Example 4.2 Kinematic model of simplified Puma

In our second example, we will attempt to solve the kinematic model for simplified version of a popular industrial manipulator (Figure 4.19) using homogeneous transforms (Lee, 1982). This robot includes two type 4 links (see Figure 4.3). The axes of the two joints in these links intersect in the proximal joint, not in the distal joint like the other links in the manipulators discussed in earlier parts of the text. A consequence of this is that the two coordinate frames coincide, even though in the actual robot there is a physical distance between the two joints. In the analysis, this distance is added to the distance between the joints of the link attached to the distal joint.

The correct assignment of coordinate frames (following Algorithm 4.2) is shown in Figure 4.19(c). The zero position for kinematic analysis

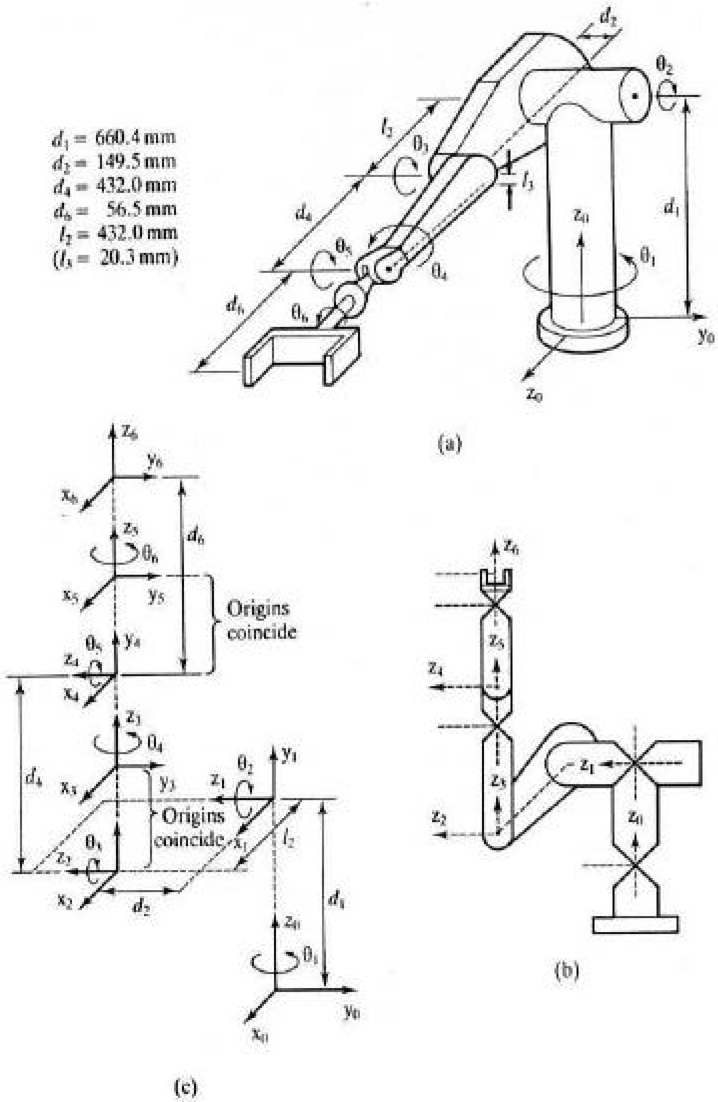

Figure 4.19
Six degree of freedom Puma Robot (simplified by neglecting l_3).
(a) Puma manipulator;
(b) Manipulator in zero position;
(c) Assignment of coordinate frames.

(Figure 4.19(b)) is different from the zero position used by Paul *et al.* (1981), where the left-arm position is used, which is the zero position used by the Unimation controller. The algorithm for assigning coordinate frames does not include a method for selecting the direction of the axis of a revolute joint. The choice of axis direction used here resulted in a right-arm configuration, where the choice used by Paul *et al.* (1981) resulted in a left-arm configuration. This difference changes the sign of the twist angles between the links,

Link	Joint variable	Angle θ_n	Displacement d_n	Length l_n	Twist α_n	Range
1	θ_1	θ_1	$d_1=660.4$	0	+90°	−160 to +160
2	θ_2	θ_2	$d_2=149.5$	$l_2=432.0$	0°	−225 to +45
3	θ_3	θ_3	0	0	−90°	−45 to +225
4	θ_4	θ_4	$d_4=432.0$	0	+90°	−110 to +170
5	θ_5	θ_5	0	0	−90°	−100 to +100
6	θ_6	θ_6	$d_6=56.5$	0	0°	−266 to +266

(a)

$$^0\mathbf{A}_1 = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^1\mathbf{A}_2 = \begin{bmatrix} C_2 & -S_2 & 0 & l_2C_2 \\ S_2 & C_2 & 0 & l_2S_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^2\mathbf{A}_3 = \begin{bmatrix} C_3 & 0 & -S_3 & 0 \\ S_3 & 0 & +C_3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^3\mathbf{A}_4 = \begin{bmatrix} C_4 & 0 & +S_4 & 0 \\ S_4 & 0 & -C_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^4\mathbf{A}_5 = \begin{bmatrix} C_5 & 0 & -S_5 & 0 \\ S_5 & 0 & +C_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^5\mathbf{A}_6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

Figure 4.20
(a) Link parameters and (b) **A** matrices for the simplified puma robot arm.

and results in a kinematic model with the same components in the equations, but with some components having different signs. Also, the zero position for the kinematic model is different from the zero position used by the manufacturer to calibrate the robot – all links pointing in a vertical direction.

The six **A** matrices (Figure 4.20) are found by substituting the link parameters for this robot into Equation 4.6, and the manipulator transform is found by multiplying these matrices. A complete kinematic model of this robot includes an l_3 parameter of 20.32mm due to asymmetry of the forearm. The axes of joints 3 and 4 do not intersect. In the following analysis, the robot kinematics are simplified by assuming a symmetric arm.

$$^R\mathbf{T}_H = {}^0\mathbf{A}_1\,{}^1\mathbf{A}_2\,{}^2\mathbf{A}_3\,{}^3\mathbf{A}_4\,{}^4\mathbf{A}_5\,{}^5\mathbf{A}_6 = \begin{bmatrix} x_x & y_x & z_x & p_x \\ x_y & y_y & z_y & p_y \\ x_z & y_z & z_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.93}$$

where the elements of the matrix are:

$$x_x = C_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \tag{4.94}$$

$$x_y = S_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \tag{4.95}$$

$$x_z = S_{23}(C_4C_5C_6 - S_4S_6) + C_{23}S_5C_6 \tag{4.96}$$

$$y_x = C_1[-C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \tag{4.97}$$

$$y_y = S_1[-C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6) \tag{4.98}$$

$$y_z = -S_{23}(C_4C_5S_6 + S_4C_6) - C_{23}S_5S_6 \tag{4.99}$$

$$z_x = -C_1(C_{23}C_4S_5 + S_{23}C_5) + S_1S_4S_5 \tag{4.100}$$

$$z_y = -S_1(C_{23}C_4S_5 + S_{23}C_5) - C_1S_4S_5 \tag{4.101}$$

$$z_z = -S_{23}C_4S_5 + C_{23}C_5 \tag{4.102}$$

$$p_x = C_1[-C_{23}d_6C_4S_5 - S_{23}(d_6C_5 + d_4) + l_2C_2] + S_1(d_6S_4S_5 + d_2) \tag{4.103}$$

$$p_y = S_1[-C_{23}d_6C_4S_5 - S_{23}(d_6C_5 + d_4) + l_2C_2] + C_1[d_6S_4S_5 + d_2] \tag{4.104}$$

$$p_z = -S_{23}d_6C_4S_5 + C_{23}(d_6C_5 + d_4) + l_2S_2 + d_1 \tag{4.105}$$

Using the solutions derived for the orientation transform (Section 4.9), we find the orientation of the coordinate frame located at the end of this robot (frame 6).

$$\phi = \operatorname{atan2}(x_x, x_y)$$

$$= \tan^{-1}\left(\frac{C_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] - S_1(S_4C_5C_6 + C_4S_6)}{S_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] + C_1(S_4C_5C_6 + C_4S_6)}\right) \tag{4.106}$$

$$\theta = \operatorname{atan2}[-x_z, x_xC(\phi) + x_yS(\phi)]$$

$$= \tan^{-1}\left(\frac{-S_{23}(C_4C_5C_6 - S_4S_6) - C_{23}S_5C_6}{[C_1C_\phi + S_1S_\phi][C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] + [-S_1C_\phi + C_1S_\phi](S_4C_5C_6 + C_4S_6)}\right) \tag{4.107}$$

$$\psi = \operatorname{atan2}(y_z, z_z) = \tan^{-1}\left(\frac{-S_{23}(C_4C_5S_6 + S_4C_6) - C_{23}S_5S_6}{-S_{23}C_4S_5 + C_{23}C_5}\right) \tag{4.108}$$

The next step is to attempt to find the inverse transform using Algorithm 4.4. The forward solution involves complex terms, and joint variables are difficult to find, so we will premultiply both sides of the transformation equation (Equation 4.93) by the inverse of the first matrix.

$${}^0\mathbf{A}_1^{-1}\,{}^R\mathbf{T}_H = {}^1\mathbf{A}_2\,{}^2\mathbf{A}_3\,{}^3\mathbf{A}_4\,{}^4\mathbf{A}_5\,{}^5\mathbf{A}_6 = \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_x & y_x & z_x & p_x \\ x_y & y_y & z_y & p_y \\ x_z & y_z & z_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_1x_x + S_1x_y & C_1y_x + S_1y_y & C_1z_x + S_1z_y & C_1p_x + S_1p_y \\ x_z & y_z & z_z & p_z - d_1 \\ S_1x_x - C_1x_y & S_1y_x - C_1y_y & S_1z_x - C_1z_y & S_1p_x - C_1p_y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
= \begin{bmatrix}
C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6 & -C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6 \\
S_{23}(C_4C_5C_6 - S_4S_6) + C_{23}S_5C_6 & -S_{23}(C_4C_5S_6 + S_4C_6) - C_{23}S_5C_6 \\
-S_4C_5C_6 - C_4S_6 & S_4C_5S_6 - C_4C_6 \\
0 & 0
\end{bmatrix.
$$

$$
\left. \begin{matrix}
-C_{23}C_4S_5 - S_{23}C_5 & -C_{23}d_6C_4S_5 - S_{23}(d_6C_5 + d_4) + l_2C_2 \\
-S_{23}C_4S_5 + C_{23}C_5 & -S_{23}d_6C_4S_5 + C_{23}(d_6C_5 + d_4) + l_2S_2 \\
S_4S_5 & d_6S_4S_5 + d_2 \\
0 & 1
\end{matrix} \right] \quad (4.109)
$$

If we equate matrix elements (1, 4) and (3, 4) from the two matrices, simplify the resulting equations using trigonometric identities, and divide the equations, we can solve for the first joint variable θ_1.

$$\phi = \tan^{-1}(p_y/p_x) \qquad r = +\sqrt{p_y^2 + p_x^2} \quad (4.110)$$

$$\text{where } p_y = r\sin(\phi) \qquad p_x = r\cos(\phi) \quad (4.111)$$

$$C_1p_x + S_1p_y = C_1 r\cos(\phi) + S_1 r\sin(\phi) \qquad = r\cos(\theta_1 - \phi) \quad (4.112)$$

$$= -C_{23}d_6C_4S_5 - S_{23}(d_6C_5 + d_4) + l_2C_2 \text{ [element 1, 4]} \quad (4.113)$$

$$S_1p_x - C_1p_y = S_1 r\cos(\phi) - C_1 r\sin(\phi) = r\sin(\theta_1 - \phi) = -d_6S_4S_5 + d_2 \quad (4.114)$$

$$\frac{r\sin(\theta_1 - \phi)}{r\cos(\theta_1 - \phi)} = \tan(\theta_1 - \phi) = \frac{-d_6S_4S_5 + d_2}{-C_{23}d_6C_4S_5 - S_{23}(d_6C_5 + d_4) + l_2C_2} \quad (4.115)$$

$$\theta_1 = \tan^{-1}\left(\frac{p_y}{p_x}\right) + \tan^{-1}\left(\frac{-d_6S_4S_5 + d_2}{-C_{23}d_6C_4S_5 - S_{23}(d_6C_5 + d_4) + l_2C_2}\right) \quad (4.116)$$

Premultiply the transformation equation again:

$$
{}^1\mathbf{A}_2^{-1}\,{}^0\mathbf{A}_1^{-1}\,{}^R\mathbf{T}_H = {}^2\mathbf{A}_3\,{}^3\mathbf{A}_4\,{}^4\mathbf{A}_5\,{}^5\mathbf{A}_6
$$

$$
= \begin{bmatrix}
C_2(C_1x_x + S_1x_y) + S_2x_z & C_2(C_1y_x + S_1y_y) + S_2y_z \\
-S_2(C_1x_x + S_1x_y) + C_2x_z & -S_2(C_1y_x + S_1y_y) + C_2y_z \\
S_1x_x - C_1x_y & S_1y_x - C_1y_y \\
0 & 0
\end{bmatrix.
$$

$$
\left. \begin{matrix}
C_2(C_1z_x + S_1z_y) + S_2z_z & C_2(C_1p_x + S_1p_y) + S_2(p_z - d_1) \\
-S_2(C_1z_x + S_1z_y) + C_2z_z & -S_2(C_1p_x + S_1p_y) + C_2(p_z - d_1) \\
S_1z_x - C_1z_y & S_1p_x - C_1p_y - d_2 \\
0 & 1
\end{matrix} \right]
$$

$$= \begin{bmatrix} C_3C_4C_5C_6 - C_3S_4S_6 - S_3S_5C_6 & -C_3C_4C_5S_6 - C_3S_4C_6 + S_3S_5S_6 & -C_3C_4S_5 - S_3C_5 & -C_3d_6C_4S_5 - S_3(d_6C_5 + d_4) \\ S_3C_4C_5C_6 - S_3S_4S_6 + C_3S_5C_6 & -S_3C_4C_5S_6 - S_3S_4S_6 - C_3S_5S_6 & -S_3C_4S_5 + C_3C_5 & -S_3d_6C_4S_5 + C_3(d_6C_5 + d_4) \\ -S_4C_5C_6 - C_4S_6 & S_4C_5S_6 - C_4C_6 & S_4S_5 & d_6S_4S_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(4.117)}$$

$$C_2(C_1p_x + S_1p_y) + S_2(p_z - d_1) = -C_3d_6C_4S_5 - S_3(d_6C_5 + d_4) \quad \textbf{(4.118)}$$

$$-S_2(C_1p_x + S_1p_y) + C_2(p_z - d_1) = -S_3d_6C_4S_5 + C_3(d_6C_5 + d_4) \quad \textbf{(4.119)}$$

$$\theta_2 = \sin^{-1}\left(\frac{(p_z - d_1)[-C_3d_6C_4S_5 - S_3(d_6C_5 + d_4)] + [S_3d_6C_4S_5 + C_3(d_6C_5 + d_4)](C_1p_x + S_1p_y)}{[(C_1p_x + S_1p_y)^2 + (p_z - d_1)]^2}\right) \quad \textbf{(4.120)}$$

Two equations involving θ_2 and θ_3, have been obtained with no obvious way of solving for either angle in terms of the atan2 function. A solution for θ_2 is calculated using the inverse sine function, and a solution for θ_3 can be found in a similar way, but it is not independent of θ_2. If we premultiply the transformation equation again, we can solve for θ_6:

$$^2A_3^{-1}\,{}^1A_2^{-1}\,{}^0A_1^{-1}\,{}^RT_H = {}^3A_4\,{}^4A_5\,{}^5A_6 \quad \textbf{(4.121)}$$

Equating elements (3, 1) and (3, 2) gets:

$$\theta_6 = \tan^{-1}\left(\frac{S_3[C_2(C_1y_x + S_1y_y) + S_2y_z] - C_3(S_1y_x - C_1y_y)}{-S_3[C_2(C_1x_x + S_1x_y) + S_2x_z] + C_3(S_1x_x - C_1x_y)}\right) \quad \textbf{(4.122)}$$

Premultiplying the transformation again, to solve for θ_4 and θ_5:

$$^3A_4^{-1}\,{}^2A_3^{-1}\,{}^1A_2^{-1}\,{}^0A_1^{-1}\,{}^RT_H = {}^4A_5\,{}^5A_6 \quad \textbf{(4.123)}$$

Equating elements (1, 4) and (2, 4) to get:

$$\theta_5 = \tan^{-1}\left(\frac{-C_4[C_3(C_2(C_1p_x + S_1p_y) + S_2(p_z - d_1)) + S_3(-S_2(C_1p_x + S_1p_y) + C_2(p_z - d_1))] + S_4[S_1p_x - C_1p_y - d_2]}{S_3[C_2(C_1p_x + S_1p_y) + S_2(p_z - d_1)] - C_3[-S_2(C_1p_x + S_1p_y) + C_2(p_z - d_1)] - d_4}\right)$$

(4.124)

By equating element (3, 4) we get:

$$\theta_4 = \tan^{-1}\left(\frac{S_1p_x - C_1p_y - d_2}{\begin{array}{c}C_3[C_2(C_1p_x + S_1p_y) + S_2(p_z - d_1)] \\ + S_3[-S_2(C_1p_x + S_1p_y) + C_2(p_z - d_1)]\end{array}}\right) \tag{4.125}$$

Equations have now been found for all the joint angles, but they are of little use because all of the equations involve other joint angles. No equations have been found for any joint angle in terms of the Cartesian space description only. To find values for the joint coordinates an iterative algorithm must be used (Window 4.3). Discussion of such algorithms is beyond the scope of this book. The unsolvability of the manipulator is due to its geometric design, where a deliberate attempt was made to mimic a human. There are three places where redundant configurations can occur: the shoulder can be a left or a right shoulder, the elbow can be up or down, and the wrist pitch can be up or down (see Figure 2.23). Thus, the manipulator has eight solutions, but only six joint variables. Also the wrist configuration has a degeneracy of the type shown in Figure 4.14.

Paul *et al.* (1981) simplify the equations by shifting the first and last displacements (d_1 and d_6) out of the manipulator transform into the transforms on either side. There is some justification for doing this with d_1, because the Puma controller locates the origin of the world coordinate system at the point where the shoulder and waist axes intersect.

$$p_x = C_1(S_{23}d_4 + l_2C_2) + S_1d_2 \tag{4.126}$$

$$p_y = S_1(S_{23}d_4 + l_2C_2) - C_1d_2 \tag{4.127}$$

$$p_z = -C_{23}d_4 + l_2S_2 \tag{4.128}$$

Lee (1983) calculates an inverse solution for this manipulator by dividing it into two 3 link manipulators, and solving for them separately. First, he solves a transform form the base to the wrist (end of link 3), given a vector from the wrist to the hand, to find equations for the first three joint angles, and then he uses this solution in conjunction with the transform from the wrist to the hand. His solution also handles the decisions which have to be made between different configurations. The details of this method of inverse kinematics are beyond the scope of this book. Manseur and Doty (1988) obtain a reduced set of equations by choosing the same frames as Paul *et al.* (1981) and applying rotational orthogonality. A manipulator is termed 'orthogonal' when all twist angles are 0° or 90°. They decompose the forward transform into a position vector and a rotation transform. Using the fact that dot products are invariant under rotation transforms, because rotation transforms are orthogonal, they obtain a set of four inverse equations. This method is a simplification of previous work by Tsai and Morgan (1984).

Pieper (1968) shows that one advantage of a spherical wrist (a wrist design where the three axes intersect at a point) is that it has a closed form solution. However, the above solutions only handle the case of a perfectly manufactured robot. As soon as a signature for the particular robot is used, the errors in the kinematics significantly complicate the model, again requiring the use of iterative solutions.