely determine a *finite rotation*.

We can give a compact description of a rotation at once if we leave the projective nature of the line description and we consider $\omega$ as a vector with a magnitude and a direction. We can then talk about a unit line by considering an $\hat{\omega}$ such that $||\hat{\omega}|| = 1$ and then call a unit line an element of the form:

$$\begin{pmatrix} \hat{\omega} \\ r \wedge \hat{\omega} \end{pmatrix}. \tag{1.17}$$

It is then possible to describe what is called a *rotor* as a triple $(l, \hat{\omega}, ||\omega||)$ with an oriented axis and a magnitude. A pure rotation of $\theta$ around an axis oriented[3] and directed by $\hat{\omega}$ and passing through $r$ can be expressed with the following rotor:

$$\theta \begin{pmatrix} \hat{\omega} \\ r \wedge \hat{\omega} \end{pmatrix} = \begin{pmatrix} \theta\hat{\omega} \\ r \wedge \theta\hat{\omega} \end{pmatrix} = \begin{pmatrix} \omega \\ r \wedge \omega \end{pmatrix}, \tag{1.18}$$

which should be interpreted as a triple $(l, d, \theta)$ where $l$ is a line, $d$ is a chosen positive direction and $\theta$ is an associated magnitude. This can also be interpreted as a line vector: a vector $\omega$ that can move along its spanning line.

In the second case, for a line at infinity, we obtain a representation of the form

$$\begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{1.19}$$

where $v := p_1 \wedge (p_2 - p_1)$. This can also be nicely interpreted as a rotation at infinity which turns out to be representative of a pure translation as explained hereafter. Suppose we want to consider a rotation around an axis which we want to move toward infinity: $\alpha r$ with $\alpha \to \infty$. In order to find a sensible representation, we need to consider a line set parameterized by:

$$\begin{pmatrix} \frac{1}{\alpha}\omega \\ (\alpha r) \wedge (\frac{1}{\alpha}\omega) \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha}\omega \\ r \wedge \omega \end{pmatrix} \tag{1.20}$$

For any value of $\alpha$, the moment $r \wedge \omega$ must be interpreted as the linear motion of a point passing through the origin of the coordinate system and rigidly attached to the space which is rotating around the axis passing through $\alpha r$. The larger $\alpha$, the closer will be the linear velocities of the points close to the origin. In the extreme, for $\alpha \to \infty$, all the finite points will have a linear motion equal to $v := r \wedge \omega$. This implies that a rotation at infinity corresponds to a pure translation and it is represented by the following line at infinity:

$$\begin{pmatrix} 0 \\ v \end{pmatrix}.$$

### The pitch

It is possible to show that there is an intrinsic measure for rotations (the Killing form of $SO(3)$) and therefore it is possible to quantify a rotation around the axis with a number $\theta$ (with physical, dimensionless units *radians*). In the same way, being in a Euclidean space, we can measure a translation displacement $t$ of a motion intrinsically without using any coordinates, but only the metric of the space; its dimensions are, for example, in meters. The ratio between the rotation and the translation is called the pitch $\lambda$ of the finite motion:

$$\lambda := \frac{t}{\theta}.$$

It's a coordinate-independent property of the motion.

[3]Clearly, implicitly, we have associated a positive direction with a positive rotation; this can be done in two ways by considering a right or left convention and corresponds to the orientation of space appearing in the term $r \wedge \omega$.

#### The Screw

Geometrically, a screw $S$ is nothing else than a line $l$ together with a scalar pitch $\lambda$:

$$S := (l, \lambda).$$

Since the dimension of the space of lines is four, the dimension of the space of screws is five. Note that a screw, as a line, does *not* have any module information a priori. If this module information is given as we did with the line, we get a six-dimensional space of measurable screws.

#### Chasles' theorem and finite twists

Chasles' theorem states that any finite rigid body motion can be expressed as a rotation around an axis plus a translation along the same axis. We will now prove this theorem by first relating the finite motions to Lie groups. One of the properties of Lie groups is the possibility to relate vectors belonging to the Lie algebra of a Lie group to elements of the Lie group (see Appendix B). For our goal, it is sufficient to realize that for *any* matrix $H \in SE(3)$, there exists always at least one[4] matrix $\tilde{T} \in se(3)$ such that:

$$H = e^{\tilde{T}},$$

where the exponential is the usual exponential of matrices, defined as:

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

This implies that for any motion which can be represented by a matrix $H \in SE(3)$ (see Sect. 1.3.1), we can consider a new *matrix* representation that expresses an element of $\mathfrak{se}(3)$ in the form:

$$\tilde{T} = \begin{pmatrix} \tilde{\omega} & v \\ 0 & 1 \end{pmatrix}. \tag{1.21}$$

We can define an equivalent *vector* representation of $\tilde{T}$:

$$T := \begin{pmatrix} \omega \\ v \end{pmatrix}.$$

Now suppose for the moment that $\omega \neq 0$. In this case, it is easy to see that we can always find a vector $r$ and a scalar $\lambda$ such that:

$$\begin{pmatrix} \omega \\ v \end{pmatrix} = \begin{pmatrix} \omega \\ r \wedge \omega \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ \omega \end{pmatrix}, \tag{1.22}$$

where $\lambda\omega$ is the $v$ component along $\omega$ and $r \wedge \omega$ is the one orthogonal to $\omega$.

It is also easy to find an expression for $r$ and $h$ in the following way. The second part of the Chasles decomposition can be written as:

$$v = r \wedge \omega + \lambda\omega, \tag{1.23}$$

and therefore, taking the vector product with $\omega$ we obtain:

$$\omega \wedge v = \omega \wedge (r \wedge \omega). \tag{1.24}$$

[4] In certain cases there could be two solutions.

With some straightforward geometric considerations, it is furthermore possible to see that $\omega \wedge (r \wedge \omega) = r||\omega||$, and therefore we find an expression for $r$:

$$r = \frac{\omega \wedge v}{||\omega||^2}. \tag{1.25}$$

The calculation of the pitch $\lambda$ is even simpler. By taking the scalar product of $\omega$ with Eq. (1.23) we obtain:

$$\omega^T v = \omega^T (r \wedge \omega) + \lambda \omega^T \omega, \tag{1.26}$$

and since the first term on the right side is zero (from the property of the mixed product), we obtain:

$$\lambda = \frac{\omega^T v}{||\omega||^2}. \tag{1.27}$$

Equivalently to Eq. (1.22), using the matrix form, it is always possible to decompose $\tilde{T}$ as:

$$\tilde{T} = \tilde{T}_r + \lambda \tilde{T}_t,$$

where

$$\tilde{T}_r := \begin{pmatrix} \tilde{\omega} & -\tilde{\omega} r \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{T}_t := \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}.$$

The elements $\tilde{T}_r$ and $\tilde{T}_t$ are both matrices belonging to $\mathfrak{se}(3)$. It is therefore possible to check whether these two matrices are commutative and therefore whether:

$$\tilde{T}_r \tilde{T}_t = \tilde{T}_t \tilde{T}_r,$$

which is true if and only if the Lie algebra commutator (see Appendix B) is zero, i.e.:

$$[\tilde{T}_r, \tilde{T}_t] = 0.$$

This can be easily seen to be satisfied for the given decomposition and therefore, thanks to the Campbell-Baker-Hausdorff formula (Selig 1996), it is possible to see that we can write:

$$e^{\tilde{T}} = e^{\tilde{T}_r} e^{\lambda \tilde{T}_t} = e^{\lambda \tilde{T}_t} e^{\tilde{T}_r}.$$

With some calculations, it is furthermore possible to see that:

$$H_r := e^{\tilde{T}_r} = \begin{pmatrix} R & -(Rr - r) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad H_t := e^{\lambda \tilde{T}_t} = \begin{pmatrix} I & \lambda \omega \\ 0 & 1 \end{pmatrix}.$$

where $R = e^{\tilde{\omega}} \in SO(3)$ is a rotation matrix. We finally obtain:

$$H = H_r H_t = H_t H_r,$$

which shows that the two motions are commutative as expected. Using wha has been explained in Sect. 1.3.1, it is possible to see that $H_r$ represents a pure rotation of $\theta := ||\omega||$ around an axis passing through $r$ and along $\omega$, and $H_r$ represents a pure translation of $\theta\lambda$ along the direction $\omega$. We have therefore proven Chasles' theorem, since we have shown that any motion can be described as a rotation around an axis after a translation along the same axis or vice versa.

In the case in which $\omega = 0$, following the same arguments as above, we can see that this represents a pure translation along $v$.

Furthermore, we have also seen that the six-dimensional vector $T$ can be interpreted exactly as the sum of the Plücker ray coordinate vector of a finite line and an infinite line:

$$T = T_r + \lambda T_t,$$

where $T_t$ is clearly related to $T_r$ by Eq. (1.22). The infinite line $T_t$ is called the *polar* of the finite line $T_r$.

This implies that we can express *any* motion by specifying a finite line, and a scalar $\lambda$ which is exactly the interpretation of a screw with an associated module!

For pure translations we only have a line at infinity which can be easily interpreted as a screw with infinite pitch using an argument similar to the one used in Eq. (1.20).

For a pure rotation we only have a finite line which can be interpreted as a screw with pitch equal to zero.

We have also shown that the linear combination of lines seen as elements of $\mathbb{PR}^5$ is in general *not* a line, but a screw! We can therefore express any linear combination of lines as a screw and therefore as a finite line plus the pitch times its polar.

Note that, as with the line, screws are projective entities which implies that they do not in general have a direction and a module. Once we associate a direction and a module, we can use them directly to express what is called a finite twist in order to express any rigid body motion. In this case, a five-dimensional screw becomes six-dimensional, due to the extra information about the magnitude.

## 1.4 Velocities for a rigid body: Twists

Once again, also for velocities, we can give either a Lie group or a screw interpretation. We will analyse them both and find relations as in the case of finite motions.

### 1.4.1 Twists as elements of $\mathfrak{se}(3)$

Elements of $\mathfrak{se}(3)$ (see Eq. (1.13)) are called twists in mechanics and they represent the velocity of a rigid body motion geometrically (i.e., coordinate-free). In order to understand this, we must look at the action of elements of $SE(3)$ on points of $\mathbb{R}^3$. Consider *any* point $p$ not moving with respect to the reference frame $\Psi_i$. If we indicate with $p^i$ its numerical representation in $\Psi_i$, this means that $\dot{p}^i = 0$. Take now a second reference $\Psi_j$ possibly moving with respect to $\Psi_i$. By looking at the change of coordinates and differentiating, we obtain:

$$\dot{P}^j = \dot{H}_i^j P^i, \quad \text{where we denoted} \quad P^k := \begin{pmatrix} p^k \\ 1 \end{pmatrix}, \qquad k = i, j,$$

and $H_i^j \in SE(3)$. We can now transport $\dot{H}_i^j$ to the identity either by left or right translation. If we do so, we obtain the two possibilities reported in Table 1.1. In the case we

| Transport | $\tilde{T}$ | $\dot{H}_i^j$ | $\dot{P}^j$ | Notation |
|---|---|---|---|---|
| Left | $\tilde{T} := H_j^i \dot{H}_i^j$ | $\dot{H}_i^j = H_i^j \tilde{T}$ | $\dot{P}^j = H_i^j(\tilde{T}P^i)$ | $\tilde{T}_i^{i,j}$ |
| Right | $\tilde{T} := \dot{H}_i^j H_j^i$ | $\dot{H}_i^j = \tilde{T} H_i^j$ | $\dot{P}^j = \tilde{T}(H_i^j P^i)$ | $\tilde{T}_i^{j,j}$ |

Table 1.1: The used notation for twists.

consider the left translation, working out the terms, we obtain:

$$\dot{P}^j = \underbrace{H_i^j(\tilde{T}}_{\dot{H}_i^j} P^i) = \begin{pmatrix} \dot{p}^j \\ 0 \end{pmatrix} = \begin{pmatrix} R_i^j & p_i^j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p^i \\ 1 \end{pmatrix} \Rightarrow \dot{p}^j = R_i^j(\omega \wedge p^i) + R_i^j v$$

and using the right translation instead we obtain:

$$\dot{P}^j = \underbrace{\tilde{T}(H_i^j}_{\dot{H}_i^j} P^i) = \begin{pmatrix} \dot{p}^j \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_i^j & p_i^j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^i \\ 1 \end{pmatrix} \Rightarrow \dot{p}^j = \omega \wedge (R_i^j p^i + p_i^j) + v$$

From the previous two expressions and Table 1.1, it is possible to see that $T_a^{b,c}$ represents the motion of $\Psi_a$ with respect to $\Psi_c$ expressed in frame $\Psi^b$. Note that $v$ is *not* the relative velocity of the origins of the coordinate systems! (This would not give rise to a geometrical entity.) $v$ is the velocity of an *imaginary* point connected to the moving body and instantaneously passing through the origin at the instant under consideration.

### Changes of coordinates for twists

Using the left and right map, we have seen that:

$$\tilde{T}_i^{i,j} = H_j^i \dot{H}_i^j \text{ and } \tilde{T}_i^{j,j} = \dot{H}_i^j H_j^i. \tag{1.28}$$

It can be easily seen that

$$\tilde{T}_i^{j,j} = H_i^j \tilde{T}_i^{i,j} H_j^i, \tag{1.29}$$

and this gives an expression for changes of coordinates of twists. This clearly corresponds to the adjoint group representation introduced in Appendix B on pag. 70. It is easier to work with the six-dimensional vector form of twists and it is possible to see that we can find a matrix expression for the adjoint representation:

$$T_i^{j,j} = Ad_{H_i^j} T_i^{i,j}. \tag{1.30}$$

It is possible to proof that this matrix representation is:

$$Ad_{H_i^j} := \begin{pmatrix} R_i^j & 0 \\ \tilde{p}_i^j R_i^j & R_i^j \end{pmatrix}.$$

By differentiation of the previous matrix as a function of $H_i^j$ we can also find an expression for its time derivative and the adjoint representation of the algebra $ad$ (see Appendix B). It can be shown that:

$$\dot{\left(Ad_{H_i^j}\right)} = Ad_{H_i^j}\ ad_{T_i^{i,j}} \qquad \text{with} \qquad T_i^{i,j} := H_j^i \dot{H}_i^j, \tag{1.31}$$

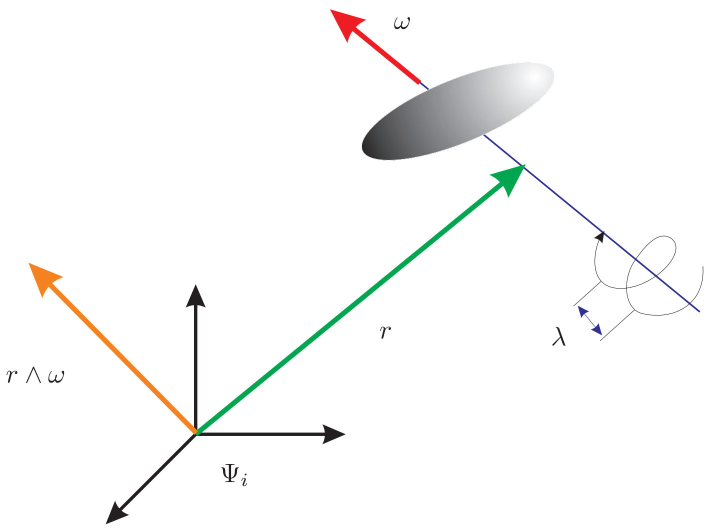



Figure 1.6: Intuition of a Twists

where

$$ad_{T_i^{k,j}} := \begin{pmatrix} \tilde{\omega}_i^{k,j} & 0 \\ \tilde{v}_i^{k,j} & \tilde{\omega}_i^{k,j} \end{pmatrix}$$

is the adjoint representation we were looking for. This can be easily checked by testing the relation proved in Eq. (B.9) for a general matrix Lie group. It is furthermore easily possible to prove that:

$$ad_{T_i^{k,j}} T_l^{k,i} = [T_i^{k,j}, T_l^{k,i}],$$

which can be useful to analyse mechanisms.

### 1.4.2 Twists as applications on screws

With the same line of reasoning as we did for finite motions, we can give an interpretation to elements of $\mathfrak{se}(3)$ as a twist applied on a geometrical screw. In a similar way, we could state Chasles' theorem, but now applied to instantaneous velocities instead of to finite twists. Similarly as before, the theorem says that any element of $\mathfrak{se}(3)$ can be described as a pure instantaneous rotation around an axis plus a pure instantaneous translation along the same axis:

$$\begin{pmatrix} \omega \\ v \end{pmatrix} = ||\omega|| \underbrace{\begin{pmatrix} \hat{\omega} \\ r \wedge \hat{\omega} \end{pmatrix}}_{\text{rotation}}, +\alpha \underbrace{\begin{pmatrix} 0 \\ \hat{\omega} \end{pmatrix}}_{translation}$$

where $\omega = ||\omega||\hat{\omega}$. Analyzing the previous formula and with reference to Figure 1.6, it is possible to see that $v$ is the velocity of an imaginary point passing through the origin of the coordinate system in which the twist is expressed and moving together with the object. The six-vector representing the rotation is what we called previously a *rotor* and can be associated to a geometrical line, namely the line passing through $r$ and spanned by $\omega$ which is left invariant by the rotation.

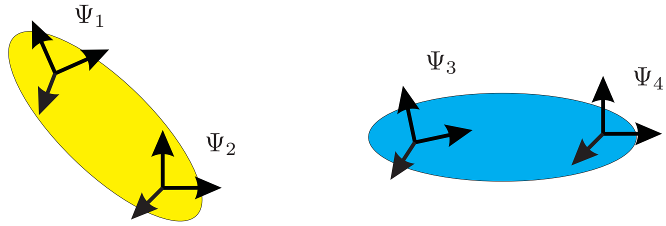



Figure 1.7: Interpretation of the indices for the notation used in twists.

Once again, the theorem of Chasles is one of the two theorems on which screw theory is based because it gives to elements of $\mathfrak{se}(3)$ a real tensorial geometrical interpretation. This interpretation is the one of a *motor* or *screw* which are entities characterized by a geometrical line and a scalar called the pitch. This pitch relates the ratio of translation and rotation along and around the line.

If we furthermore consider the change of coordinates expressed in Eq. (1.30), this can be interpreted as a change of Plücker ray coordinates of the geometrical line associated to the twist using Chasles' theorem.

This also implies that in the notation $T_i^{k,j}$ we use, the frames $\Psi_j$ can be any frame fixed to the considered body of reference and this holds also for $\Psi_i$.

With reference to Figure 1.7, we can prove what has just been said by noticing that since $H_1^2$ and $H_3^4$ are constant, the following is true:

$$\tilde{T}_4^{1,1} = \dot{H}_4^1 H_1^4 = \dot{H}_3^1 \underbrace{H_4^3 H_3^4}_{I} H_1^3 = \tilde{T}_3^{1,1},$$

and that

$$\tilde{T}_3^{2,1} = H_1^2 \tilde{T}_3^{1,1} H_2^1 = H_1^2 \dot{H}_3^1 H_1^3 H_2^1 = (\dot{\overline{H_1^2 H_3^1}})(H_1^3 H_2^1) = \tilde{T}_3^{2,2}. \tag{1.32}$$

We can therefore talk about $T_i^j$ as a geometric entity modeled as a screw without the necessity to express it numerically in a certain frame $\Psi_k$.

## 1.5 Forces applied to rigid bodies: Wrenches

The concept of duality is fundamental when working in mechanics. Within the theory of Lie groups, we know that $\mathfrak{se}(3)$ is a vector space, and as such, we can consider the dual $\mathfrak{se}^*(3)$ which is the vector space of *linear operators* on $\mathfrak{se}(3)$. This corresponds to the space of *wrenches* which are a six-dimensional generalization of three-dimensional forces to rigid bodies.

### 1.5.1 Wrenches as elements of $\mathfrak{se}(3)^*$

Twists are the generalization of velocities and are elements of $\mathfrak{se}(3)$. The dual vector space of $\mathfrak{se}(3)$ is called the dual Lie algebra and denoted with $\mathfrak{se}^*(3)$. It is the vector space of linear operators from $\mathfrak{se}(3)$ to $\mathbb{R}$. This space represents the space of 'forces' for rigid bodies which are called *wrenches*. The application of a wrench on a twists gives a scalar representing the power supplied by the wrench. A wrench in vector form will be a $6$ dimensional row vector since it is a co-vector (linear operator on vectors) instead than a vector.

$$W = \begin{pmatrix} m & f \end{pmatrix}$$

where $m$ represents a torque and $f$ a linear force. For what just said we have:

$$\text{Power} = WT$$

where $T$ is a twist of the object on which the wrench is applied. Clearly, to calculate the power, the wrench and the twist have to be numerical vectors expressed in the same coordinates and result

$$\text{Power} = WT = m\omega + fv.$$

Another representation of a wrench in matrix form is:

$$\tilde{W} = \begin{pmatrix} \tilde{f} & m \\ 0 & 0 \end{pmatrix}.$$

How do wrenches transform changing coordinate systems ? We have seen that for twists:

$$T_\bullet^{j,\bullet} = Ad_{H_i^j} T_\bullet^{i,\bullet}$$

where

$$Ad_{H_i^j} := \begin{pmatrix} R_i^j & 0 \\ \tilde{p}_i^j R_i^j & R_i^j \end{pmatrix}.$$

Suppose to supply power to one body attached to $\Psi_j$ by means of a wrench which represented in $\Psi_j$ is $W^j$.

Changing coordinates from $\Psi_j$ to $\Psi_i$ the expression for the supplied power should stay constant and this implies that:

$$W^j T_j^{j,i} = W^j Ad_{H_i^j} T_j^{i,i} = (Ad_{H_i^j}^T (W^j)^T)^T T_j^{i,i} = W^i T_j^{i,i}$$

which implies that the transformation of wrenches expressed in vector form is:

$$(W^i)^T = Ad_{H_i^j}^T (W^j)^T.$$

Note that if the mapping $Ad_{H_i^j}$ was mapping twists from $\Psi_i$ to $\Psi_j$, the transposed maps wrenches in the opposite direction: from $\Psi_j$ to $\Psi_i$! This is a direct consequence of the fact that wrenches are duals to twists.

### 1.5.2 Wrenches as applications on screws

The concept of duality is instead not directly used in the theory of screws. This has often brought to useless discussions which could be avoided just by realizing the duality structure between twists and wrenches. For reasons which will not be explained here (Stramigioli, Maschke & Bidard 2000) and depending on the existence of what is called the Klein form, the decomposition result for twists given by Chasles' theorem,

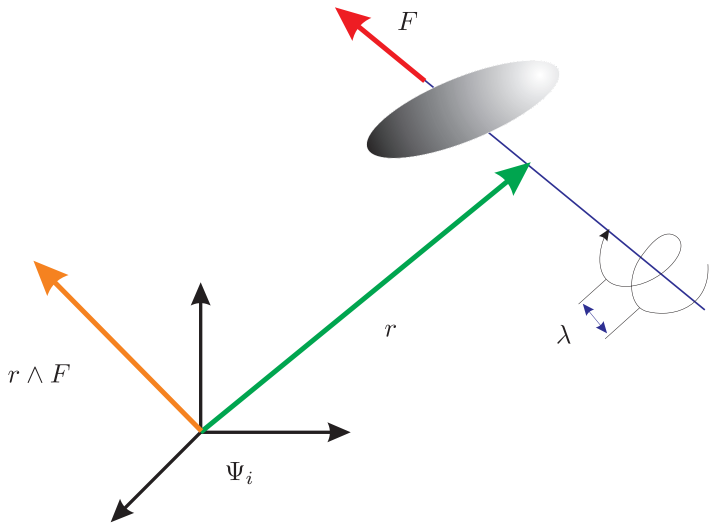



Figure 1.8: Intuition of a Wrench

can be given for wrenches by its dual analogous theorem called Poinsot's theorem. This theorem states that any element of $\mathfrak{se}^*(3)$ can be split as a sum of two terms:

$$\begin{pmatrix} m \\ f \end{pmatrix} = \begin{pmatrix} r \wedge f \\ f \end{pmatrix} + \lambda \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{1.33}$$

It can be directly seen that we have somehow inverted the the role of the first three and last three components with respect to Chasles theorem as shown in Eq. (1.22). This fact as profound explanations which can be found in (Stramigioli, Maschke & Bidard 2000). The components of Eq. (1.33) can still be interpreted as line components which are now called *Plücker axis* coordinates. For simplicity, the axis coordinates can be just considered as ray coordinates in which we inverted the first and the last three components. There are actually much deeper projective geometrical reasons for the relation between ray and axis coordinates which have to do with the self duality of lines in projective space (Lipkin 1985).

Also with reference to Figure 1.8, the first element of Eq. (1.33) is representing a pure force applied along the line passing through $r$. The second is instead a pure momenta which does not need to be associated to a finite line but to an infinite line which is again the polar of the line representing the pure force.

As we did with twists, we can describe a wrench as applied on a screw. This is true because once we have defined a line, a pitch and a magnitude, any wrench can be expressed as a line with a magnitude plus the pitch times its polar as shown in Eq. (1.33).

Note that the role of angular velocities for twists is taken by linear forces in wrenches since both elements are associable to a pure line: the first as a rotation axis and the second as an application line of a force. In a similar way, the role of pure translations is taken by pure moments.

This also implies that a wrench applied on a screw with zero pitch corresponds to a pure force and a wrench applied on a screw with infinite pitch corresponds to a pure moment.

## 1.6 Dynamics of a rigid body

In this section we analyse the dynamics of a rigid body in detail.

### 1.6.1 Kinetic co-energy and energy

Kinetic energy must be expressed as a function of the relative motion of a body with respect to an inertial reference. From now on we will consider an inertial reference frame $\Psi_0$.

Even basic courses of mechanics teach that the co-energy with respect to $\Psi_0$ of a point of mass $m$ and velocity $v$ with respect to an inertial reference is:

$$E_k^*(v) = \frac{1}{2}mv^2.$$

One could wonder why the previous quantity is called co-energy. Formally speaking, $E_k^*$ is a function of a velocity, and as a function it differs from:

$$E_k(p) = \frac{1}{2m}p^2$$

which is called kinetic energy of the mass, where $p$ denotes the momentum (Paynter 1960, Breedveld 1984).

We can consider a reference $\Psi_i$ rigidly connected to a body $B_i$. Talking about rigid bodies, we can than consider a constant mass distribution function $\rho : B_i \to \mathbb{R}^+$ which associates to each point a mass density. Furthermore, each Euclidean space has an $n-$form $\Omega$ corresponding to the volume form. The mass of a body $B_i$ can thus be defined as:

$$m_i = \int_{B_i} \rho_i(p)\Omega_i(p). \tag{1.34}$$

Now consider a relative motion of $B_i$ with respect to $\Psi_0$. The kinetic co-energy of an infinitesimal volume element $dV$ in a certain point $p \in B_i$ which has velocity $v$ with respect to $\Psi_0$ is:

$$dE_k^* = \frac{1}{2}\rho(p)v^2 dV.$$

The total energy is nothing else than the integral over $B_i$. To handle the energy easily, we look for an expression of the velocity $v$ which is a function of $T_i^{i,0}$, the twist of $B_i$ with respect to $\Psi_0$. Using coordinates we clearly have:

$$\begin{pmatrix} P_0 \\ 1 \end{pmatrix} = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P_i \\ 1 \end{pmatrix}.$$

This implies that the velocity of the fixed point $p_i \in B_i$ ($\dot{P}_i = 0$) for the reference $\Psi_0$ in the chosen coordinates is:

$$\begin{pmatrix} \dot{P}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_i \\ 1 \end{pmatrix} = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_i \\ 1 \end{pmatrix}$$

where the matrix containing $\tilde{\omega}$ and $v$ is $\tilde{T}_i^{i,0}$, the twist of the relative motion of $B_i$ with respect to $\Psi_0$ expressed in $\Psi_i$. It is possible to see that, after few manipulations:

$$\dot{P}_0 = \begin{pmatrix} -R\tilde{P}_i & R \end{pmatrix} \begin{pmatrix} \omega \\ v \end{pmatrix}.$$

We have shown therefore that for any fixed point $p_i \in B_i$ there is a linear map from $\mathfrak{se}(3)$ to the velocity of the point $p_i$ with respect to the inertial reference $\Psi_0$. This also implies that the square of the magnitude of $\dot{P}_0$ can be expressed as a quadratic form:

$$||\dot{P}_0||^2 = T_i^T M(P_i) T_i$$

where

$$M(P_i) := \begin{pmatrix} \tilde{P}_i^T R^T R \tilde{P}_i & \tilde{P}_i R^T R \\ -R^T R \tilde{P}_i & R^T R \end{pmatrix} = \begin{pmatrix} \tilde{P}_i^T \tilde{P}_i & \tilde{P}_i \\ \tilde{P}_i^T & I_n \end{pmatrix}$$

and $T_i := \begin{pmatrix} \omega^T & v^T \end{pmatrix}^T$. The matrix $M(P_i)$ can be singular for a single point $P_i$, but the sum of $M(P_i)$ for a sufficient number of noncollinear $P_i$ gives a nonsingular matrix. A consequence of this is that the integral of this matrix over a nonzero volume would always give a nonsingular matrix. Hence the total kinetic co-energy is:

$$E_k^*(T_i) = \frac{1}{2} T_i^T I T_i \tag{1.35}$$

where

$$I := \int_{B_i} M(P_i) \rho(P_i) \Omega_i(P_i).$$

It can be easily seen that this matrix which is called *inertia tensor* is symmetric with the following form:

$$I = \begin{pmatrix} Q^i & \tilde{P}^i \\ (\tilde{P}^i)^T & mI \end{pmatrix}$$

where $m$ is the mass of the inertial body.

### 1.6.2 Coordinate changes for inertias

The previous inertia is a tensor and therefore we can analyse how its numerical representation changes if we change the reference system. If we change the coordinates of $T_i^{i,0}$.

$$T_i^{i,0} = Ad_{H_k^i} T_i^{k,0}$$

we get:

$$T^* = \frac{1}{2} (T_i^{k,0})^T I^{k,i} T_i^{k,0}$$

where

$$I^{k,i} = (Ad_{H_k^i})^T I Ad_{H_k^i}$$

is the inertia tensor of body $i$ expressed in frame $\Psi_k$. Note that if $H_k^i$ is constant also the tensor $I^{k,i}$ will be constant. It is then well known in mechanics that there exists a frame $\Psi_k$ centered in the center of gravity and properly oriented such that the corresponding $H_k^i$ gives:

$$I^{k,i} = \begin{pmatrix} J_i & 0 \\ 0 & m_i I \end{pmatrix}$$

where

$$J_i = \begin{pmatrix} j_x & 0 & 0 \\ 0 & j_y & 0 \\ 0 & 0 & j_z \end{pmatrix}.$$

The frame $\Psi_k$ is called *principal inertia frame* for the rigid body. It is easy to see that if $\Psi_k$ is the principal inertia frame we obtain:

$$T^* = \frac{1}{2}(T_i^{k,0})^T \mathcal{I}^{k,i} T_i^{k,0} = \frac{1}{2}m_i||v_i^{k,0}||^2 + \frac{1}{2}(\omega_i^{k,0})^T J_i \omega_i^{k,0} = \frac{1}{2}m_i||v_i^{k,0}||^2 + \frac{1}{2}j_x\omega_x^2 + \frac{1}{2}j_y\omega_y^2 + \frac{1}{2}j_z\omega_z^2.$$

From this we can conclude that any rigid body $B_i$ of any shape, density and material behaves as a uniform ellipsoid centered and oriented in/as $\Psi_k$ of mass $m_i$ and principal inertias $j_x, j_y, j_z$ !!

In any reference, due to the non-singularity of the integral of $M(\cdot)$ and because of physical reasons, $I$ is always a positive definite matrix. The physical reason is that if we had a singular $I$, a non-zero twist would exist with no energy associated with it. For an object with a finite volume, this is clearly impossible.

It can be seen that $I$ represents a quadratic form defined in the Lie algebra $\mathfrak{se}(3)$. It is therefore possible to define intrinsically:

$$\langle , \rangle_I : \mathfrak{se}(3) \times \mathfrak{se}(3) \to \mathbb{R} \ ; \ (T_1, T_2) \mapsto T_1^T I T_2 \tag{1.36}$$

We can then see that the co-energy is equal to:

$$E_k^*(T_i^{i,0}) = \frac{1}{2}\langle T_i^{i,0}, T_i^{i,0}\rangle_I.$$

The quadratic form $I$ uniquely defines a bijection between elements of $\mathfrak{se}(3)$ and elements of $\mathfrak{se}^*(3)$ such that $N^i \in \mathfrak{se}^*(3)$ corresponds to $T_i^{i,0} \in \mathfrak{se}(3)$ if and only if:

$$N^i(T_i^*) = \langle T_i^{i,0}, T_i^* \rangle_I \quad \forall T_i^* \in \mathfrak{se}(3).$$

The quantity $N^i$, corresponding through the inertia tensor $I$ to $T_i^{i,0}$, is called the *momentum*, and it corresponds to the twist $T_i^{i,0}$.

**Remark 5** *Note that the quadratic form $I$ in the Lie algebra is a constant quadratic form, independent of the relative position of $B_i$ with respect to any other space. Therefore it is a quantity intrinsically associated with $B_i$. It is possible to see that, by the group operation of left transport, this quadratic form defines a left invariant Riemannian metric in the Lie group $SE(3)$.*

Since $I$ is nonsingular, we can define the following function of $N^i$, which will be called *kinetic energy* of the body:

$$E_k(N^i) = \frac{1}{2}\langle N^i, N^i \rangle_Y. \tag{1.37}$$

Where $\langle , \rangle_Y$ denotes the quadratic form on $\mathfrak{se}^*(3)$ corresponding to $Y = I^{-1}$. Numerically, we have:

$$E_k(N^i) = E_k^*(T_i^{i,0}) = \frac{1}{2}N^i T_i^{i,0}.$$

### 1.6.3 Euler equation of a rigid body

For the reasons seen in the previous section, the momenta of a rigid body is defined as:

$$(N^i)^T := I^i T_i^{i,0} \tag{1.38}$$

We defined $N$ as a row vector because it transforms like a wrench with changes of coordinates. This will be clearly shown by the second law of dynamics. It is once again a co-vector: a linear operator on elements of $\mathfrak{se}(3)$. Newton's law for a point mass says

$$\dot{p}^0 = F^0.$$

It can be shown that integrating this equation for a complete rigid body, it is generalized to

$$\dot{N}^{0,i} = W^{0,i}, \tag{1.39}$$

where $N^{0,i}$ is the momenta of body $i$ expressed numerically in frame $\Psi_0$ and $W^{0,i}$ the total wrench applied to body $i$ expressed in frame $\Psi_0$. Since the time derivative of the momenta is equal to a wrench, both terms of the equation transform identically with changes of coordinates and this is why the wrench is a co-vector. If we write the previous equation as an equality of column vectors, we have

$$(\dot{N}^{0,i})^T = (W^{0,i})^T,$$

and writing out the changes of coordinates we can rewrite it as

$$\dot{\left(Ad^T_{H^i_0}(N^i)^T\right)} = \dot{\left(Ad^T_{H^i_0}\right)}(N^i)^T + Ad^T_{H^i_0}(\dot{N}^i)^T = (W^{0,i})^T$$

Using the result of Eq. (1.31) for the time derivative of the matrix $Ad_{H^i_0}$ we can then write:

$$ad^T_{T^{0,i}_0} Ad^T_{H^i_0}(N^i)^T + Ad^T_{H^i_0}(\dot{N}^i)^T = (W^{0,i})^T,$$

and multiplying on the left for $Ad^T_{H^0_i}$ we get

$$\underbrace{Ad^T_{H^0_i} ad^T_{T^{0,i}_0} Ad^T_{H^i_0}}_{ad^T_{T^{i,i}_0}}(N^i)^T + \underbrace{Ad^T_{H^0_i} Ad^T_{H^i_0}}_{Id}(\dot{N}^i)^T = \underbrace{Ad^T_{H^0_i}(W^{0,i})^T}_{W^i}$$

which can be written as

$$(\dot{N}^i)^T = -ad^T_{T^{i,i}_0}(N^i)^T + (W^i)^T$$

and since $-T^{i,i}_0 = T^{i,0}_i$ we have that:

$$-ad_{T^{i,i}_0} = ad_{-T^{i,i}_0} = ad_{T^{i,0}_i}.$$

We have therefore obtained the final result:

$$(\dot{N}^i)^T = ad^T_{T^{i,0}_i}(N^i)^T + (W^i)^T. \tag{1.40}$$

If we furthermore express the previous equation in a principal inertial frame $\Psi_k$ we get:

$$\mathcal{I}^k \dot{T}^{k,0}_i = \begin{pmatrix} -\tilde{\omega}^{k,0}_k & -\tilde{v}^{k,0}_k \\ 0 & -\tilde{\omega}^{k,0}_k \end{pmatrix} \mathcal{I}^k T^{k,0}_k + (W^k)^T$$

which can be written component-wise as:

$$\begin{aligned} J\dot{\omega}^{k,0}_k &= (J\omega^{k,0}_k) \wedge \omega^{k,0}_k + (m^{k,i})^T \\ mv^{k,0}_k &= mv^{k,0}_k \wedge \omega^{k,0}_k + (f^{k,i})^T \end{aligned}$$

which is indeed a more common form.

## 1.7 Exponential of Lie algebras

In what will follow, it is very important to be able to calculate the matrix exponential of matrices belonging either to $\mathfrak{so}(3)$ or even more important to $\mathfrak{se}(3)$. This is relevant to use the presented techniques in practice. We recall that a matrix exponential is defined as:

$$e^A := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots \tag{1.41}$$

It is possible to see (prove it as an exercise) that the following identities hold:

1. $\left(e^A\right)^{-1} = e^{-A}$
2. $e^{HAH^{-1}} = He^AH^{-1}$
3. $e^Ae^B = e^Be^A$ if and only if $AB - BA = 0$

### 1.7.1 Exponential of elements of $\mathfrak{so}(3)$

We now start analyzing how the exponential of a $3 \times 3$ skew-symmetric matrix looks like. We will get to the well known *Rodriguez formula*. Clearly, we can always write an element $\omega$ of $\mathfrak{so}(3)$ as:

$$\omega = \theta\hat{\omega}$$

where $\theta \in \mathbb{R}$ and $\hat{\omega}$ is a vector of unit length. By straight forward calculation, it is possible to check that

$$\tilde{\omega}^2 = \omega\omega^T - I||\omega||^2 \tag{1.42}$$

$$\tilde{\omega}^3 = -\tilde{\omega}||\omega||^2 \tag{1.43}$$

Using the previous equations recursively, it is possible to see that:

$$e^{\theta\tilde{\hat{\omega}}} = I + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)}_{\sin(\theta)}\tilde{\omega} + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \dots\right)}_{1-\cos(\theta)}\tilde{\omega}^2$$

And therefore we finally obtain Rodriguez formula:

$$e^{\theta\tilde{\hat{\omega}}} = I + \tilde{\hat{\omega}}\sin\theta + \tilde{\hat{\omega}}^2(1 - \cos\theta). \tag{1.44}$$

### 1.7.2 Exponential of elements of $\mathfrak{se}(3)$

It is now possible to calculate the exponential of an element of $\mathfrak{se}(3)$ which in matrix form we have seen to be looking like:

$$\tilde{T} = \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \tag{1.45}$$

We will consider two cases. The first one is the exponential of a twist whose axis is at infinity which we know to be corresponding to a pure translation ($\omega = 0$). In this case, it is easy to check that

$$\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}^n = 0 \qquad \forall n > 1$$

and therefore we have that:

$$e^{\tilde{T}} = \begin{pmatrix} I & v \\ 0 & 1 \end{pmatrix}. \tag{1.46}$$

From what we have seen in Sect. 1.4.2, a twist can be interpreted as applied on a screw. It is to be expected, that the exponential of a twist expressed in a reference frame such that the corresponding screw axis is passing through its origin will be easier to calculate. This is exactly what happens. We can therefore translate the coordinate system of $r$ where $r$ is the position of the axis of the screw on which the twist is applied. We can therefore write:

$$\begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\omega} & -\tilde{\omega} r + \lambda\omega \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -r \\ 0 & 1 \end{pmatrix}, \tag{1.47}$$

where the middle matrix on the right side is an expression of the initial twist in a frame shifted of $r$ and therefore placed on the axis of the screw. By using one of the properties of the exponential, it is therefore sufficient to calculate the exponential of

$$\begin{pmatrix} \tilde{\omega} & -\tilde{\omega} r + \lambda\omega \\ 0 & 0 \end{pmatrix}. \tag{1.48}$$

By straight forward calculation it is possible to see that the exponential of the previous matrix is equal to

$$\begin{pmatrix} e^{\tilde{\omega}} & \lambda\omega \\ 0 & 1 \end{pmatrix}, \tag{1.49}$$

and re-transforming this expression to the original coordinate system we obtain

$$\begin{pmatrix} I & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\tilde{\omega}} & \lambda\omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & -r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\tilde{\omega}} & (I - e^{\tilde{\omega}})r + \lambda\omega \\ 0 & 1 \end{pmatrix}. \tag{1.50}$$

Eventually, substituting the expressions for $r$ and $\lambda$ reported in Eq. (1.25) and Eq. (1.27) we finally obtain:

$$e^{\begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} e^{\tilde{\omega}} & \frac{1}{||\omega||^2}\left((I - e^{\tilde{\omega}})(\omega \wedge v) + \omega^T v \omega\right) \\ 0 & 1 \end{pmatrix} \tag{1.51}$$

where $e^{\tilde{\omega}}$ can be calculated using the Rodrigues formula reported in Eq. (1.44).

2

# Serial kinematic chains

*The previous Chapter has shown how to use Lie groups and the geometrical interpretation of screws in the study of the motion of one single rigid body. This Chapter applies that theory to serial kinematic chains of multiple rigid bodies. Lie theory of screws provides a very concise and complete description, offering more insight than the coordinate-based discussions of traditional textbooks.*

## 2.1 Configuration kinematics

### 2.1.1 Kinematic pairs

In Chapter 1 we have seen how to describe the kinematics of a single rigid body. We will now describe the effect of interconnecting two rigid bodies by means of an ideal geometric constraint. Such a constraint is usually called a *kinematic pair*. For reasons of simplicity we will only consider so-called *lower pairs*, which are motion constraints with the feature that their configuration is a Lie subgroup of $SE(3)$. Typical lower pairs are prismatic or rotational joints, which are the most common in robotics. More complex pairs can be described using differential geometric *distributions*, but this is beyond the scope of this course.

Important for the description of a constraint between two rigid bodies is what *relative* configurations and motions are allowed, but not what their *absolute* position is with respect to any other body or reference. So, constrained motions are described by *relative twists*; for a lower pair, these relative twists form a constant involutive subspace of $\mathfrak{se}(3)$. (HB: ????)

Consider two bodies $B_i$ and $B_j$, and frames $\Psi_i$ and $\Psi_j$ fixed to $B_i$ and $B_j$, respectively. Then consider a relative twist $T_i^{j,j}$ of $B_j$ with respect to $B_i$ (indicated by the leftmost $j$ and $i$ indices), as seen by the body $B_j$ (indicated by the rightmost $j$ superscript). All possible relative twists $T_i^{j,j}$ form a subspace of $se(3)$. A lower pair is *defined* to have the property that this subspace is the same for all allowed relative configurations $H_i^j$ of $B_j$ with respect to $B_i$. A *coordinate basis* of the relative motion subspace consists of a matrix $\mathcal{T}_F \in \mathbb{R}^{6\times n}$ ($n \leq 6$ is the number of *degrees of freedom* of the lower pair):

$$T_i^{j,j} \in \mathrm{range}\left(\mathcal{T}_F\right). \tag{2.1}$$

In screw theory terminology, the image of the matrix $\mathcal{T}_F$ represents a *screw system of order* $n$. The revolute and prismatic joints are examples of one-dimensional screw systems; a prismatic joint is a two-dimensional screw system; and a plane–plane contact is a three-dimensional screw system.

In geometric terminology, twists belong to the Lie algebra $\mathfrak{se}(3)$, and Lie algebras always

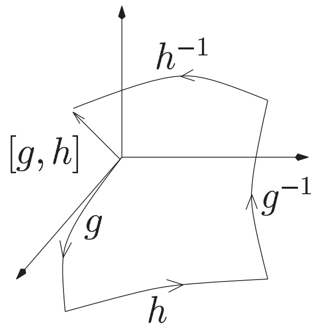



Figure 2.1: Commutator of two twists (or rather, infinitesimal displacements).

have a so-called *commutator* operator defined on them, in addition to the more traditional *summation* operator. (The commutator is often also called the *Lie derivative* or *Lie bracket*.) In the case of rigid body motion, the commutator $[T_l, T_k]$ of two twists $T_l$ and $T_k$ is shown in Figure 2.1. Give the rigid body $B$ a twist $T_l$, and move it during a short period of time; call this motion $g$. Then give it a different twist $T_k$, and move it again during a short period; call this motion $h$. The third motion is the inverse of the first one: move with the inverse twist $-T_l$. Finally, execute the inverse of the motion generated by $T_k$. Figure 2.1 sketches this four-motion operation. This composition of four operations $h^{-1}g^{-1}hg$ is the *commutator* of the finite displacements $g$ and $h$. In general, this commutator will not bring $B$ back to its original position and orientation. Now imagine that the motions $g$ and $h$ tend to infinitesimally small motions, or, in other words, take the limit, for the time going to zero, of the commutator divided by the short time period during which the motions are executed. This means that the infinitesimal displacements $g$ and $h$ become tangent vectors to the trajectories, i.e., the above-mentioned *twists* $T_k$ and $T_l$. Because of the limit process, these velocities apply at the "identity element," i.e., the undisplaced pose of the body $B$. Hence, $[T_l, T_k]$ is a mapping from two tangent vectors at the identity element to a third vector at the identity element: $[\cdot,\cdot]$ : se(3) $\times$ se(3) $\rightarrow$ se(3) : $v, w \mapsto [v, w]$. In matrix representation, the commutator is represented by a vanishing *matrix product*:

$$\widetilde{[T_l, T_k]} = [\tilde{T}_l, \tilde{T}_k] = \tilde{T}_l\tilde{T}_k - \tilde{T}_k\tilde{T}_l.$$

The "tilde" notation for twists has been introduced in Eq. (1.21).

The twist commutator has the physical units of an *acceleration*. Hence, $[T_l, T_k]$ is often also called the *Lie derivative* of $T_k$ with respect to $T_l$. (Sometimes denoted with $\mathcal{L}_{T_l}T_k$.) A Lie derivative has all properties of a derivative (linearity and Leibniz condition), but can only be defined for smooth *vector fields*; a vector field is the assignment of a vector (in this case, a twist) at each point of the (relative) configuration space.

If one has a *parallel transport* defined on the manifold, this parallel transport also transports the definition of the Lie bracket from the tangent space at the identity to the tangent space at any other element $g$ of SE(3): first transport the two tangent vectors $T_k$ and $T_l$ to the identity; then apply the Lie bracket to these two transported tangent vectors; bring the resulting tangent vector at the identity back to the original tangent space; and *define* this last vector to be the Lie bracket of $T_k$ and $T_l$. Note that the Lie bracket in the tangent space at the identity element of a Lie group is an intrinsic feature of that Lie group; the Lie bracket at an arbitrary other element of the Lie group depends on a rule of parallel transport. For example, left and right translation define *different* Lie brackets.

Some lower pairs (e.g., a plane-plane contact consisting of two “prismatic” joints with axis in the plane and one revolute joint with axis orthogonal to the plane) have the property that the *commutation* of any two possible twists $T_l$ and $T_k$ vanishes. *Physically*, this means that moving body $B_j$ over the kinematic constraint of the lower pair doesn't change its relative motion freedom with respect to body $B_i$. Other lower pairs (e.g., a spherical joint, being three revolute joints with intersecting axes) do not have vanishing commutators.

Every ideal motion constraint between two rigid bodies has two so-called *dual* subspaces associated to it:

- Its $n$-dimensional *twist* screw system, with basis $\mathcal{T}_F$.
- Its $(6-n)$-dimensional *wrench* screw system, with basis $\mathcal{W}_F$.

$\mathcal{T}_F$ is a subspace of $se(3)$, and $\mathcal{W}_F$ is a subspace of $\mathfrak{se}^*(3)$. Both spaces are *dual*, in the sense of linear vector spaces, (Stramigioli 2001). This duality is often called “*reciprocity*” in screw theory. The wrenches in $\mathcal{W}_F$ do not generate any power against the twists in $\mathcal{T}_F$. Or, equivalently, a wrench in $\mathcal{W}_F$ cannot transmit energy to the body. Or, in still other words, they are the *constraint wrenches* that the pair can ideally resist without actuation.

Twists of the form

$$\hat{T} = \begin{pmatrix} \hat{\omega} \\ \bullet \end{pmatrix}, \quad \text{or} \quad \hat{T} = \begin{pmatrix} 0 \\ \hat{v} \end{pmatrix}$$

are called *unit twists*. A one degree of freedom kinematic pair or joint constraints the relative motion of two objects with a unique twist:

$$T_i^{j,j} = \theta\, \hat{T}_i^{j,j}, \quad \theta \in \mathbb{R},$$

where $\hat{T}_i^{j,j}$ is a constant unit twist. For a rotational joint

$$\hat{T}_i^{jj} = \begin{pmatrix} \hat{\omega} \\ r \wedge \hat{\omega} \end{pmatrix},$$

for any $r$ that connects the origin of the reference frame to a point on the joint axis. If the rotation axis passes through the origin of $j$, than $r = 0$, and $\hat{T}_i^{j,j} = (\hat{\omega}^T \; 0)^T$.

Eq. (1.28) has shown the relationship between the rate of change in the position representation $H_i^j$ and the twist $\tilde{T}_i^{j,j}$ (in “tilde” matrix representation). If we apply this to the motion generated by moving about a joint axis with unit twist $\hat{T}_i^{j,j}$ and joint velocity $\dot{\theta}_i$, and solve the corresponding differential equation, we get:

$$H_i^j(\theta_i) = \exp(\tilde{\hat{T}}_i^{j,j}\, \theta_i)\; H_i^j(0). \tag{2.2}$$

Hence, the trajectory $H_i^j(t)$ generated by applying a constant twist $\hat{T}_i^{j,j}$ during a certain time interval corresponds to the *exponentiation* of $T$ (Chapter 1). As such, this exponential will appear in the kinematics treatment of robotic structures, which are a (mostly serial) connection of lower kinematic pairs.

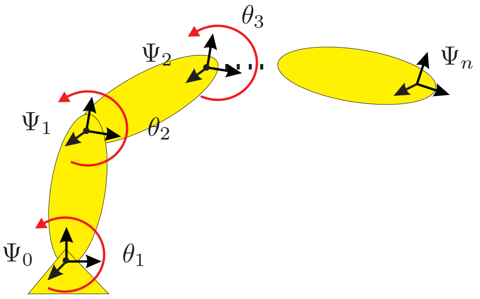



Figure 2.2: A serial kinematic chain.

### 2.1.2 Forward position kinematics of serial chains and Brockett's Product of Exponentials

Robots are most often used to *position* a tool (the "end-effector") in the workspace ("Cartesian space," "operational space"). The desired position (including *orientation*) has to be achieved by moving the motors (mostly rotating drives) to the correct angle (these angles form the "configuration space"). The relationship between motor angles and end-effector position and orientation is in general *non-linear*. All non-linear mappings have an "easy" direction, and a "difficult" direction. For *serial* robots, it is "easy" to find the end-effector position when the motor angles are given; that direction of the mapping is called the *forward position kinematics* (or "direct position kinematics"), and is subject of this subsection. The direction from end-effector position to motor angles is more difficult; this is the *inverse position kinematics*, discussed in the following section.

The forward position kinematics are traditionally described by means of an iterative numerical procedure: the end-effector position and orientation are found by applying the *chain rule* Eq. (1.11), where each homogeneous transformation matrix corresponds to the exponential formula Eq. (2.2): the twist $T$ in $\exp(\tilde{T}t)$ corresponds to a pure rotational velocity around the motor axis, and $\tilde{T}t$ is the matrix that corresponds to appying the joint twist during a time period $t$. It is a simple matter to find the (virtual) "time" $t$ that the motor had to turn from its zero position to the current position represented by the joint angle $\theta$.

Applying the chain rule

$$H_n^0 = H_1^0 H_2^1 \dots H_n^{n-1},$$

and the exponentiation of Eq. (2.2) to the typical serial kinematic chain of Figure 2.2 yields:

$$H_n^0(\theta_1, \theta_2, \dots, \theta_n) = \underbrace{e^{\tilde{\hat{T}}_1^{0,0}\theta_1} H_1^0(0)}_{H_1^0(theta_1)} \underbrace{e^{\tilde{\hat{T}}_2^{1,1}\theta_2} H_2^1(0)}_{H_2^1(theta_2)} \dots \underbrace{e^{\tilde{\hat{T}}_n^{(n-1),(n-1)}\theta_n} H_n^{n-1}(0)}_{H_n^{n-1}(\theta_n)},$$

where $H_i^{i-1}(0)$ is the position for $\theta_i = 0$.

$$H_n^0(\theta_1, \theta_2, \dots, \theta_n) = e^{\tilde{\hat{T}}_1^{0,0}\theta_1} \underbrace{H_1^0(0) e^{\tilde{\hat{T}}_2^{1,1}\theta_2} H_0^1(0)}_{e^{\tilde{\hat{T}}_2^{0,1}\theta_2}} H_1^0(0) H_2^1(0) \dots e^{\tilde{\hat{T}}_n^{(n-1),(n-1)}\theta_n} H_n^{n-1}(0),$$

or

$$H_n^0(\theta_1, \theta_2, \dots, \theta_n) = \exp(\tilde{\hat{T}}_1^{0,0}\theta_1) \exp \tilde{\hat{T}}_2^{0,1}\theta_2) \dots \exp \tilde{\hat{T}}_n^{0,(n-1)}\theta_n) H_n^0(0). \quad (2.3)$$

This result is called the *Products of Exponentials* (POE) formula, made popular by Roger Brockett. This POE formula is attractive from a geometrical point of view, because it fully describes the forward kinematics, in a coordinate-independent and concise way.

### 2.1.3 Inverse position kinematics

Formally speaking, the *inverse position kinematics* problem finds the joint angles $\theta_i$ from the non-linear equation Eq. (2.3), when the end-effector position and orientation $H_0^n$ is given. This text doesn't expand much on this topic, because the most appropriate solution technique depends heavily on the particular geometric form of the robot. For a robot with six revolute joints, in an arbitrary relative placement, the solution can be reduces to solving a 16th order polynomial; most industrial robots have special geometrical designs (three intersecting "wrist" axes; parallel second and third axes; etc.) that reduce this complexity to simple trigonometric and quadratic problems. Anyway, the non-linearity of the inverse problem gives rise to *multiple solutions*: for each $H_0^n$, multiple, discrete solution sets $(\theta_1, \ldots, \theta_n)$ exist. If the robot has more than 6 joints, one single $H_0^n$ gives rise to a *continuous* family of solution sets. Finally, only a limited number of $H_0^n$ is reachable by the robot; this reachable set is called the *workspace* of the robot.

## 2.2 Differential kinematics of serial chains

Very often, not only the *position* of the robot end-effector is important for the task, but also its *velocity*. The relationship between the velocities of the $n$ motors and the velocity (*twist*) of the end-effector has got the name of *differential kinematics*, or (forward and inverse) *velocity kinematics*.

This Section explains the relevant geometric properties: the Jacobian matrix, the singularities of the robot, and how to cope with redundancy in the task specification.

### 2.2.1 The geometric Jacobian

Mathematically speaking, going from the position kinematics (Eq. (2.3)) to the differential kinematics is just a matter of taking the time derivative. The derivative of the non-linear POE function results in a *linear* relationship between the joint velocities $\dot{\theta}_i = \partial\theta_i/\partial t$ and the end-effector twist $T_i^n$. The matrix that describes this linear relationship is called the *Jacobian matrix*, denoted by $J(\theta(t))$.

$$T_1^n = J \begin{pmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{pmatrix}. \tag{2.4}$$

Although the differential kinematics *at any given instant* $t$ is a linear relationship, the Jacobian matrix itself is a *non-linear* function of the joint angles.

It's not difficult to find the differential kinematics from Brockett's Product of Exponentials Eq. (2.3):

$$\frac{\partial H_0^n}{\partial t} = H_0^1 \left\{ \left( \tilde{T}^1 \exp(\tilde{T}^1 t) \right) \ \exp(\tilde{T}^2 t) \ \ldots \ \exp(\tilde{T}^n t) \right\} \\ + \cdots + \left\{ \exp(\tilde{T}^1 t) \ \ldots \ \left( \tilde{T}^n \exp(\tilde{T}^n t) \right) \ \right\}, \tag{2.5}$$

where $\tilde{T} = \tilde{\hat{T}}\theta(t)$. And Eq. (1.28) shows the relationship between the twist $T_1^n$ of the end-effector and the time rate of $H_0^n$, i.e., post-multiplication by $H_n^0 = (H_0^n)^{-1}$. Eq. (1.29) then allows to write Eq. (2.5) as (with respect to the first body):

$$\tilde{T}_1^n = \tilde{T}^1 + H_1^2 \tilde{T}^2 H_2^1 + \cdots + H_1^n \tilde{T}^2 H_n^1. \tag{2.6}$$

So, we finally arrive at the Jacobian matrix $J$ in Eq. (2.4):

$$T_1^n = \hat{T}^1 \dot{\theta}_1 + \hat{T}^2 \dot{\theta}_2 + \cdots + \hat{T}^n \dot{\theta}_n = \underbrace{\begin{pmatrix} \hat{T}^1 & \hat{T}^2 & \ldots & \hat{T}^n \end{pmatrix}}_{J} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{pmatrix}. \tag{2.7}$$

This equation just means that the end-effector twist $T_1^n$ is the *sum* of the contributions of all joints. The Jacobian is a *coordinate basis* of the $n$-dimensional sub-space of the tangent space that can be covered by moving the robot's joints (at this particular point in time). Its $i$th column corresponds to the end-effector twist generated by applying a unit angular velocity at the $i$th joint, and no velocities at the other joints.

### 2.2.2 Singularities

A serial robot with $n$ motors spans an $n$-dimensional subspace, with the Jacobian matrix as coordinate basis. However, because the Jacobian is a non-linear function of the configuration parameters $\theta_i$, there will be, in general, configurations where the rank of $J$ drops, or, in other words, the twists of the different joints become linearly dependent. The physical meaning of a singular configuration is that the robot *looses a velocity degree of freedom*, or, because of the duality between twist and wrench spaces, it gaines a *passive* degree of constraint. The latter phenomen means that one can exert an arbitrary large wrench along a certain line without the robot needing any joint torques to withstand this load.

In differential-geometric terms, a singularity is a point where the *vector field* of joint twists "folds." Another way to describe a singularity is as the configuration where the set of differential equations of which the joint twist vector fields are solutions, reaches a so-called "catastrophy."

### 2.2.3 Redundancy

The "opposite" of a singularity also exists: the robot has more actuated degrees-of-freedom than it needs to move its end-effector. (Typically, one calls an $n$-jointed robot redundant, if $n > 6$; however, even a robot with only 4 joints can be redundant for its task, e.g., when the task is to place a microphone at a desired position in the work space, while the orientation is not important.)

Formally, a redundant situation is reflected in the Jacobian matrix having more columns than rows. This means that the same end-effector twist $T_1^n$ can be generated by an infinite number of joint velocities $\dot{\theta}_i$. Or, that the inverse $J^{-1}$ of the Jacobian $J$ is not uniquely defined. The concept of the so-called *pseudo-inverse* $J^{\dagger}$ is well known in linear algebra, and often used in robotics. Its analytical form is

$$J^{\dagger} = (J^T J)^{-1} J^T. \tag{2.8}$$

It is not difficult to show that the pseudo-inverse is the solution of the constrained quadratic optimization problem:

$$\min ||\dot{\theta}^T \dot{\theta}||, \quad \text{s.t.} \quad T = J\dot{\theta}. \tag{2.9}$$

This means: of all possible joint velocity vectors $\theta$ that generate the desired end-effector twist $T$, one chooses the vector with the smallest "norm."

However, in the form of Eq. (2.8), the pseudo-inverse has no intrinsic geometrical meaning, because the elements of the matrix $J^T J$ contain components with different physical units, i.e., $m^2/s^2 + rad/s^2$! This inconsistency corresponds to the fact that no natural norm exists on the space of rigid body motions. On the other hand, most readers will intuitively understand that *nature* solves redundancy in a unique and unambiguous way: for example, if one removes all actuation on a 7 degrees-of-freedom robot, then it will "fall" according to the laws of physics, and these generate one single solution at each different configuration. We will delay the geometric discussion about solving redundancy until after the following Section, because that requires knowing more about the dynamics of serial robots.

## 2.3 Dynamics of serial chains

The previous Section dealt with the *kinematics* of serial connections of rigid bodies, i.e., looking only at the position and velocity descriptions, irrespective of the forces that generate these motions. This Section looks at this relationship between forces and motion (i.e., acceleration), and extends the discussion about the dynamics of one single rigid body to a chained sequence of $n$ rigid bodies.

### 2.3.1 The dynamic equations of a serial manipulator

Sect. 1.6 explained that the dynamics of a single, unconstrained rigid body is the six-dimensional generalisation of the three-dimensional dynamics of a point mass; the important difference is that the scalar mass property of a point mass is replaced by a full $6 \times 6$ inertia matrix.

When connecting the different links of a serial robot, each of these links is *constrained* by the dynamics of the other links. Therefore, the *forward and inverse dynamics* of an $n$-jointed serial chain are a bit more complicated than the independent concatenation of $n$ rigid body dynamics. This section considers the simplest case, as shown in Figure 2.3. The constraint generated by the revolute joint can be expressed in two ways:

- *Acceleration constraint*: the *relative* acceleration of both links must be about the common joint axis:

$$\dot{T}_1 - \dot{T}_2 = Z\ddot{\theta}_2, \tag{2.10}$$

with $Z$ the six-dimensional basis vector of the joint ("unit twist"), and $\ddot{\theta}$ the (as yet unknown) acceleration of the joint. (In a frame with its $z$ axis on the joint axis, $Z$ has the simple coordinate representation $(0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)^T$.)

- *Wrench constraint*: the transmitted wrench $W_2$ cannot have a component about the revolute joint axis, hence:

$$W_2 Z = 0. \tag{2.11}$$

(Recall that the coordinate representation for a wrench is a *row vector*.)

Because $(N^2)^T = I^2 \dot{T}^2$ (Euler's equation for the dynamics of a single rigid body, Eq. (1.38)), with $I^2$ the inertia tensor of link 2, one finds after premultiplying Eq. (2.10) with $Z^T I^2$ that:

$$Z^T I^2 \dot{T}^1 = Z^T I^2 Z \ddot{\theta}_2. \tag{2.12}$$

From this equation, $\ddot{\theta}$ can be found, because $Z^T I^2 Z$ is always a full-rank matrix:

$$\ddot{\theta}_2 = \left(Z^T I^2 Z\right)^{-1} Z^T I^2 \dot{T}^1. \tag{2.13}$$

If both bodies would have been rigidly connected, the force $W^1$ would have had to accelerate the combined inertia $I^1 + I^2$ of both bodies, and both bodies would accelerate with the same acceleration $\dot{T}^1$. However, the joint filters out part of the acceleration, namely the part $Z\ddot{\theta}_2$. Hence, the relationship between wrench $W^1$ and acceleration $\dot{T}^1$ is:

$$W^1 = I^1 \dot{T}^1 + \left(I^2 - I^2 Z \left(Z^T I^2 Z\right)^{-1} Z^T I^2\right) \dot{T}^1, \tag{2.14}$$

$$= I_a^1 \dot{T}^1, \tag{2.15}$$

$$\text{with} \quad I_a^1 = I^1 + I^2 - I^2 Z \left(Z^T I^2 Z\right)^{-1} Z^T I^2. \tag{2.16}$$

$I_a^1$ the so-called *articulated body inertia*, (Featherstone 1987), i.e., the increased inertia of link 1 due to the fact that it is connected to link 2 through an "articulation" which is the revolute joint. The mass of link 2 is "projected" onto link 1 through the joint between both links. The corresponding $6 \times 6$ *projection operator* $P^2$ is:

$$P^2 = 1 - I^2 Z \left(Z^T I^2 Z\right)^{-1} Z^T. \tag{2.17}$$

The matrix $P^2$ is indeed a projection operator, because

$$P^2 P^2 = P^2. \tag{2.18}$$

The total articulated inertia of link 1 is the sum of its own inertia $I^1$ and the projected part $P^2 I^2$ of the inertia of the second body:

$$I_a^1 = I^1 + P^2 I^2. \tag{2.19}$$

Note that the projection operator has the form of the *pseuso-inverse* in Eq. (2.8), which is also a well-known fact from linear algebra. The geometrical interpretation of this fact is that the (articulated) inertia acts as a *natural metric* on the configuration space of a serial kinematic chain: it defines the "orthogonal projection" of physical quantities. It also acts as a natural *identification* of the (acceleration) twist and wrench spaces: each wrench gives rise to a well-defined acceleration twist, and vice versa.

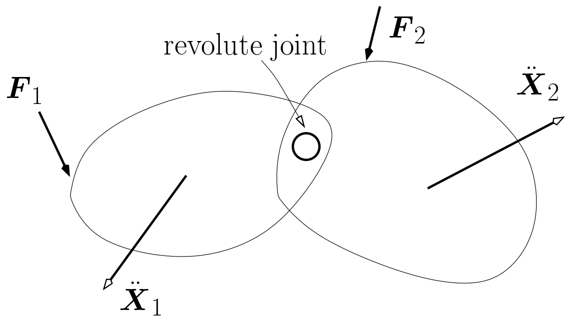



Figure 2.3: A rigid body is connected to another rigid body by a revolute joint. The joint cannot transmit a pure torque component about its axis, generated by the external forces.

The previous paragraphs explained the projection of the inertia tensor. The corresponding projections of acceleration and wrenches follow straigthforwardly:

$$\dot{T}^2 = \left(1 - Z\left(Z^T I^2 Z\right)^{-1} Z^T I^2\right)\dot{T}^1, \tag{2.20}$$

$$= \left(P^2\right)^T \dot{T}^1. \tag{2.21}$$

and

$$W^1 = W^2 - I^2 Z(Z^T I^2 Z)^{-1} Z^T W^2, \tag{2.22}$$

$$= P^2 W^2. \tag{2.23}$$

These link-to-link relationships are the basis of recursive algorithms to find the forward and inverse dynamics of serial robot, but this fall beyond the scope of this text.

### 2.3.2 Redundancy resolution

The kinematic section on redundancy resolution, Sect. 2.2.3, introduced the pseudo-inverse formalism for redundancy resolution. But the $(J^T J)^{-1} J^T$ form is physically inconsistent. However, the inertia-weigthed projection matrices of the previous Section *do* have a clear physical interpretation, and it is not difficult to show that they occur as the solution of the constrained and *weighted* quadratic optimization problem that is a generalization of Eq. (2.9):

$$\min ||\dot{\theta}^T I^\theta \dot{\theta}||, \quad \text{s.t.} \quad T = J\dot{\theta}, \tag{2.24}$$

where $I^\theta$ is the robot's inertia tensor, expressed in joint coordinates. The physical interpretation of Eq. (2.24) is: of all possible joint velocity vectors $\theta$ that generate the desired end-effector twist $T$, one chooses the vector with the smallest *kinetic co-energy*. (Compare Eq. (2.24) to Eq. (1.35), which expresses the same co-energy in Cartesian space.) This kinetic co-energy weighting is often used in robotics publications, but few understand that it is the *unique choice that nature makes itself*.

### 2.3.3 Ideal constraints

The previous Sections assumed that the robot's end-effector was moving unconstrained in free space. What changes if the end-effector is making contact with a rigid surface, and hence looses one or more degrees of freedom? (This section considers *ideal* constraints only, i.e., frictionless and infinitely rigid constraints.) Conceptually, the discussion runs along exactly the same lines as in the Sect. 2.3.1: the constraint on the end-effector is represented by a *wrench space* of ideal constraint forces, and the $Z$ vector is to be replaced by a *basis* of this wrench space. Formally, nothing else changes in the expression of the projection operators.

So, when in contact with an environment that constrains its motion according to a wrench space with basis $Z$ (we abuse the same notation here to represent a more than one-dimensional subspace), the robot's end-effector twist $T$ should be such that it minimizes its internal kinetic co-energy while respecting the motion constraint:

$$\min ||T^T I T||, \quad \text{s.t.} \quad ZT = 0, \tag{2.25}$$

where $I$ is the robot's total inertia tensor.

The projection operator $P$ that can be constructed from this optimization problem (just as in Sect. 2.3.1) has been used in so-called *hybrid force/position* controllers, (De Schutter, Torfs, Dutré & Bruyninckx 1997, Raibert & Craig 1981). These controllers are based on the observation that an ideal constraint reduces the motion degrees of freedom of the end-effector from 6 to $n < 6$. Hence, only $n$ degrees of freedom should be "position" controlled, and the other $6 - n$ degrees of freedom are "force" controlled. The "hybrid" controller then consists of the sum of contributions from independent, lower-than-six-dimensional position and force controllers. And the six-dimensional force and motion *measurements* are projected on these lower-dimensional subspaces by means of the above-mentioned projection matrices.

3

# Interaction and Control

*This Chapter introduces what is called the IPC-Supervisor scheme for robot control (Stramigioli 2001). Within this control paradigm, robot control is not any longer a classical input-output system, but rather an energy-exchanging interconnection of the robot and the controller (which is itself interpretable as a physical system). The advantages of the method are robustness and intuition in control design. The mathematical foundations of the control approach are, on the one hand, classical Hamiltonian theory of dynamic systems, and on the other hand, the interconnection theory of mechanical networks.*

## 3.1 Ports and Interconnection

In this section we will describe the basic ideas and tools which are used in the proposed approach. We introduce power ports and Port Controlled Generalized Hamiltonian Systems, as alternatives to the more traditional *input-output* control paradigm. The latter has produced controllers such as *hybrid position/force* control, which relies on the projection operators of Sect. 2.3.3. The disadvantage of this type of controllers for systems that are *interconnected* to each other is that it is often not possible to specify, for example, desired velocities as inputs, because the robot might be constrained by an (unknown) rigid wall. The control approach in this Chapter is not of the input-output family, but starts from the energy flow in the network of interconnected systems; forces and motions will be the *result* of specifying the *behaviour* of the total system. These interaction controllers have very thorough geometrical descriptions, that generalize the concept of a Lie group/algebra of the previous Chapters.

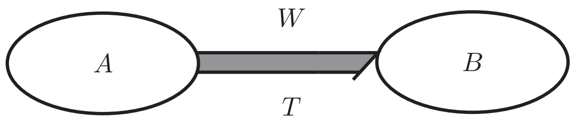
### 3.1.1 Power Ports

A basic concept which is needed to talk about interconnection of physical systems is the one of a power port (Maschke, van der Schaft & Breedveld 1992),(Stramigioli 1998). With reference to Figure 3.1, a power port is the entity which describes the media by means of which subsystems can mutually exchange physical energy. Analytically, a power port is the Cartesian product of a vector space $V$ and its dual space $V^*$:

$$P := V \times V^*.$$

Therefore, power ports are pairs $(e, f) \in P$. The values of both $e$ and $f$ (*effort* and *flow* variables) change in time and these values are shared by the two subsystems which are exchanging power through the considered port. The power exchanged at a certain time is equal to the intrinsic dual product:

$$\text{Power} = \langle e|f \rangle.$$




$$(T, W) \in \mathfrak{se}(3) \times \mathfrak{se}^*(3)$$

Figure 3.1: The interaction between two mechanical systems

This dual product is intrinsic in the sense that elements of $V^*$ are linear operators from $V$ to $\mathbb{R}$, and therefore, to express the operation, we do not need any additional structure than the vector space structure of $V$.

To talk about the interconnection of mechanical systems, a proper choice is $V = \mathfrak{se}(3)$, the space of twists and $V^* = \mathfrak{se}^*(3)$ the space of wrenches.

### 3.1.2 Generalized Port-Controlled Hamiltonian Systems

Dynamical systems have traditionally had two exchangeable theoretical approaches, the *Lagrangian* and the *Hamiltonian* approach. In the context of this Chapter, where the *exchange of energy* is the key physical phenomenon, the Hamiltonian framework is the more natural, because it uses the total energy of the system as its key component.

In standard Hamiltonian theory, the starting point is the existence of a generalized configuration manifold $\mathcal{Q}$. Based on $\mathcal{Q}$, its co-tangent bundle $T^*\mathcal{Q}$ is introduced which represents the state space (or "*phase space*") of *configuration* ("position") and *momentum* pairs $(q, p)$. $T^*\mathcal{Q}$ can be given a natural so-called *symplectic* structure, on which the *Hamiltonian dynamics* can be expressed (Arnold 1989).

A symplectic structure on a manifold $\mathcal{M}$ is a *closed* "2-form". A *one-form* is another name for an element of a co-tangent space, i.e., a mapping that takes a tangent vector as input, and gives a real number as output; a two-form is an anti-symmetric mapping that takes two tangent vectors as inputs, and gives a real number as output. An $n$-form is *closed* if it is the derivative of an $n-1$-form. A wrench is "closed" (and therefore exact using Poincare' lemma) if it is the derivative of a potential function; in other words, it is *conservative*.

Without going into details, the reader should recall from these paragraphs that *every* co-tangent space has a *natural* symplectic form; the corresponding tangent space does *not* have this structure. This is another fundamental difference between twists (i.e., tangent vectors) and wrenches and momenta (i.e., co-tangent vectors).

A limitation of the Hamiltonian approach is that, (by construction), the dimension of the state space $T^*\mathcal{Q}$ is always even. Moreover, it can be shown that, in general, the interconnection of Hamiltonian systems in this form does not result in a system of the same form.

These problems can be overcome with the more general *Poisson* structure, (Olver 1993), or, even more generally, using *Dirac structures* (Dalsmo & van der Schaft 1999). A Poisson structure is an additional structure on any state manifold. The Poisson structure is the mathematical representation of a kind of *network structure* and the manifold is the total state space of a system. With network structure it is meant that this structure specifies how energy can flow from one part of the system to another.

If this state space is a dual Lie algebra on which a left invariant metric is defined, as in the case of the momenta (the state) of a rigid body and its inertia tensor (the left invariant metric), a *natural* Poisson structure can be defined which is called a *Lie-Poisson* structure.

A *Poisson bracket* $\{\cdot,\cdot\}$ is a binary operation $\{f,g\}$ on *functions* $f$ and $g$ of the state space. For such an operation to be a Poisson bracket, it should satisfy the following properties:

1. *(bi-)linearity*: $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$.
2. *anti-symmetry*: $\{f,g\} = -\{g,f\}$.
3. *Jacobi identity*: $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$.
4. *Leibniz identity*: $\{fg,h\} = f\{g,h\} + g\{f,h\}$.

If all the previous properties are satisfied except Jacobi identity, we talk about a *generalized Poisson structure*. It can be shown (van der Schaft 2000) that this structure is needed for modeling mechanical systems with nonholonomic constraints. The information about the constraints is included in the structure and this is different than in the Lagrangian approach.

Even if the Poisson structure is properly represented by a Poisson bracket on functions, the best way to understand it is by using what is called the corresponding Poisson tensor. As a quadratic form maps two vectors (fields) to a scalar (function), a Poisson tensor maps two co-vectors (fields) to a scalar (function). A typical co-vector is the differential of a function. In our case this function is the energy of the modeled system which is clearly a function of the state of the system. Furthermore, for the anti-symmetry property, this Poisson tensor should be skew-symmetric. Once we have chosen a chart for the state manifold under consideration, we can represent the Poisson bracket as follows:

$$\{F(x), H(x)\}(x) = \frac{\partial F}{\partial x}^T J(x) \frac{\partial H}{\partial x}.$$

Clearly, since $\{F(x), G(x)\}(x)$ is a function and since $\frac{\partial F}{\partial x}$ is a 1-form (co-vector field), the vector

$$J(x)\frac{\partial H}{\partial x}$$

will be a vector field. If $H(x)$ is the Hamiltonian of the system (the energy function), this vector field is called the Hamiltonian vector field of the system. Clearly this vector field leaves the Hamiltonian invariant (constant) due to the skew-symmetry of $J(x)$:

$$\dot{H} = \{H, H\} = \frac{\partial H}{\partial x}^T J(x) \frac{\partial H}{\partial x} = 0.$$

An autonomous Hamiltonian system is therefore completely described by an Hamiltonian function and a Poisson structure and the corresponding equations using the Hamiltonian vector-fields just explained are:

$$\dot{x} = J(x)\frac{\partial H}{\partial x}. \tag{3.1}$$

Even more general structures than the Poisson structures called the Dirac structure have been defined. The Dirac structure can cope with the modeling of implicit Hamiltonian system. The Poisson structure only with explicit Hamiltonian systems. Some introductory material on Dirac structures for modeling Hamiltonian systems can be found in (van der Schaft 2000). For more specific applications to mechanical systems and robotics the reader is referred to (Stramigioli 2001). In Eq. (3.1) the system is autonomous and therefore it cannot exchange energy with the rest of the world. It is then possible, using the concept of power port introduced in Sect. 3.1.1 to extend Eq. (3.1) to what is called a Port Controlled Hamiltonian System (PCHS) (Maschke et al. 1992) :

$$\dot{x} = J(x)\frac{\partial H(x)}{\partial x} + g(x)u \tag{3.2}$$

$$y = g^T(x)\frac{\partial H(x)}{\partial x}, \tag{3.3}$$

where $u$ is an element of a vector space $V$ and the external input to the previously autonomous system, $J(x) = -J^T(x)$ is the skew-symmetric Poisson tensor which can be seen to describe in general the network structure and interconnection of the energy flows within the system, $g(x)$ is a representation of the interface of the system with the rest of the world, and $y$ is the representation of an element belonging to the dual vector space $V^*$. Clearly $(u, y)$ is a power port since it belongs to $V \times V^*$ and it will be shown shortly that $y^T u$ corresponds to the power supplied to the system..

The Jacoby identity for the Poisson bracket corresponds to the following condition for the Poisson tensor:

$$\sum_{l=1}^{n}\left[J^{il}\frac{\partial J^{jk}}{\partial x^l} + J^{kl}\frac{\partial J^{ij}}{\partial x^l} + J^{jl}\frac{\partial J^{ki}}{\partial x^l}\right] \equiv 0.$$

It can be shown that conservative mechanical systems with non-holonomic constraints can be nicely modeled by Eq. (3.2) where $J(x)$ does not satisfy the previous condition. The most general form of an explicit, conservative, Hamiltonian system is furthermore:

$$\dot{x} = J(x)\frac{\partial H(x)}{\partial x} + g(x)u \tag{3.4}$$

$$y = g^T(x)\frac{\partial H(x)}{\partial x} + S(x)u \tag{3.5}$$

where $S(x)$ is a skew-symmetric quadratic form on the vector space $V$ depending on the state $x$.

To account for dissipating elements, it is possible to see that we can generalize the previous form considering a symmetric, semi-positive definite, two covariant tensor $R(x)$ which can be subtracted from $J(x)$:

$$\begin{aligned}\dot{x} &= (J(x) - R(x))\frac{\partial H(x)}{\partial x} + g(x)u \\ y &= g^T(x)\frac{\partial H(x)}{\partial x} + S(x)u.\end{aligned} \tag{3.6}$$

With this new term, it can be seen that the change in internal energy is:

$$\dot{H} = \underbrace{y^T u}_{\text{supplied power}} - \underbrace{\left(\frac{\partial H}{\partial x}\right)^T R(x)\frac{\partial H}{\partial x}}_{\text{dissipated power}}.$$

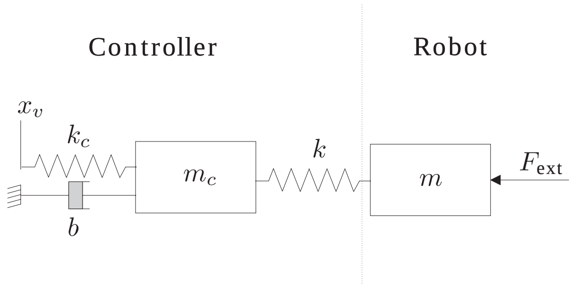



Figure 3.2: A simple example of interconnection

Since $R(x)$ is positive semi-definite, this implies that the internal energy can only increase if power is supplied through the ports. Eventually, it has been shown in (Stramigioli, van der Schaft, Maschke, Andreotti & Melchiorri 2000), that an even more general form is necessary for tele-manipulation applications. This form is:

$$\begin{aligned}\dot{x} &= (J(x) - R(x))\frac{\partial H(x)}{\partial x} + g(x)u \\ y &= g^T(x)\frac{\partial H(x)}{\partial x} + (S(x) - B(x))u.\end{aligned} \tag{3.7}$$

where the new positive semi-definite quadratic form $B(x)$ on $V$ is necessary in order to model the impedance adaptation to a communication line.

As an example, consider the interconnection shown in Figure 3.2 of a mass representing a robot with the physical equivalent of a controller implementing damping injection as introduced in (Stramigioli, Maschke & van der Schaft 1998).

The Generalized Hamiltonian model of the "robot" is:

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{p}\end{pmatrix} &= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0\\ p/m\end{pmatrix} + \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}F_{\text{ext}}\\ F_c\end{pmatrix},\\ \begin{pmatrix}\dot{x}\\ \dot{p}\end{pmatrix} &= \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ p/m\end{pmatrix},\end{aligned}$$

where the port $(u_1, y_1) = (F_{\text{ext}}, \dot{x})$ represents the interaction port of the robot with the environment and $(u_2, y_2) = (F_c, \dot{x})$ the interaction port with the controller. The energy function is $H(p) = \frac{1}{2m}p^2$ where $p$ is the momentm[1]. The physical system representing the "controller" of Figure 3.2 can be instead represented using the following Generalized Hamiltonian equations with dissipation:

$$\begin{aligned}\begin{pmatrix}\dot{\Delta x_c}\\ \dot{p}_c\\ \dot{\Delta x}\end{pmatrix} &= \begin{pmatrix}0 & 1 & 0\\ -1 & -b & 1\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}k_c\Delta x_c\\ p_c/m_c\\ k\Delta x\end{pmatrix} + \begin{pmatrix}0 & -1\\ 0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{x}_v\end{pmatrix},\\ \begin{pmatrix}-F_c\\ -F_v\end{pmatrix} &= \begin{pmatrix}0 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}k_c\Delta x_c\\ p_c/m_c\\ k\Delta x\end{pmatrix},\end{aligned}$$

where the port $(u_1, y_1) = (\dot{x}, -F_c)$ is used to express the interconnection with the "robot" and the port $(u_2, y_2) = (\dot{x}_v, -F_v)$ is used to express the interconnection with another

[1] It is important to notice that $\frac{1}{2}mv^2$ is properly speaking called co-energy instead of energy because it is a function of $v$ which is not a physical state extensive variable (Breedveld 1984).

system which turns out to be the supervisory module. The stored energy is:

$$H(\Delta x_c, p_c, \Delta x) = \frac{1}{2}k_c \Delta x_c^2 + \frac{1}{2}k\Delta x^2 + \frac{1}{2m}p_c^2.$$

It is shown in (Stramigioli et al. 1998) that by choosing $m_c \ll m$ and $k \gg k_c$ for the controller, damping injection can be implemented with pure position measurements. Furthermore, actuator's saturation can be handled in a physical way by choosing a non linear spring $k$.

Any physical system can be modeled in the same way and this is the power of GPCHS and their importance to describe the proposed architecture.

### 3.1.3 Interconnection of GPCHSs

A very important feature of GPCHSs is that their interconnection is still a GPCHS. To show this, consider two GPCHSs:

$$\dot{x}_i = (J_i - R_i)\frac{\partial H_i}{\partial x_i} + \begin{pmatrix} g_i^I & g_i^O \end{pmatrix} \begin{pmatrix} u_i^I \\ u_i^O \end{pmatrix}, \tag{3.8}$$

$$\begin{pmatrix} y_i^I \\ y_i^O \end{pmatrix} = \begin{pmatrix} (g_i^I)^T \\ (g_i^O)^T \end{pmatrix} \frac{\partial H_i}{\partial x_i} \qquad i = 1, 2. \tag{3.9}$$

The two systems can be interconnected through the interconnection ports $(u_i^I, y_i^I)$ by setting:

$$u_1^I = y_2^I \qquad \text{and} \qquad u_2^I = -y_1^I \tag{3.10}$$

Note that the minus sign in the previous equation is necessary to be consistent with power: $P_i = \langle u_i^I, y_i^I \rangle$ is the input power of system $i$ and, interconnecting the system through the "$I$" ports, we clearly need $P_1 = -P_2$. It is possible to see that the interconnected system results in:

$$\dot{x} = (J(x) - R(x))\frac{\partial H}{\partial x} + g(x) \begin{pmatrix} u_1^O \\ u_2^O \end{pmatrix}, \tag{3.11}$$

$$\begin{pmatrix} y_1^O \\ y_2^O \end{pmatrix} = g^T(x)\frac{\partial H}{\partial x}, \tag{3.12}$$

where $x = (x_1, x_2)^T$, $H(x) = H_1(x_1) + H_2(x_2)$ (the sum of the two energies),

$$J(x) = \begin{pmatrix} J_1 & g_1^I (g_2^I)^T \\ -(g_2^I)(g_1^I)^T & J_2 \end{pmatrix},$$

$$R(x) = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

and

$$g(x) = \begin{pmatrix} g_1^O \\ g_2^O \end{pmatrix},$$

where all dependencies of the matrices have been omitted for clarity.

It is possible to conclude that the interconnected system is therefore again a GPCHS, with as ports the remaining (i.e., not yet interconnected) ports. Furthermore the total energy is the sum of the energies of the two systems.

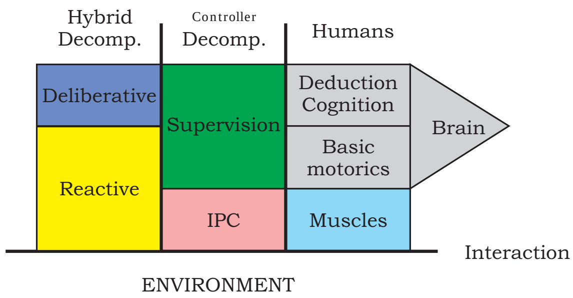



Figure 3.3: A biological analogy of the control strategy

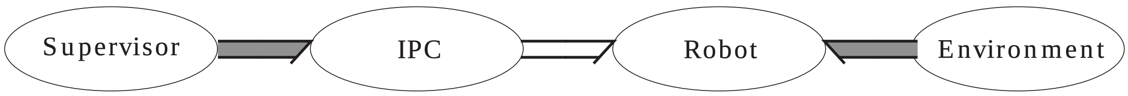



Figure 3.4: The System Interconnection

## 3.2 The proposed control architecture

The main idea of the proposed scheme is to divide the control in two parts, one which controls the real time interaction *passively* and is called *Intrinsically Passive Control* (IPC), and one which takes care of the task decomposition and other planning issues. As shown in Figure 3.3, the IPC has the role of the muscle spindles in biological systems, and the supervisor the neurological role of the brain.

### 3.2.1 The IPC

The IPC is interconnected with the supervisor and the robot through power ports as shown in Figure 3.4 where bond-graph notation is used. Each bond is representing two elements belonging to a vector space (effort) and to its dual (flow) as explained in Sect. 3.1.1. With reference to Sect. 3.1.1, the power port corresponding to the interconnection with the robot is characterized by the vector space $V$ being $T_q\mathcal{Q}$ where $\mathcal{Q}$ is the robot configuration manifold and $q$ is the current configuration and therefore the elements of the ports will be pairs of the form $(\dot{q}, \tau)$ where $\tau$ are the joint torques.

The other port of the IPC will be connected to the supervisor and in general will have a geometric structure such that:

$$V = \mathfrak{se}(3) \times \ldots \times \mathfrak{se}(3)$$

corresponding to a set of twists. This is the case in both (Stramigioli, van der Schaft, Maschke, Andreotti & Melchiorri 2000) and (Stramigioli, Melchiorri & Andreotti 1999).

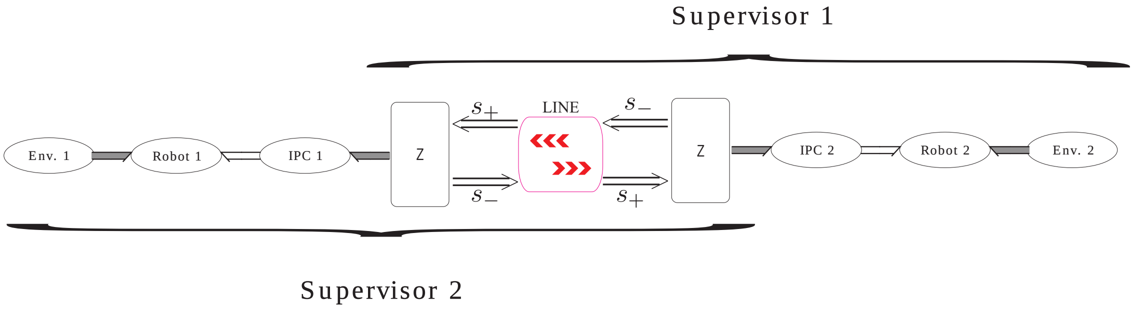



Figure 3.5: A tele-manipulation setting.

The IPC will therefore be characterized by the following differential equations:

$$\dot{x} = (J(x) - R(x))\frac{\partial H_c}{\partial x} + g(x)\begin{pmatrix} \dot{q} \\ T_1 \\ \vdots \\ T_n \end{pmatrix} \tag{3.13}$$

$$\begin{pmatrix} \tau \\ W_1 \\ \vdots \\ W_n \end{pmatrix} = g^T(x)\frac{\partial H_c}{\partial x} + B(x)\begin{pmatrix} \dot{q} \\ T_1 \\ \vdots \\ T_n \end{pmatrix} \tag{3.14}$$

where $J(x)$ is skew symmetric, $R(x)$ positive semi-definite, $T_1, \ldots, T_n$ are a set of interaction twists and $W_1, \ldots, W_n$ a set of the dual interaction wrenches. It has been shown in (Stramigioli, van der Schaft, Maschke, Andreotti & Melchiorri 2000) that the feed-through term $B(x)$ is needed in tele-manipulation to adapt the impedance of the line.

### 3.2.2 The Supervisor

The Supervisor must *plan* the task, and *schedule* the subsequent sub-tasks, and plays the role of “brain” in a human analogy.

Following the grasping example of the previous section, to grasp and move an object, we have to

1. Open the hand
2. Move around the object
3. Close the hand (grasp)
4. Move the object
5. Open the hand

All this subtasks can be implemented by means of control signals to the IPC by the Supervisor which can supply a controlled amount of energy to the system in order to perform useful tasks.

#### A Tele-manipulation setting

In a tele-manipulation setting, the presented architecture is still valid, but the role of the supervisor is taken over by a human on the other side of a transmission line.

A perfectly bilateral tele-manipulation system using the presented architecture is reported in Figure 3.5. This system has been studied and implemented experimentally in (Stramigioli, van der Schaft, Maschke, Andreotti & Melchiorri 2000).

The environment on one side is the environment to be manipulated and the one on the other side is the human which manipulates the system remotely.

The port variables are transformed to scattering variables (block $Z$ in the figure) to preserve passivity even with time-varying time delays due to the transmission line (Stramigioli, van der Schaft, Maschke, Andreotti & Melchiorri 2000). The supervisor on one side is in this case composed of the transmission line, the IPC, the Robot and the “Environment” of the other side.

Once again, due to the consistent framework using power ports, passivity is preserved in any situation.

Some inconsistencies where present in (Stramigioli, van der Schaft, Maschke, Andreotti & Melchiorri 2000) due to a non intrinsic choice. These inconsistencies have been solved in (Stramigioli 2001).

## 3.3 IPC Grasping

In this section we present and discuss, as an example, the application of an Intrinsically Passive Control (IPC) strategy to robotic grasping and manipulation tasks. Major advantages of the presented control strategy are the physical intuition on which it is based, its passive nature, and the ensured stability for the overall system in all the situations, including in particular the transition from no-contact to contact and *vice-versa*. One of the main features of the proposed control is that only joint position measurements are needed. This means that no velocity or force measurements are required in the control loop, simplifying the sensorial complexity and enlarging the possibilities of application of the scheme.

In general, the environment a robot interacts with must be characterized from a geometrical point of view, i.e. the dimensions and location of the object to be handled must be known in advance. This knowledge is used for planning the grasp or manipulation phases, e.g. approach, contact, force application and so on. On the other hand, from a mechanical point of view the information on the object/environment, if available, are often poor or not precise, e.g. mass and friction properties are in general not exactly known *a priori* and therefore cannot be used for the task planning.

For this reason, additional sensors (force, tactile, ...) must be introduced and used in real time, and the planned grasp should be robust enough to ensure a proper behavior, i.e. the safe achievement of the grasp/manipulation, for different materials and grasping configurations.

From the control point of view, several control and task planning strategies have been proposed in the literature in order to execute grasps and manipulations with a robotic

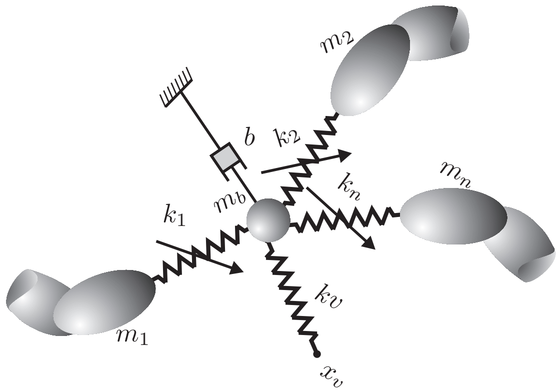



Figure 3.6: Basic idea of the proposed IPC.

system, see e.g. (Murray, Li & Sastry 1994) among many others. Concerning these control schemes, it may be noticed that very few among them consider explicitly the problem of controlling, or even defining, the dynamics of *interaction*. In addition, one of the most problematic phenomena in force control strategies is that stability cannot be ensured if not assuming as known important features of the object to be grasped, like its stiffness and friction. Furthermore, a force control strategy alone is not suitable to properly control the transition between no-contact and contact. This is due to the obvious fact that force control is meaningful only in contact, since no force can be exerted in free space. For these reasons, a control strategy for grasp and manipulation based on physically-based observations and on passivity concepts seems worth to be pursued.

This strategy does not have the shortcomings of other grasping control techniques. In particular, it is strictly passive in steady-state while during manipulation the supplied energy is directly controllable. Moreover, the related compliance control of each finger allows for rolling, slipping, and whole-hand grasps in a natural way.

In other terms, the technique illustrated in this section allows to shape the potential energy of the robot/object system in order to achieve a desired compliance, and injects damping to ensure both asymptotic stability and proper transient behavior. The main point here is to study the interaction between the robot and an object (the environment) in such a way that the overall system is stable independently on partially unknown geometrical or mechanical properties, and also achieves desired performances during task executions.

This section is concerned with the design of the IPC part of the controller. Since the IPC has been conceived in order to have a physically interpretable behavior, it will be described here in terms of spatial interconnection of physical elements like springs, dampers and inertias. As a matter of fact, the basic idea of the IPC is shown in Figure 3.6, where bodies $m_1, m_2 \dots, m_n$ represent the last links of some kinematic chains, e.g. the fingertips of a robotic hand or the distal links of robot arms. Both the body with mass $m_b$ (assumed to be rigid) and the springs (with stiffnesses $k_1, k_2, \dots, k_n$ and $k_v$) are implemented directly by means of control and are exploited, along with the damping $b$, in order to define the dynamic behavior of the IPC and consequently of the overall system. In the control algorithm, the dynamics of mass $m_b$ (called here the *virtual object*) is simulated in real-time. In particular, since it is subject to the forces generated by springs $k_1, k_2, \dots, k_n$ and by the damper $b$, its motion in 3D is computed in real time. Although this mass does not physically exist, it is of major importance for the control, since its "kinetic energy" can be dissipated by the damper $b$, achieving in this manner a passive behavior for the control.

Note that the computation of the damping force due to $b$ needs only the knowledge of the velocity of the virtual mass $m_b$, that is known since it is computed in the controller. If dampers had been attached to masses $m_1, \dots, m_n$, their velocities would have been necessary to simulate their behavior, with the necessity of joint velocity measurements for the robot.

The stiffnesses $k_1, \dots, k_n$ and $k_v$ of the springs, their rest lengths, the mass $m_b$ of the virtual object and the position $x_v$ are parameters to be used in real-time to achieve the desired behavior from the system. Obviously, it should be taken into account that careless changes of these parameters can result in a non passive behavior due to an apparent energy change of the system. More sophisticated methods to use variable springs can be found in (Stramigioli & Duindam 2001).

In the following sections, formal models of these physical entities and of their interconnection is discussed. This model is the basis of the proposed control strategy.

### 3.3.1 The Springs - Spatial Compliance

A spatial (3D) compliance is a geometric spring connecting two rigid bodies $B_i$ and $B_j$. Lončarić in (Lončarić 1985, Lončarić 1987) studied geometric springs represented by potential energy functions of the relative position of the rigid bodies to which they are attached. Successively (Fasse 1997, Fasse & Breedveld 1998*a*, Fasse & Breedveld 1998*b*) extended this work giving some useful geometrical parameterizations. More recently, (Stramigioli 1998) extended the formulation to a completely coordinate-free setting and also in order to consider elastic elements with more than two ports and variable lengths.

A spring between two rigid bodies $B_i$ and $B_j$ is characterized by a positive definite function representing the stored potential energy[2] with the following form:

$$V_{i,j} : SE(3) \to \mathbb{R}; H_i^j \mapsto V_{i,j}(H_i^j)$$

where $H_i^j \in SE(3)$ is the matrix representing the isometry which brings a chosen reference frame $\Psi_j$ (fixed to body $B_j$) to another reference frame $\Psi_i$ (fixed to body $B_i$)[3].

Once a potential energy has been defined, the corresponding "force" generated by the elastic potential can be computed by considering the differential of $V_{i,j}$:

$$dV_{i,j} : SE(3) \to T^*SE(3)$$

which is the local force that body $B_i$ applies on the spring $V_{i,j}$ in the relative position $H_i^j$: $dV_{i,j}(H_i^j) \in T^*_{H_i^j}SE(3)$. This local force can be seen as the generalized force corresponding to a parameterization of $SE(3)$ like Euler angles and translation positions.

The wrench applied on the spring connecting $B_i$ to $B_j$ by body $B_i$ and expressed in frame $\Psi_j$, with a relative position $H_i^j$ is:

$$\tilde{W}_{i,j}^j(H_i^j) = \begin{pmatrix} \tilde{f}_{i,j}^j & n_{i,j}^j \\ 0 & 0 \end{pmatrix} = R^*_{H_i^j} dV_{i,j}(H_i^j)$$

[2]This function defines implicitly the unit of energy.

[3]Note that $H_i^j$ is also the matrix expressing the change of coordinates from $\Psi_i$ to $\Psi_j$, but the corresponding motion is given by its inverse $(H_i^j)^{-1}$.

where $\tilde{f}^j_{i,j} \in \mathbb{R}^{3\times3}$ is a skew-symmetric matrix corresponding to the force $f^j_{i,j} \in \mathbb{R}^3$ using the tilde notation introduced in Eq. (1.1). In the previous notation, we have indicated with $W^k_{i,j}$ the wrench applied to a spring element connecting $\Psi_i$ to $\Psi_j$ on the side of $\Psi_i$ expressed as a numerical vector expressed in $\Psi_k$.

We can also associate to the wrench matrix $\tilde{W}^j_{i,j} \in \mathbb{R}^{4\times4}$ the corresponding vector representation which we indicate with $W^j_{i,j} := (n^j_{i,j} \quad f^j_{i,j}) \in \mathbb{R}^6$. It can be seen that$(W^i_i)^T = -(W^i_{i,j})^T = -Ad^T_{H^j_i}(W^j_{i,j})^T$ is the wrench that the spring applies to body $B_i$ expressed in the frame $\Psi_i$ and that $(W^j_j)^T = -(W^j_{j,i})^T = -Ad^T_{H^i_j}(W^i_{j,i})^T$ is nothing else than the wrench that the spring applies to body $B_j$ expressed in frame $\Psi_j$ and furthermore $W_{i,j} = -W_{j,i}$ due to the nodicity of a spring.

A desired energy function can be defined (and implemented using control) such that the relative configuration $H^j_i = I_4$ corresponds to a minimum of the potential energy $V_{i,j}(\cdot)$ (Fasse 1997). In this configuration, the frames $\Psi_j$ and $\Psi_i$ will coincide. The energy function can be chosen such that the common origins of $\Psi_i$ and $\Psi_j$ in the equilibrium position represent the *center of stiffness* (Stramigioli 1998). Expressed in the equilibrium frame ($\Psi_i = \Psi_j$), we can then choose three $3 \times 3$ desired stiffness matrices $K_o$, $K_t$ and $K_c$, corresponding respectively to the orientational, translational and coupling stiffnesses. From these stiffness matrices, we can calculate the so-called co-stiffness matrices (Fasse 1997) $G_o$, $G_t$ and $G_c$ related to the stiffnesses matrices by:

$$G_\alpha = \frac{1}{2}\,\mathrm{tr}(K_\alpha)I - K_\alpha$$

where $\alpha = o, t, c$ and where $\mathrm{tr}()$ is the tensor trace operator. It is then possible to give an expression of the wrench $W^i_i$ as a function of the relative configuration $H^j_i = \begin{pmatrix} R^j_i & p^j_i \\ 0 & 1 \end{pmatrix}$ (see (Fasse 1997)):

$$W^i_i(H^j_i) = \begin{pmatrix} \tilde{f}^i_i & n^i_i \\ 0 & 0 \end{pmatrix} \text{ where}$$

$$\begin{aligned} n^i_i &= -2\,\mathrm{as}(G_o R^j_i) - \mathrm{as}(G_t R^i_j \tilde{p}^j_i \tilde{p}^j_i R^j_i) - 2\,\mathrm{as}(G_c \tilde{p}^j_i R^j_i) \\ \tilde{f}^i_i &= -R^i_j\,\mathrm{as}(G_t \tilde{p}^j_i) R^j_i - \mathrm{as}(G_t R^i_j \tilde{p}^j_i R^j_i) - 2\,\mathrm{as}(G_c R^j_i) \end{aligned} \tag{3.15}$$

and where $\mathrm{as}(\cdot)$ is an operator which takes the skew-symmetric part of a square matrix and the 'tilde operator' is defined in Eq. (1.1).

Eq. (3.15) is an expression that can be directly used for the implementation, but as previously remarked, it is possible to give a GPCHS form of a spring (Maschke 1996):

$$h^j_i = R_{h^j_i} T^{j,j}_i \tag{3.16}$$

$$W^j_{i,j} = R^T_{h^j_i} \frac{\partial V_{i,j}}{\partial h^j_i}. \tag{3.17}$$

where $h^j_i$ is a six dimensional local coordinate of $SE(3)$, $J(h^j_i) = 0$, and $R_h(\cdot)$ represents the Lie group right translation in the chosen coordinates $h$. It is also possible to consider an additional power port which can be used to vary the effective rest length of a spring as shown schematically in Figure 3.7. By using an additional port, the relative position of $b$ and $i$ can be modified. The obtained effect is to change the rest length of the spring between $b$ and $j$. Applying in the general 3D case the concept of Figure 3.7, the twist relation becomes:

$$T^{j,j}_i = T^{j,j}_b + Ad_{H^j_b} T^{b,b}_i \tag{3.18}$$

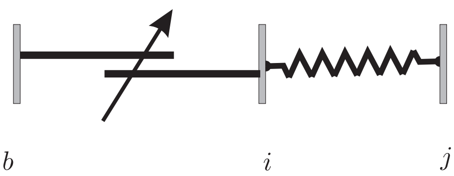



Figure 3.7: Schematic drawing of a power consistent variable spring.

In this case, its GPCHS representation is:

$$h_i^j = \begin{pmatrix} R_{h_i^j} & R_{h_i^j} Ad_{h_b^j} \end{pmatrix} \begin{pmatrix} T_b^{j,j} \\ T_i^{b,b} \end{pmatrix} \tag{3.19}$$

$$\begin{pmatrix} (W_{b,j}^j)^T \\ (W_{i,b}^b)^T \end{pmatrix} = \begin{pmatrix} R_{h_i^j}^T \\ Ad_{h_b^j}^T R_{h_i^j}^T \end{pmatrix} \frac{\partial V_{i,j}}{\partial h_i^j}.$$

where the port $(T_i^{b,b}, W_{i,b}^b)$ can be effectively used to change the rest length of the spring. More details on the above concepts and mathematical derivations can be found in (Stramigioli 1998) and (Fasse 1997).

### 3.3.2 Masses

As it was shown in Sect. 1.6, the dynamic properties of a rigid body are uniquely described by its inertia tensor $I^b$. Let us consider a uniform sphere $B_b$, and a reference system $\Psi_b$ fixed to it with origin coincident to the center of the sphere. We have seen in Eq. (1.40) that the equation of the rigid body can be expressed as:

$$(\dot{N}^b)^T = ad_{T_b^{b,0}}^T (N^b)^T + (W^b)^T. \tag{3.20}$$

It can be shown that:

$$(\dot{N}^i)^T = ad_{T_i^{i,0}}^T (N^i)^T + (W^i)^T \Rightarrow$$

$$(\dot{N}^i)^T = \begin{pmatrix} -\tilde{\omega}_k^{k,0} & -\tilde{v}_k^{k,0} \\ 0 & -\tilde{\omega}_k^{k,0} \end{pmatrix} \begin{pmatrix} (N_\omega^i)^T \\ (N_v^i)^T \end{pmatrix} + (W^i)^T \Rightarrow ...$$

$$(\dot{N}^i)^T = \begin{pmatrix} \tilde{N}_\omega^i & \tilde{N}_v^i \\ \tilde{N}_v^i & 0 \end{pmatrix} \begin{pmatrix} \omega_i^{i,0} \\ v_i^{i,0} \end{pmatrix} + (W^i)^T \Rightarrow ...$$

$$(\dot{N}^i)^T = (N^i \wedge) T_i^{i,0} + (W^i)^T$$

and therefore we can write the equations as

$$(\dot{N}^b)^T = (N^b \wedge) T_b^{b,0} + (W^b)^T$$

where where $(N^b)^T = I^b T_b^{b,0} \in \mathbb{R}^6$ is the generalized momentum, $T_b^{b,0} = ((\omega_b^{b,0})^T \quad (v_b^{b,0})^T)^T$ is the twist of the sphere respect to an inertial frame $\Psi_0$, $W_{b\,\mathrm{tot}}^b$ is the total wrench applied to the sphere and $N^b \wedge$ comes from the Lie-Poisson bracket and in the body coordinates $\Psi_b$ is represented by a $6 \times 6$ matrix of the following form:

$$(N^b \wedge) := \begin{pmatrix} \tilde{N}_\omega^b & \tilde{N}_v^b \\ \tilde{N}_v^b & 0 \end{pmatrix}$$

where $N^b = (N_\omega \quad N_v)$. For the specific case of the sphere, $I^b = \begin{pmatrix} jI_3 & 0 \\ 0 & mI_3 \end{pmatrix}$ where $I_3$ is the $3 \times 3$ identity matrix, $j$ is the rotational inertia of the sphere and $m$ its mass.

Also in this case, it is possible to give a GPCHS representation of the inertia's dynamics:

$$\begin{aligned} (\dot{N}^b)^T &= (N^b \wedge)\frac{\partial E_k}{\partial N^b} + (W_b^b)^T \\ T_b^{b,0} &= \frac{\partial E_k}{\partial N^b} \end{aligned} \tag{3.21}$$

where $E_k(N^b) = \frac{1}{2}N^b(I^b)^{-1}(N^b)^T$ is the kinetic energy which is a function of $N^b$ instead that a function of $T_b^{b,0}$ as usually thought. The corresponding function $E_k^*(T_b^{b,o})$ is called co-energy.

### 3.3.3 Dampers - Energy Dissipation

The easiest manner to model a linear spatial damping effect is to use an element which generates a wrench directly proportional to the twist of the body whose free-energy has to be dissipated. In this presentation we use:

$$(W_b^b)^T{}_{\text{diss}} = RT_b^{b,0} \tag{3.22}$$

where $R \in \mathbb{R}^{6\times 6}$ is a positive definite matrix representing a dissipation tensor in the frame $\Psi_b$.

Note that this element does not have a state and it will appear in the complete GPCHS of the interconnected part, as a part of the tensor $R(x)$ of Eq. (3.6).

### 3.3.4 The control Scheme

Following the steps presented in (Stramigioli 1999, Stramigioli 2001), it is possible to give a GPCHS description of the controller in the following form:

$$\begin{aligned} \begin{pmatrix} \dot{x}_B \\ \dot{x}_S \end{pmatrix} &= \begin{pmatrix} J_B - R_B & -\phi_B\phi_v^* \\ \phi_v\phi_B^* & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H_C}{\partial x_B} \\ \frac{\partial H_C}{\partial x_S} \end{pmatrix} + \\ & \begin{pmatrix} 0 & 0 & 0 \\ \phi_{\text{r}} & \phi_{v(b)} & \phi_{\text{var}} \end{pmatrix} \begin{pmatrix} t_{\text{r}}^0 \\ t_{v(b)}^0 \\ t_{\text{var}}^b \end{pmatrix} \\ \begin{pmatrix} w_{\text{r}}^0 \\ w_{v(b)}^0 \\ w_{\text{var}}^b \end{pmatrix} &= \begin{pmatrix} 0 & \phi_{\text{r}}^* \\ 0 & \phi_{v(b)}^* \\ 0 & \phi_{\text{var}}^* \end{pmatrix} \begin{pmatrix} \frac{\partial H_C}{\partial x_B} \\ \frac{\partial H_C}{\partial x_S} \end{pmatrix}. \end{aligned} \tag{3.23}$$

where $x_B, x_S$ are respectively the states of the virtual object and the springs as presented in Eq. (3.21) and Eq. (3.19), $H_C(x_B, x_S)$ is the the sum of the kinetic energy of the virtual object plus the potential energies of the springs, and $\phi_i, J_B$ and $R_B$ are properly defined matrices (Stramigioli 1998). The interaction ports are the pairs $(t_{\text{r}}^0, w_{\text{r}}^0)$,$(t_{v(b)}^0, w_{v(b)}^0)$ and $(t_{\text{var}}^b, w_{\text{var}}^b)$ where

$$t_{\text{r}}^0 = \begin{pmatrix} T_1^{0,0} \\ \vdots \\ T_n^{0,0} \end{pmatrix}$$

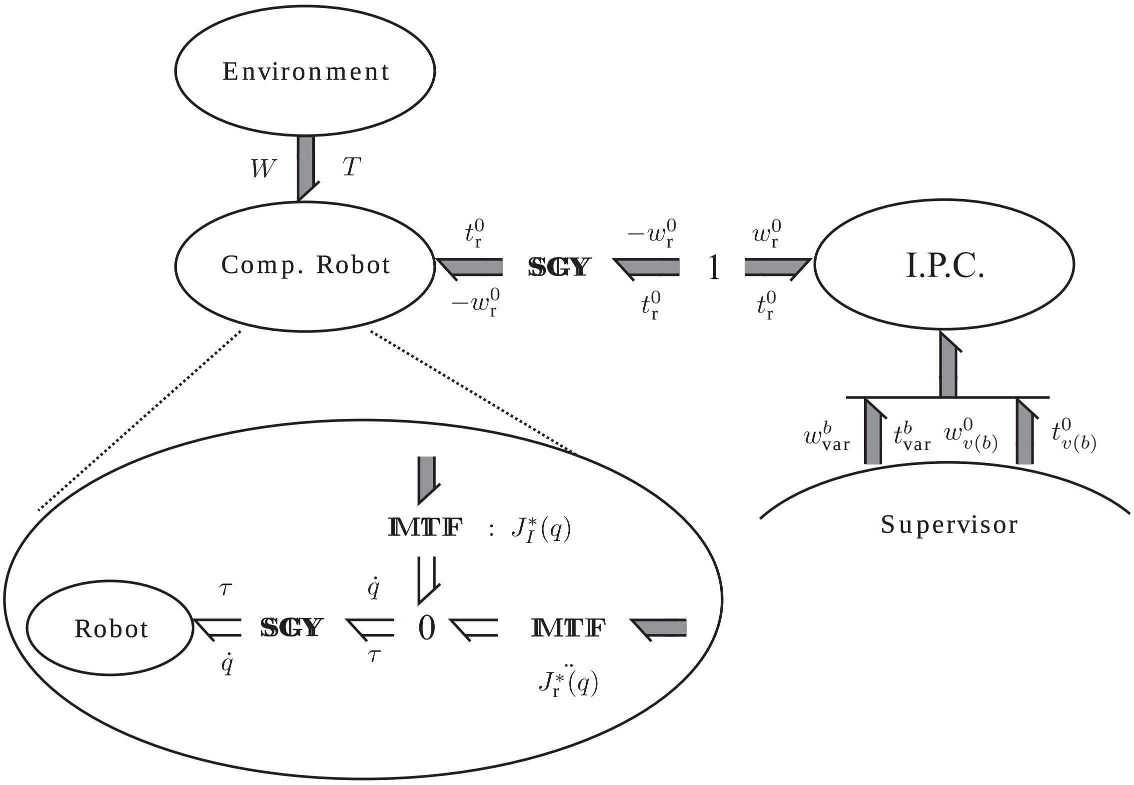



Figure 3.8: The interconnection between the robot and the intrinsically passive controller.

is the vector of twists of the fingertips with respect to the inertial frame, $t_{v(b)}^0$ is the twist corresponding to the motion of the configuration $x_v$ in Figure 3.6 and it is used by the supervisor to change the virtual position of the hand and

$$t_{var}^b = \begin{pmatrix} T_1^{b_1,b_1} \\ \vdots \\ T_n^{b_n,b_n} \end{pmatrix}$$

is the vector of twists that can be set by the supervisor to change the minimum potential configuration of the springs as shown schematically in Figure 3.7 when changing the relative configuration of $i$ and $b$. This last subsystem represents the IPC of Figure 3.8. In the figure, bond graphs notation is used: each power port of Eq. (3.23) corresponds to a power bond in the figure.

In order to use the IPC for the real control of a robotic system, we need a way to map $w_r^0$ to the actuation of the robot. This is done by using the robot differential kinematics, expressed by the Jacobian matrix $J_r(q)$, that maps a configuration velocity $\dot{q}$ to the twists $t_r^0$ of the tips of the hand:

$$t_r^0 = J_r(q)\dot{q}$$

which implies

$$\tau = J_{\mathrm{r}}^T(q)\begin{pmatrix}-W_1^0\\ \vdots \\ -W_n^0\end{pmatrix} = -J_{\mathrm{r}}^T(q)w_{\mathrm{r}}^0 \tag{3.24}$$

where $W_i^0$ is the total wrench applied to tip $i$ by the controller's spring $i$. From the above equations, it could be deduced at first that, for the given GPCHS representation of the controller, the velocities of the joints are needed for the proposed scheme. This is not the case, since the controller's actions (i.e. the torques) applied to the robot are a consequence of the springs attached to the fingertips, as shown in Figure 3.6. These elastic forces are function only of the relative configuration of the tips with respect to the virtual object. In particular, this implies that only the pose of the tips are necessary to calculate these wrenches, and not their velocities. The configurations can be directly computed by using joints measurements and the forward kinematics.

Clearly, this has the advantage that only position measurements are necessary to implement the scheme: no velocity and not even force measurements are in fact required. On the other hand, note that this scheme can therefore be used if the robot is back-drivable, which means in particular that any external force applied to the robot will result in a motion of the robot itself, that can be therefore detected with position measurements alone.

This scheme does not change the driving point mass of the robot, but only its interactive compliance, differently than the well known impedance scheme presented in (Hogan 1985*b*). It should be noticed though that even if different from (Hogan 1985*a*, Hogan 1985*b*, Hogan 1985*c*), the adopted philosophy is exactly the same as it has been commented in (Won, Stramigioli & Hogan 1997).

#### Energy dissipation

Even if only positions measurements are needed, overall dissipation is achieved because of the presence of the viscous term connected to the virtual object. The viscous force is applied to the virtual object as shown in Figure 3.6. Any energy entering the control system which causes a motion of the virtual object is dissipated, and this has a nice physical interpretation. Again, note that this is achieved without the necessity to measure velocities of the robotic system.

A control design philosophy very similar to this one has been developed by Ortega and others, see for example (Ortega, Loria, Kelly & Praly 1994). However, it has to be pointed out that in this case the formulation is based on a Lagrangian framework. Here, the problem is considered in the more general Hamiltonian setting as the power consistent interconnection of GPCHSs.

## 3.4 Summary

A novel architecture has been presented which slightly resembles the human physiology. Real-time behavior is controlled by the Intrinsically Passive Controller which corresponds to a virtual physical system. The IPC more or less resembles the role of muscle and spindles in biological systems. The IPC together with the robot can be seen as a pre-compensated robot. Due to the implementation of the IPC control as a power consistent interconnection with the robot to be controlled, passivity is ensured in *any* situation if no power is injected by the Supervisor. The structure of the IPC is that of a Port Controlled Hamiltonian System with dissipation. An example of such an IPC has been given. An important point is that the IPC can be really designed using mechanical analogies like springs, dampers and masses and then implemented using the theory of interconnection of Port Controlled Hamiltonian Systems. Furthermore, it has been shown in (Stramigioli et al. 1998) using the theory of Casimir functions that it is possible to implement the IPC with only measurements of positions $q$ and not of velocities. With the presented strategy, it is not appropriate to talk about position or force control anymore, because what it is controlled is actually the *behavior* of the system and not the position or force at its interconnection port. This has the advantages of being very robust with respect to different materials and object with which the robot interact; just think about shaking hand to somebody, we never have a perfect model of the person we shake hand with, but the interaction is always well behaved.

A

# Projective geometry and kinematics

An extention to the conception we have of the three dimensional Eucledian spaces can be found in projective geometry[1]. To talk about the 3D Eucledian world in a projective setting, we need three ingredients:

- A real vector space $\mathcal{V}^4$ called the *supporting vector space* of dimension 4 from which we exclude the origin.
- An equivalence relation on $\mathcal{V}^4 - \{0\}$: $v_1 \sim v_2 \Leftrightarrow \exists \alpha \in \mathbb{R} \neq 0$ s.t. $v_1 = \alpha v_2$.
- A *polarity* $P$, which is a 2 covariant, symmetric tensor defined on $\mathcal{V}^4$ which in the sequel will be taken semipositive defined and of rank 1.

The basic transformations between points of projective spaces are defined as injective linear transformations between the supporting vector spaces. These transformations must be injective to prevent that the subspace corresponding to the kernel of the transformation is mapped to the $0$ element of the codomain which is NOT a valid element of the projective space[2]. These kinds of transformations are called *homographies* or *collineations* and in our case are mappings from $\mathcal{V}^4$ to $\mathcal{V}^4$. Maps from $\mathcal{V}^4$ to the dual $\mathcal{V}^{4*}$ are instead called *correlations*. A symmetric correlation as $P$ is called *polarity*.

### Improper hyperplane

Using the polarity $P$, it is possible to consider the vectors $p^i$ belonging to the quadric defined by the polarity $P$ which is called *the absolute*:

$$P_{ij} p^i p^j = 0 \qquad p^i \in \mathcal{V}^4.$$

The absolute, is a three dimensional subspace of $\mathcal{V}^4$ which is called the *improper hyperplane* and it is indicated with $\mathcal{I}^3 \subset \mathcal{V}^4$. This hyperplane represents the “points at infinity”.

The improper hyperplane splits $\mathcal{V}^4$ in two disjointed semi-spaces which we will call respectively *positive* semi-space and indicate it with $\mathcal{I}^+$ and *negative* semi-space and indicate it with $\mathcal{I}^-$.

[1] Caley in 1859, by introducing the concept of an “absolute”, showed that projective geometry is the most general.

[2] It will be shown later that to have a proper definition of the projective space, the $0$ element must be excluded.

The three dimensional projective space is defined as the quotient space of $\mathcal{V}^4$ excluding the origin with respect to the defined equivalent relation:

$$\mathcal{P}^3 := \frac{\mathcal{V}^4 - \{0\}}{\sim}$$

In a purely projective setting, without considering the polarity, all points are of the same type. Considering the polarity, we can make a distinction between finite points and infinite points. Infinite points are those whose representative in $\mathcal{V}^4$ belong to $\mathcal{I}^3$ and finite points are the others. We indicate with $\mathcal{P}_F$ the finite points and with $\mathcal{P}_\infty$ the points at infinity. Clearly we have that $\mathcal{P} = \mathcal{P}_F \cup \mathcal{P}_\infty$. Furthermore, for each finite point $p \in \mathcal{P}_F$ ($P_{ij}p^ip^j \neq 0$), there are representatives $v \in \mathcal{V}^4$ belonging either to $\mathcal{I}^+$ or to $\mathcal{I}^-$. We can define the sign function $\sigma$ for elements $v \in \mathcal{V}^4$:

$$\sigma(v) := \begin{cases} +1 & v \in \mathcal{I}^+ \\ 0 & v \in \mathcal{I}^3 \\ -1 & v \in \mathcal{I}^- \end{cases} \tag{A.1}$$

## Adjoint polarity

Associated with the polarity $P$, one defines its adjoint[3] $Q$ which is a 2 contravariant, symmetric, semipositive tensor of rank 3. Once a proper base $\{e_x, e_y, e_z, e_0\}$ for $\mathcal{V}^4$ is chosen, the representations of $P$ and $Q$ become:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.2}$$

In the same coordinates, a vector of $\mathcal{V}^4$ has the form $(x, y, z, \alpha)^T$ and if $\alpha = 0$ the vector belongs to $\mathcal{I}^3$.

## Points and Free vectors

It is possible to associate to each pair of finite vectors in $\mathcal{P}_F$ a unique element of $\mathcal{I}$:

$$f : \mathcal{P}_F \times \mathcal{P}_F \to \mathcal{I} \; ; \; (p, q) \mapsto \frac{p}{||p||\sigma(p)} - \frac{q}{||q||\sigma(q)} \tag{A.3}$$

where $p, q$ are any representatives and $||\cdot||$ represents the $P$-norm. It is possible to see that the previous operation is indeed independent of the representatives of the points and therefore well defined. Note that the difference of two points can be calculated using the vector structure of $\mathcal{V}^4$. Furthermore, the improper hyperplane without the equivalence relation and with the origin of $\mathcal{V}^4$, gets the meaning of the vector space of free-vectors. In the usual coordinates, this means that if

$$p = \begin{pmatrix} \alpha_p x_p \\ \alpha_p y_p \\ \alpha_p z_p \\ \alpha_p \end{pmatrix} \text{ and } q = \begin{pmatrix} \alpha_q x_q \\ \alpha_q y_q \\ \alpha_q z_q \\ \alpha_q \end{pmatrix}$$

[3] If instead of 0 on the diagonal elements of $P$ and $Q$ we substitute $\epsilon$, we see that $QP = PQ = \epsilon I$ where $I$ is the identity matrix.

we have $||p||\sigma(p) = \alpha_p$ and $||q||\sigma(q) = \alpha_q$ and therefore

$$f(p,q) = \begin{pmatrix} x_p - x_q \\ y_p - y_q \\ z_p - z_q \\ 0 \end{pmatrix}$$

Note that indeed the last component is equal to zero which confirms the fact that the element belongs to $\mathcal{I}$. It is usual to use the notation:

$$p - q := f(p,q)$$

for obvious reasons, but the normalisation of Eq. (A.3) before the subtraction is essential to make the operation intrinsically defined.

### Lines

A line $l \subset \mathcal{P}^3$ in a projective context is nothing else than a one dimensional subspace which corresponds to a two dimensional subspace $L \subset \mathcal{V}^4$ (without the origin) of the supporting vector space $\mathcal{V}^4$. In a projective context it is also possible to talk about lines at infinity when $L \subset \mathcal{I}^3$. As a consequence, a line can be described both as the subspace spanned by two points of $\mathcal{V}^4$ or as the intersection of two hyperplanes of $\mathcal{V}^4$. As in every $2n$ dimensional vector space an $n$ dimensional subspace is self-dual, so is a line in $\mathcal{P}^3$ a self dual entity.

It is possible to show that using the previous coordinates, given two distinct points $x, y$ belonging to a line where

$$x = \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} \bar{y} \\ 1 \end{pmatrix} \text{ where } \bar{x}, \bar{y} \in \mathbb{R}^3$$

the corresponding line can be identified homogeneously to a vector of the form

$$\alpha \begin{pmatrix} \bar{y} - \bar{x} \\ \bar{x} \wedge \bar{y} \end{pmatrix} \text{ where } \alpha \in \mathbb{R} - \{0\}. \tag{A.4}$$

Eq. (A.4) can be clearly written as:

$$\begin{pmatrix} \bar{y} - \bar{x} \\ \bar{x} \wedge (\bar{y} - \bar{x}) \end{pmatrix} = \begin{pmatrix} \bar{u} \\ \bar{r} \wedge \bar{u} \end{pmatrix}. \tag{A.5}$$

It is easy to see that choosing instead of $\bar{x}$ and $\bar{y}$ other two distinct points on the same line in the usual geometrical sence, the vector $l$ defined as in Eq. (A.4) is the same if a proportional multiplication constant is used. This implies that the space of lines has somehow a projective nature by itself, but it should be noticed that the linear combination of two lines as expressed in Eq. (A.4) is in general NOT a line.

### Euclidean product on the improper hyperplane

It is now possible to define a proper non-singular, internal product for $\mathcal{I}^3$ which gives rise to the scalar product which characterises a proper Euclidean Space. In the previous canonical coordinates, an element $v_i \in \mathcal{I}^3$ is characterised by the last component equal to zero. We can associate to $v_i \in \mathcal{I}^3$ the subspace of hyperplanes $\mathfrak{H}_{v_i}$ for which $v_i$ is what is called the *polar* with respect to $Q$:

$$h \in \mathfrak{H}_{v_i} \Leftrightarrow v_i = Qh$$

where $Q$ is the adjoint of $P$. We can now define the scalar product of two vectors $v_1, v_2$ in $\mathcal{I}^3$ as:

$$\langle v_1, v_2 \rangle := \sqrt{h_1^T Q h_2} \qquad h_1 \in \mathfrak{H}_{v_1}, h_2 \in \mathfrak{H}_{v_2} \tag{A.6}$$

It is easy to see that the previous definition is well posed since it is independent on the elements $h_1$ and $h_2$ due to the structure of $Q$.

**B**

# Introduction to Lie groups

A manifold is intuitively a smooth space which is locally homeomorphic to $\mathbb{R}^n$ and brings with itself nice differentiability properties. Proper definitions of manifolds can be found on (Dubrovin, Fomenko & Novikov 1985, Boothby 1975). A group is an algebraical structure defined on a set. Definitions of groups can be found on any basic book of algebra.

A Lie group is a group, whose set on which the operation are defined is a manifold $\mathcal{G}$. This manifold $\mathcal{G}$ has therefore a special point '$e$' which is the identity of the group.

Using the structure of the group, and by denoting the group operation as:

$$o\ : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}\ ;\ ;\ (h, g) \mapsto h\ o\ g,$$

we can define two mappings within the group which are called respectively left and right mapping:

$$L_g : \mathcal{G} \rightarrow \mathcal{G}\ ;\ h \mapsto g\ o\ h \tag{B.1}$$

and

$$R_g : \mathcal{G} \rightarrow \mathcal{G}\ ;\ h \mapsto h\ o\ g \tag{B.2}$$

As we will see later the differential of these mappings at the identity, plays an important role in the study of mechanics.

The tangent space $T_e\mathcal{G}$ to $\mathcal{G}$ at $e$, which is indicated with $\mathfrak{g}$, has furthermore the structure of a Lie algebra which is nothing else than a vector space $\mathfrak{g}$ together with an internal, skew-symmetric operation called the commutator:

$$[,] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}\ ;\ (g_1, g_2) \mapsto [g_1, g_2] \tag{B.3}$$

For $\mathfrak{g}$ to be a Lie algebra, the commutator should furthermore satisfy what is called the Jacoby identity:

$$[g_1, [g_2, g_3]] + [g_2, [g_3, g_1]] + [g_3, [g_1, g_2]] = 0 \quad \forall g_1, g_2, g_3 \in \mathfrak{g} \tag{B.4}$$

Lie groups are important because we can use them as acting on a manifold $\mathcal{M}$, which in our case will be the Euclidean space. An action of $\mathcal{G}$ on $\mathcal{M}$, is a smooth application of the following form:

$$a : \mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$$

such that

$$a(e, x) = x \qquad \forall x \in \mathcal{M},$$

and

$$a(g_1, a(g_2, x)) = a(g_1 g_2, x) \qquad \forall x \in \mathcal{M}, g_1, g_2 \in \mathcal{G}.$$

This means that an action is somehow compatible with the group on which it is defined.

## B.1 Matrix Lie groups

For a lot of fundamental reasons like Ado's theorem (Olver 1993), matrix algebras are excellent representatives for any finite dimensional group like the ones we need for rigid body mechanisms.

A matrix Lie group is a group whose elements are square matrices and in which the composition operation of the group corresponds to the matrix product. The most general real matrix group is $GL(n)$ which represents the group of non singular $n \times n$ real matrices. This is clearly a group since the identity matrix represents the identity element of the group, for each matrix, there is an inverse, and matrix multiplication is associative. We will now analyse more in detail features and operations of matrix Lie groups.

### Left and Right maps

If we consider a matrix Lie group $\mathcal{G}$, the operations of left and right translation clearly become:

$$L_G(H) = GH \qquad \text{and} \qquad R_G(H) = HG.$$

We can now consider how velocities are mapped using the previous maps. Suppose that we want to map a velocity vector $\dot{H} \in T_H\mathcal{G}$ to a velocity vector in $T_{GH}\mathcal{G}$ using the left translation and to a vector in $T_{HG}\mathcal{G}$ using right translation. We obtain:

$$(L_G)_*(H, \dot{H}) = (GH, G\dot{H}) \quad \text{and} \quad (R_G)_*(H, \dot{H}) = (HG, \dot{H}G)$$

In particular, if we take a reference velocity at the identity, we obtain:

$$(L_G)_*(I, T) = (G, GT) \quad \text{and} \quad (R_G)_*(I, T) = (G, TG)$$

where $T \in \mathfrak{g}$. With an abuse of notation, we will often indicate:

$$(L_G)_*\, T = GT \qquad \text{and} \qquad (R_G)_*\, T = TG$$

when it is clear that we consider mappings from the identity of the group. On a Lie group, we can define *left invariant* or *right invariant* vector fields. These vector fields are such that the differential of the left invariant and right invariant map leaves them invariant. If we indicate with

$$V : \mathcal{G} \to T\mathcal{G}\ ;\ x \mapsto (x, v)$$

a smooth vector field on the Lie group $\mathcal{G}$, we say that this vector field is left invariant if:

$$V(L_g(h)) = (L_g)_* V(h) \quad \forall g, h \in \mathcal{G},$$

and similarly it is right invariant if:

$$V(R_g(h)) = (R_g)_* V(h) \quad \forall g \in \mathcal{G}.$$

For a matrix group, if we take in the previous definitions $h = I$ we obtain respectively:

$$V(G) = GT_L \quad \text{and} \quad V(G) = T_R G$$

where we indicated the representative of the left and right invariant vector fields at the identity with $T_L$ and $T_R$. We can conclude from this that any left or right invariant vector field is characterized completely by its value at the identity of the group. We could now ask ourself: what are the integrals of a left or invariant vector field? From what just said, the integral of a left invariant vector field, can be calculated as the integral of the following matrix differential equation:

$$\dot{G} = GT_L \Rightarrow G(t) = G(0)e^{T_L t} \tag{B.5}$$

where $T_L$ is the value of the vector field at the identity. In a similar way, the integral of a right invariant vector field is:

$$\dot{G} = T_R G \Rightarrow G(t) = e^{T_R t}G(0). \tag{B.6}$$

From this it is possible to conclude that if we take an element $T \in \mathfrak{g}$, its left and right integral curves passing through the identity coincide and they represent the exponential map from the Lie algebra to the Lie group:

$$e : \mathfrak{g} \to \mathcal{G}; T \mapsto e^T.$$

It is easy to show, and important to notice, that integral curves passing through points $H = e^{T_1}$ of right and left invariant vector fields which have as representative in the identity $T_2$, are coincident iff $e^{T_1}e^{T_2} = e^{T_2}e^{T_1}$ which is true iff $[T_1, T_2] = 0$, where the last operation is the commutator of the Lie algebra. But how does the commutator look like for a matrix Lie algebra ? Being a Lie group a manifold, we can compute the Lie brackets of vector fields on the manifold. Furthermore, we know that elements of the Lie algebra $\mathfrak{g}$ have a left and right vector field associated to them. We can than calculate the Lie bracket of two left or right invariant vector fields, and if the solution is still left or right invariant, consider the value of the resulting vector field at the identity as the solution of the commutator. We will start with the left invariant case first. Consider we are in a point $G(t) \in \mathcal{G}$ at time $t$. If we have two left invariant vector fields characterized by $T_1, T_2 \in \mathfrak{g}$, the Lie bracket of these two vector fields, can be calculated by moving from $G(t)$ along the vector field correspondent to $T_1$ for $\sqrt{s}$ time, than along the one correspondent to $T_2$, than along $-T_1$ and eventually along $-T_2$. In mathematical terms we have:

$$\begin{aligned} G(t+\sqrt{s}) = G(t)e^{T_1\sqrt{s}} \to G(t+2\sqrt{s}) = G(t+\sqrt{s})e^{T_2\sqrt{s}} \to \\ G(t+3\sqrt{s}) = G(t+2\sqrt{s})e^{-T_1\sqrt{s}} \to G(t+4\sqrt{s}) = G(t+3\sqrt{s})e^{-T_2\sqrt{s}} \to \\ G(t+4\sqrt{s}) = G(t)e^{T_1\sqrt{s}}e^{T_2\sqrt{s}}e^{-T_1\sqrt{s}}e^{-T_2\sqrt{s}} \end{aligned} \tag{B.7}$$

If we look at $\frac{d}{ds}G(t+4\sqrt{s})\big|_{s=0}$, we can approximate the exponentials with the first low order terms and we obtain:

$$\begin{aligned} G(t+4\sqrt{s}) \simeq G(t)\left(\left(I + T_1\sqrt{s} + \frac{T_1^2}{2}s\right)\left(I + T_2\sqrt{s} + \frac{T_2^2}{2}s\right)\right. \\ \left.\left(I - T_1\sqrt{s} + \frac{T_1^2}{2}s\right)\left(I - T_2\sqrt{s} + \frac{T_1^2}{2}s\right)\right) \\ \simeq G(t)(I + (T_1T_2 - T_2T_1)s + o(s)) \end{aligned} \tag{B.8}$$

which implies

$$\frac{d}{ds}G(t+4\sqrt{s})\bigg|_{s=0} = G(t)(T_1T_2 - T_2T_1).$$

From the previous equation, we can conclude that the resulting vector field is still left invariant and it is characterized by the Lie algebra element $T_1T_2 - T_2T_1$. We can therefore define the commutator based on left invariant vector fields as:

$$[T_1, T_2]_L = T_1T_2 - T_2T_1.$$

With similar reasoning, it is possible to show for right invariant vector fields that:

$$\frac{d}{ds}G(t+4\sqrt{s})\bigg|_{s=0} = (T_2T_1 - T_1T_2)G(t).$$

and therefore, in this case:

$$[T_1, T_2]_R = T_2T_1 - T_1T_2.$$

We have therefore that:

$$[T_1, T_2]_L = -[T_1, T_2]_R.$$

In the literature, $[,]_L$ is used as the standard commutator and we will adapt this convention.

### B.1.1 Matrix Group Actions

A group action we can consider for an $n$ dimensional matrix Lie group is the linear operation on $\mathbb{R}^n$. We can therefore define as an action:

$$a(G, P) = GP \qquad G \in \mathcal{G}, P \in \mathbb{R}^n$$

It is easy to see that this group action trivially satisfies all the properties required.

### B.1.2 Adjoint representation

Using the left and right maps, we can define what is called the conjugation map as $K_g := R_{g^{-1}}L_g$ which for matrix groups results:

$$K_G : \mathcal{G} \to \mathcal{G} \; ; \; H \mapsto GHG^{-1}.$$

But what is the importance of this conjugation map ? To answer this question, we need the matrix group action. Suppose we have a certain element $H \in \mathcal{G}$ such that $Q = HP$ where $Q, P \in \mathbb{R}^n$. What happens if we move all the points of $\mathbb{R}^n$ and therefore also $Q$ and $P$ using an element of $\mathcal{G}$ ? What will the corresponding mapping of $H$ look like ? If we have $Q' = GQ$ and $P' = GP$, it is straight forward to see that:

$$Q' = K_G(H)P'.$$

The conjugation map is therefore related to global motions or equivalently changes of coordinates. We clearly have that $K_G(I) = I$ and therefore the differential of $K_G()$ at the identity is a Lie algebra endomorphism. This linear map is called the Adjoint group representation:

$$Ad_G : \mathfrak{g} \to \mathfrak{g} \; ; \; T \mapsto GTG^{-1}.$$

The Adjoint representation of the group shows how an infinitesimal motion changes moving the references of a finite amount $G$. Eventually, it is possible to consider the derivative of the previous map at the identity

$$ad_T := \frac{d}{ds}Ad_{e^{Ts}}\bigg|_{s=0}.$$

This map is called the adjoint representation of the Lie algebra and it is a map of the form:

$$ad_T : \mathfrak{g} \to \mathfrak{g} \qquad T \in \mathfrak{g}$$

If we use the definitions we can see that:

$$\frac{d}{ds} Ad_{e^{T_1 s}} T_2 \bigg|_{s=0} = \frac{d}{ds} e^{T_1 s} T_2 e^{-T_1 s} \bigg|_{s=0} = T_1 T_2 - T_2 T_1 = [T_1, T_2]_L$$

which shows that:

$$ad_{T_1} T_2 = [T_1, T_2]_L \tag{B.9}$$

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