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DYNAMICS OF INDUSTRIAL ROBOT MANIPULATORS

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Abstract—The paper presents formulas connecting driving forces of actuators with natural coordinates describing motion dynamics of manipulators. The formulas are useful for any drive of manipulators, i.e. for actuators installed in axes of kinematic pair joining links and beyond them.

NOMENCLATURE

$\mathbb{A}_i$—homogeneous transformation describing the relation $i-1$st link and the ith link
C_i—coefficient [see system equations (20a–20b)]
D_i—coefficient represents the effect of gravity forces on the ith link
D_{ij}—coefficient represents the effect of the jth link inertia forces on the ith link
D_{ijk}—coefficient represents the effect of Coriolis forces, resulting from relative motion between jth and kth links, or, centripetal forces (for $j = k$), on the ith link
δ_{pk}—Kronecker delta
E_p—potential energy
F_i—ith link reactive force
F_{aid}—driving force generated by the ith actuator
F_{aifm}—ith actuator's Coulomb motion friction
F_{aifs}—ith actuator's Coulomb starting friction
$\mathbf{g}$—homogeneous form of gravitation vector $\mathbf{g}$
$\mathbb{J}_i$—ith link pseudoinertia matrix
$\mathbb{J}_{ai}$—ith actuator effector pseudoinertia matrix
$\mathbb{J}_{ij}$—pseudoinertia matrix of jth element of ith group transmitting drive from the ith actuator to the ith link
L—Lagrangian function
m_i—mass of the ith link
m_{ai}—mass of the ith actuators
m_{ij}—mass of jth element of ith drive unit transmitting drive from the ith actuator to the ith link
N—number of natural link coordinates
N_i—number of elements of drive unit transmitting drive from ith actuator to ith link
q_i—ith link natural coordinates
q_{ai}—ith actuator natural coordinates
$\bar{\mathbf{r}}_{ai}$—homogeneous form of vector describing gravity centres of the ith actuator effector in $x_{ai}y_{ai}z_{ai}$ coordinate system
$\bar{\mathbf{r}}_i$—homogeneous form of vector describing gravity centres of the ith link in $x_iy_iz_i$ coordinate system
$\bar{\mathbf{r}}_{ijc}$—homogeneous form of vector describing gravity centres of the ith element of the ith drive unit
$\mathbb{T}_i$—homogeneous transformation describing the relation between ith link and base link
$\mathbb{T}_{ij}$—homogeneous transformation describing the relation between jth element of ith drive unit and link
$\mathbb{T}_{ai}$—homogeneous transformation describing the relation between ith actuator effector and ith link
$\mathbb{T}_{0il}$—homogeneous transformation describing the relation between lth element of ith drive unit and base link
$x_{ai}y_{ai}z_{ai}$ —coordinate system associated with ith actuator effector
$x_iy_iz_i$—coordinate system associated with ith link
$x_{ij}y_{ij}z_{ij}$—coordinate system associated with jth element of ith drive unit
ΔD_i—change D_i
ΔD_{ij}—change D_{ij}
ΔD_{ijk}—change D_{ijk}
Δq_i—change q_i
Δt—change t

1. INTRODUCTION

In further considerations an IRM abbreviation will stand for manipulators of industrial robots. In the present paper, IRM will be considered in the form of kinematic series chains [1, 2]. The initial link is an IRM base, whereas the final link is a link to which a manipulator effector is fixed. An IRM effector may be a task, jaws clasping a welding electrode, and the like. Industrial robots are most often equipped with manipulators with V class kinematic pairs. The present paper will therefore deal solely with kinematic pairs of this type.

Right-handed rectangular coordinate systems associated with particular IRM elements will be used to describe the IRM kinematics. To describe a position and orientation of the systems, homogeneous transformations will be used [3–6]. These transformations afford a possibility for a joint description of the position and orientation, which is an inestimable advantage in describing kinematics, and, especially dynamics of IRM.

A number of the V class kinematic pairs connecting links, or, a number of the natural coordintes of links will be called a number of IRM degrees of freedom. The coordinates describing a relative motion of links will be called link natural coordinates [6, 7]. Auxiliary kinematic pairs of IRM effector, making its functioning possible (e.g. movement of task fingers) will be ignored.

One of the major problems in industrial robots control is to design algorithms planning optimal reference trajectories and servos which would be able to meet relevant requirements [7].

Effectiveness of trajectory planning algorithms which optimize quality coefficients, corresponding with IRM motion dynamics, depends largely on the accuracy of the dynamics models. Particularly important is the accuracy of models of IRM motion dynamics for algorithms simulating the work of robot control systems.

By accurate model of motion dynamics we mean a model respecting principal physics laws. These laws are described by Newton equations [1, 3, 8–10], or Lagrange equations [3–5, 8–12].

With Newton equations, forces and torques of IRM elements interaction can be determined. The interactions render a possibility for determining stresses and strains of IRM elements. However, the equations do not allow to determine a general analytic formula for an effective inertia of actuators [13]—a significant parameter which makes possible designing and selecting of servo controller settings.

Lagrange equations allow to determine driving forces and torques which should be applied in kinematic pairs of links. A significant advantage of Lagrange equations is that general formulas, connecting driving forces with the first and second derivatives of IRM natural coordinates, can be derived from them. Due to this, a description of IRM as a controlled plant is possible. It is done by means of state equations, thus enabling control system synthesis [14, 15]. Another important advantage of these equations is that general analytic formulas can be easily derived from them for effective inertia of actuators [13].

A general model of IRM motion dynamics, resulting from Newton equations, has been presented in the works [1, 3, 16, 17]. However, only interaction of IRM neighbouring links has been taken into account. Therefore these equations are useful for IRM with drive in the axes of kinematic joints, with the gravitation being taken into account. A general model of IRM motion dynamics, resulting from Lagrange equations, can be found in the handbook [4]. This model, however, is useful only for IRM with drive in the axes of kinematic pairs of links. Due to the model it is possible to derive correct analytic formulas determining effective inertia of the actuators, installed in the axes of kinematic pairs joining links. In the papers [11, 12] all factories connected with actuators move are omitted.

The dynamics models presented in the papers [1, 4, 5, 16, 17] are not useful for the IRM with drives installed beyond the axes of kinematic pairs of links. This means the models cannot be used to design algorithms for minitime and mini-cost trajectories planning, or, for simulating of servos work in such IRM IRb-6 and IRb-60 robots are equipped with the manipulators of this type. Choice of settings of controllers for manipulators is possible due to effective inertia of actuators. A description of IRb-6 and IRb-60 manipulators motion dynamics—as used in Poland so far—has been based on simplified models, in which the two last links are ignored. Their masses have been included in the mass of the 3rd link. The simplifications being introduced, the IRb-6 manipulator motion dynamics have been described by coefficients without a physical interpretation, which had been determined through identification research [18, 19].

The present author has worked out models of kinematics and dynamics [6] of IRM with actuators installed beyond and in the axes of pairs of links. Let us assume that all the IRM elements are rigid.

In the second section, Lagrange equations rendering possibilities for determining of driving forces of IRM actuators will be presented. The third section will contain results of exemplary computing, illustrating changes of natural coordinates of actuators caused by known currents of the IRb-6 actuators. The fourth section will contain conclusions.

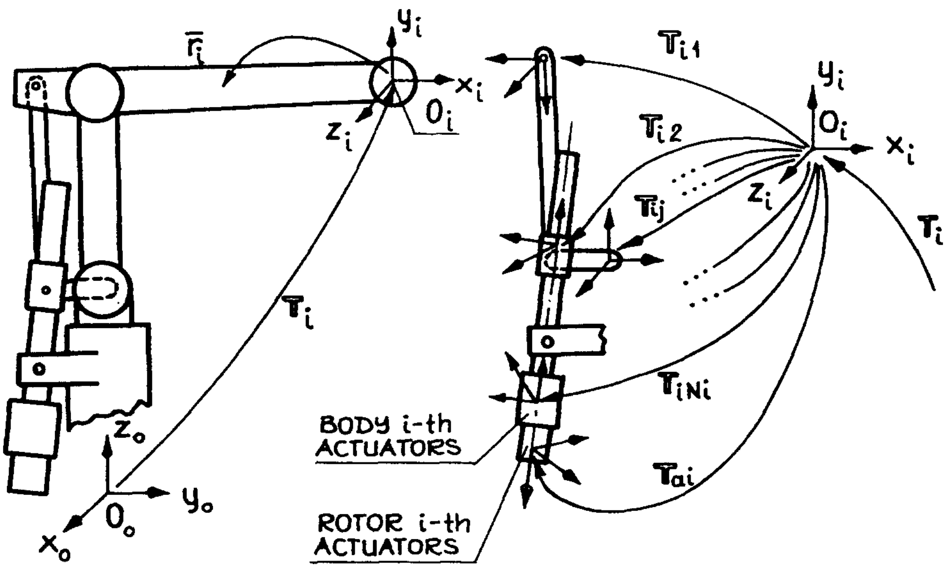

Fig. 1. Driving unit of the IRM ith link.

2. LAGRANGE EQUATION FOR IRM

The basic work task of IRM actuators [6] is generating such forces or torques in kinematic pairs which can ensure a manipulation object's motion along a reference trajectory with reference kinematics and dynamics. The forces or torques generated by actuators will be further called driving forces of actuators. The corresponding forces affecting links will be called driving forces of links.

Formulas will be derived for driving forces as functions of q_i link natural coordinates and their derivatives $\dot{q}_i$ and $\ddot{q}_i$.

To determine driving forces of links, methods of analytic mechanics, with Lagrange equations [8, 9], will be used. We will consider IRM with:

(a) actuators installed beyond pairs of links;
(b) actuators installed within pairs of links.

To calculate driving forces of links by means of Lagrange method, one must know L Lagrangian function. First, IRM with actuators installed beyond the link pairs will be considered. Lagrangian function is the difference between E_k total kinetic energy and E_p total potential energy of IRM. These energies are the sums of corresponding E_{ki} and E_{pi} energies of IRM links. E_{ki} and E_{pi} energies are, in turn, sums of corresponding energies: E_{lki} and E_{lpi} of the ith link along with the elements fixed and immobile towards it, E_{laki} and E_{lapi} of a drive unit of elements transmitting drive from the ith actuator to the ith link, E_{aki} and E_{api} of the ith actuator [6]. Figures 1 and 2 show a

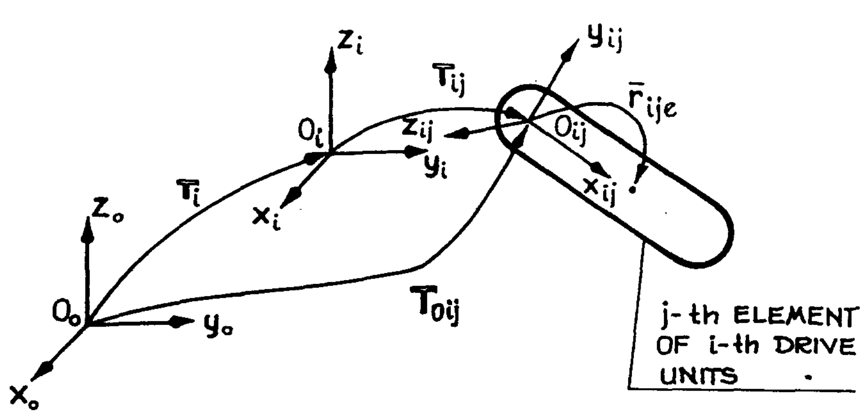

Fig. 2. Description of the jth element of a drive unit transmitting drive of the ith degree of IRM freedom.

description of links, driving groups and actuators, used in computing of Lagrangian function L. It follows from [6], that:

$$L = E_k - E_p = \frac{1}{2}\sum_{i=1}^{N}\left\{\mathrm{Trace}\left(\frac{d\mathbb{T}_i}{dt}\mathbb{J}_i\frac{d\mathbb{T}_i^T}{dt}\right) + \sum_{j=1}^{N}\left[\mathrm{Trace}\left(\frac{d\mathbb{T}_i}{dt}\mathbb{T}_{ij}\mathbb{J}_{ij}\mathbb{T}_{ij}^T\frac{d\mathbb{T}_i^T}{dt}\right)\right.\right.$$

$$\left. + 2\,\mathrm{Trace}\left(\frac{d\mathbb{T}_i}{dt}\mathbb{T}_{ij}\mathbb{J}_{ij}\frac{d\mathbb{T}_{ij}^T}{dt}\mathbb{T}_i^T\right) + \mathrm{Trace}\left(\frac{d\mathbb{T}_{ij}}{dt}\mathbb{J}_{ij}\frac{d\mathbb{T}_{ij}^T}{dt}\right)\right]$$

$$+ \mathrm{Trace}\left(\frac{d\mathbb{T}_i}{dt}\mathbb{T}_{ai}\mathbb{J}_{ai}\mathbb{T}_{ai}^T\frac{d\mathbb{T}_i^T}{dt}\right) + 2\,\mathrm{Trace}\left(\frac{d\mathbb{T}_i}{dt}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{d\mathbb{T}_{ai}^T}{dt}\mathbb{T}_i^T\right)$$

$$\left. + \mathrm{Trace}\left(\frac{d\mathbb{T}_{ai}}{dt}\mathbb{J}_{ai}\frac{d\mathbb{T}_{ai}^T}{dt}\right)\right\} + \mathbf{g}^T\sum_{i=1}^{N}\mathbb{T}_i\left(m_i\bar{\mathbf{r}}_i + \sum_{j=1}^{N_i} m_{ij}\mathbb{T}_{ij}\bar{\mathbf{r}}_{ije} + m_{ai}\mathbb{T}_{ai}\bar{\mathbf{r}}_{ai}\right). \quad (1)$$

N_i is a number of elements of drive units transmitting drive from the ith actuator to the ith link. When an actuator of the kth link is installed in the axis of kinematic pair, then $N_k = 0$, and the sum after j for $i = k$ in formula (1) is being ignored.

$\bar{\mathbf{r}}_{ai}$ is a homogeneous form of vector describing gravity centres of the ith actuator effector in $x_{ai}y_{ai}z_{ai}$ coordinate system (associated with ith actuator effector). $\bar{\mathbf{r}}_i$ and $\bar{\mathbf{r}}_{ije}$ are homogeneous forms of vector describing gravity centres of the elements shown in Figs 1 and 2. The $\mathbf{g}$ appearing in formula (1) is a homogeneous form of a gravitation vector $\mathbf{g}$ [4, 6] shown in Fig. 2. $\mathbb{T}_i$, $\mathbb{T}_{ij}$, $\mathbb{T}_{ai}$ matrices describe reciprocal position and orientation coordinates systems associated as in Figs 1 and 2.

In IRM, the mass of elements driving units transmitting drive is usually small, as compared to the mass of links. Actuator frames are usually fixed and immobile towards any links. These elements and the actuator frame velocities can be compared to the velocities of links. Velocities of the actuator effectors are usually very high as compared to the velocity of links. The element velocity to link velocity ratio corresponds to the transmission ratio. The transmission ratio is usually high, and, for planetary gear, for instance, it may reach up to 10^5 [7]. It follows from the above that in further considerations the elements transmitting drive and actuator frames energy may be ignored. Due to the reductions we obtain:

$$L = \sum_{i=1}^{N}\left[\frac{1}{2}\mathrm{Trace}\left(\frac{d\mathbb{T}_i}{dt}\mathbb{J}_i\frac{d\mathbb{T}_i^T}{dt}\right) + \mathrm{Trace}\left(\frac{d\mathbb{T}_i}{dt}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{d\mathbb{T}_{ai}^T}{dt}\mathbb{T}_i^T\right)\right.$$

$$\left. + \frac{1}{2}\mathrm{Trace}\left(\frac{\mathbb{T}_{ai}}{dt}\mathbb{J}_{ai}\frac{d\mathbb{T}_{ai}^T}{dt}\right) + m_i\mathbf{g}^T\mathbb{T}_i\bar{\mathbf{r}}_i\right]. \quad (2)$$

Differentiation of time is described in the following formulae:

$$\frac{d\mathbb{T}_i}{dt} = \sum_{j=1}^{i}\frac{d\mathbb{T}_i}{dq_j}\dot{q}_j, \quad \frac{d\mathbb{T}_{ai}}{dt} = \sum_{i=1}^{N}\frac{d\mathbb{T}_{ai}^T}{dq_j}\dot{q}_j. \quad (3)$$

The $\mathbb{T}_i$ derivative depends on natural coordinates of q_1–q_i links; it is a product of $\mathbb{A}_1$–$\mathbb{A}_i$ matrices describing reciprocal position and orientation of neighbouring links. As it follows from the paper [4, 6], each $\mathbb{A}_i$ matrix is dependent only on a correspondent natural coordinate of q_i link. $\mathbb{T}_{ai}$ coordinate, generally, depends on all natural coordinates of q_1–q_N links.

On taking into account the formulae (3), we obtain:

$$L = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_j}\mathbb{J}_i\frac{\partial\mathbb{T}_i^T}{\partial q_k}\right)\dot{q}_j\dot{q}_k + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_j}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^T}{\partial q_k}\mathbb{T}_i^T\right)\dot{q}_j\dot{q}_k$$

$$+ \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_{ai}}{\partial q_j}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^T}{\partial q_k}\right)\dot{q}_j\dot{q}_k + \sum_{i=1}^{N} m_i\mathbf{g}^T\mathbb{T}_i\bar{\mathbf{r}}_i. \quad (4)$$

$\mathbb{J}_i$ is the ith link pseudoinertia matrix, calculated in the $x_i y_i z_i$ coordinate system as follows:

$$\mathbb{J}_i = \int_0^{m_i} r_i r_i^{\mathrm{T}} \, \mathrm{d}m = \begin{bmatrix} I_{ixx} & I_{iyx} & I_{izx} & m_i \bar{x}_i \\ I_{ixy} & I_{iyy} & I_{izy} & m_i \bar{y}_i \\ I_{ixz} & I_{iyz} & I_{izz} & m_i \bar{z}_i \\ m_i \bar{x}_i & m_i \bar{y}_i & m_i \bar{z}_i & m_i \end{bmatrix},$$

$$I_{iuv} = \int_0^{m_i} uv \, \mathrm{d}m, \quad m_i \bar{u}_i = \int_0^{m_i} u \, \mathrm{d}m; \quad u, v = x_i, y_i, z_i. \tag{5}$$

m_i is a mass of the ith link together with all the elements fixed to it. $\mathbb{J}_{ij}$ is a pseudoinertia matrix of the jth element of m_{ij} mass, calculated in the $x_{ij} y_{ij} z_{ij}$ system. $\mathbb{J}_{ai}$ is a pseudoinertia matrix of the ith actuator effector of m_{ai} mass, calculated in the $x_{ai} y_{ai} z_{ai}$ coordinate system.

Reaction forces of links are derived from Lagrange equations [8–10]:

$$F_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}. \tag{6}$$

F_i are either torques (for rotary joints) or forces (for travelling joints), directed along z_{i-1} axis. In the Appendix, formulae for F_i have been determined.

$$F_i = \sum_{j=1}^{N} D_{ij} \ddot{q}_j + \sum_{j=1}^{N} \sum_{k=1}^{N} D_{ijk} \dot{q}_j \dot{q}_k + D_i. \tag{7}$$

where:

$$D_{ij} = \sum_{p=\max(i,j)}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_i} \mathbb{J}_p \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_j}\right) + \sum_{p=1}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_{ap}}{\partial q_i} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_j}\right). \tag{8}$$

$$\begin{aligned} D_{ijk} = {} & \sum_{p=\max(i,j,k)}^{N} \mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_p}{\partial q_j \partial q_k} \mathbb{J}_p \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_i}\right) + \sum_{p=i}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_i} \frac{\partial \mathbb{T}_{ap}}{\partial q_k} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_j} \mathbb{T}_p^{\mathrm{T}}\right) \\ & + \sum_{p=j}^{N} \left[\mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \frac{\partial \mathbb{T}_{ap}}{\partial q_k} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_i} \mathbb{T}_p^{\mathrm{T}}\right) - \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \frac{\partial \mathbb{T}_{ap}}{\partial q_i} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_k} \mathbb{T}_p^{\mathrm{T}}\right)\right], \end{aligned} \tag{9}$$

$$D_i = -\sum_{p=i}^{N} m_p \mathbf{g}^{\mathrm{T}} \frac{\partial \mathbb{T}_p}{\partial q_i} \bar{\mathbf{r}}_p. \tag{10}$$

These formulae are reduced for the actuators installed in kinematic pairs so, that $\partial \mathbb{T}_{ap}/\partial q_k = (\partial \mathbb{T}_{ap}/\partial q_k)\delta_{pk}$. Then D_{ij} and D_{ijk} coefficients have the following forms:

$$D_{ij} = \sum_{p=\max(i,j)}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_i} \mathbb{J}_p \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_j}\right) + \delta_{ij} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_{ai}}{\partial q_i} \mathbb{J}_{ai} \frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_i}\right) \tag{11}$$

$$D_{ijk} = \sum_{p=\max(i,j,k)}^{N} \mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_p}{\partial q_j \partial q_k} \mathbb{J}_p \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_i}\right) + \delta_{jk} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_j}{\partial q_i} \frac{\partial \mathbb{T}_{aj}}{\partial q_j} \mathbb{J}_{aj} \frac{\partial \mathbb{T}_{aj}^{\mathrm{T}}}{\partial q_j} \mathbb{T}_j^{\mathrm{T}}\right), \tag{12}$$

where δ_{ij} and δ_{ik} are Kronecker delta [8].

D_i coefficients are, as formerly, described by the formula (10). The second addent in the formula (11), multiplied by δ_{ij}, is constant and equal Ia_i (actuator inertia [4]), and independent of natural coordinates. D_{ij} coefficients represent the effect of the jth link inertia forces on the ith link. D_{ijk} coefficients represent the effect of Coriolis forces, resulting from relative motion between jth and kth links or, centripetal forces (for $j = k$), on the ith link. D_i coefficients represent the effect of gravity forces on the ith link.

F_i link reactive forces are corresponded by F_{ai} actuator reactive forces. If an actuator natural coordinate, describing changes of coordinate system $x_{ai} y_{ai} z_{ai}$ (associated with an actuator effector) in relation to $x_{Ni} y_{Ni} z_{Ni}$ (associated with the actuator body Fig. 1) has been denoted by q_{ai}, then [6, 20]:

$$F_{ai} = \sum_{j=1}^{N} F_j \frac{\partial q_j}{\partial q_{ai}}. \tag{13}$$

For the actuator installed in the pairs so that $\partial q_j/\partial q_{ai} = (\partial q_j/\partial q_{ai})\delta_{ij}$,

$$F_{ai} = F_i \frac{\partial q_i}{\partial q_{ai}}. \tag{14}$$

$\partial q_{ai}/\partial q_j$ correspond to transmission ratios of high values. It follows from the formula (13) that F_{ai} forces are inversely proportional to $\partial q_{ai}/\partial q_j$ and directly proportional to F_j forces. The formula (7) describing F_j contains expressions with $\mathbb{J}_i$—representing forces which result from links motion and expressions with $\mathbb{J}_{ai}$—representing forces which result from the motion of actuator effectors. The expressions with $\mathbb{J}_{ai}$ contain $\partial \mathbb{T}_{ai}/\partial q_j$ derivatives proportional to transmission ratios. So, when the ratio increases, the effect of links motion and gravity forces on the F_{ai} force decreases. With the transmission ratios rise, the effect of forces resulting from actuator effectors motion on F_{ai} force may either be ratio independent (for expressions containing $\mathbb{J}_{ai}$ with only one derivative $\partial \mathbb{T}_{ai}/\partial q_j$), or, it may increase (for the expressions containing $\mathbb{J}_{ai}$, with two derivatives $\partial \mathbb{T}_{ai}/\partial q_j$). Due to high transmission ratio in IRM, the present consideration proves the formula (1) reductions to be useful.

Transmission ratios of high values minimize the effect of links together with a manipulation object motion on the actuators load. With the ratios of high values, in D_{ij} and D_{ijk} coefficients [described by the formulae (8)–(12)] the most significant are expressions containing $\mathbb{J}_{ai}$.

The handbook [4] contains equations of F_i forces, derived by R. Paul for the situation when all the actuators are installed in the axis of kinematic pairs joining appropriate links. However, in the formula describing D_{ijk} coefficients, an expression with $\mathbb{J}_{ai}$ was omitted. In the same handbook (p. 180, Tables 6.5 and 6.6) mechanical parameters for the Stanford robot were mentioned. It can be seen in Table 6.6 that, for the majority of freedom degrees, actuator effectors motion energy is by one order higher than the energy of links motion. For the fourth degree of freedom, the energy of actuator effector is as much as 100 times higher than the energy of the fourth link motion. The same inaccuracies appeared in the handbook [5].

If dynamic interaction of elements of drive unit transmitting are to be taken into consideration, then D_{ij}, D_{ijk} and D_i coefficients must be completed with the following corrections:

$$\Delta D_{ij} = \sum_{l=1}^{Ni} \text{Trace}\left(\frac{\partial \mathbb{T}_{\text{O}il}}{\partial q_i} \mathbb{J}_{il} \frac{\partial \mathbb{T}_{\text{O}il}^{\text{T}}}{\partial q_j}\right), \tag{15a}$$

$$\Delta D_{ijk} = \sum_{l=1}^{Ni} \text{Trace}\left(\frac{\partial^2 \mathbb{T}_{\text{O}il}}{\partial q_j \, \partial q_k} \mathbb{J}_{il} \frac{\partial \mathbb{T}_{\text{O}il}^{\text{T}}}{\partial q_i}\right), \tag{15b}$$

$$\Delta D_i = -\sum_{l=1}^{Ni} \mathbf{g}^{\text{T}} \frac{\partial \mathbb{T}_{\text{O}il}}{\partial q_i} m_{il} \bar{\mathbf{r}}_{ilc}. \tag{15c}$$

$\mathbb{T}_{\text{O}il}$ matrix describes a coordinate system associated with the lth element of a driving unit of the ith link towards the base coordinate system (see Fig. 2).

Change of coefficients of dynamics equations (7) is also caused by grasping of a manipulation object. This leads to a change of matrix of a working link pseudoinertia by $\Delta\mathbb{J}_N$ [6]. It implies the following changes of force equations coefficients [6]:

$$\Delta D_{ij} = \text{Trace}\left(\frac{\partial \mathbb{T}_N}{\partial q_i} \Delta\mathbb{J}_N \frac{\partial \mathbb{T}_N^{\text{T}}}{\partial q_j}\right), \tag{16a}$$

$$\Delta D_{ijk} = \text{Trace}\left(\frac{\partial^2 \mathbb{T}_N}{\partial q_j \, \partial q_k} \Delta\mathbb{J}_N \frac{\partial \mathbb{T}_N^{\text{T}}}{\partial q_i}\right), \tag{16b}$$

$$\Delta D_i = -m_P \mathbf{g}^{\text{T}} \frac{\partial \mathbb{T}_N}{\partial q_i} \bar{\mathbf{r}}_P. \tag{16c}$$

m_P and $\bar{\mathbf{r}}_P$ are respectively mass and homogeneous form of vector describing the centre of gravity of a manipulation object towards $x_N y_N z_N$ coordinate system.

In our previous considerations, friction was ignored. It is difficult to give an analytical description of energy dissipated while overcoming friction. Interaction of Coulomb friction forces will be considered in relation to IRM actuators effectors. F_{air} resultant driving force of the ith actuator is a difference of F_{aid} driving force generated by the actuator and Coulomb friction force.

$$F_{air} = \begin{cases} 0 \quad \text{for } \dot{q}_{ai} = 0 \text{ and } |F_{aid}| \leqslant F_{aifs}, \\ F_{aid} - F_{aifs}\,\mathrm{sgn}(F_{aid}) \text{ for } \dot{q}_{ai} = 0 \text{ and } |F_{aid}| > F_{aifs}, \\ F_{aid} - F_{aifm}\,\mathrm{sgn}(\dot{q}_{ai}) \text{ for } \dot{q}_{ai} \neq 0. \end{cases} \tag{17}$$

F_{aifs} and F_{aifm} are the ith actuator's Coulomb starting and motion friction respectively. These parameters may be experimentally estimated [4, 6]. If $\dot{q}_{ai} = 0$, the motion start ($\dot{q}_{ai} \neq 0$) depends on the relation between F_{aid} deriving force and F_{aifs} starting friction force. F_{air} forces are equal to F_{ai} reaction forces described in the formulae (13). Apart from Coulomb friction forces already taken

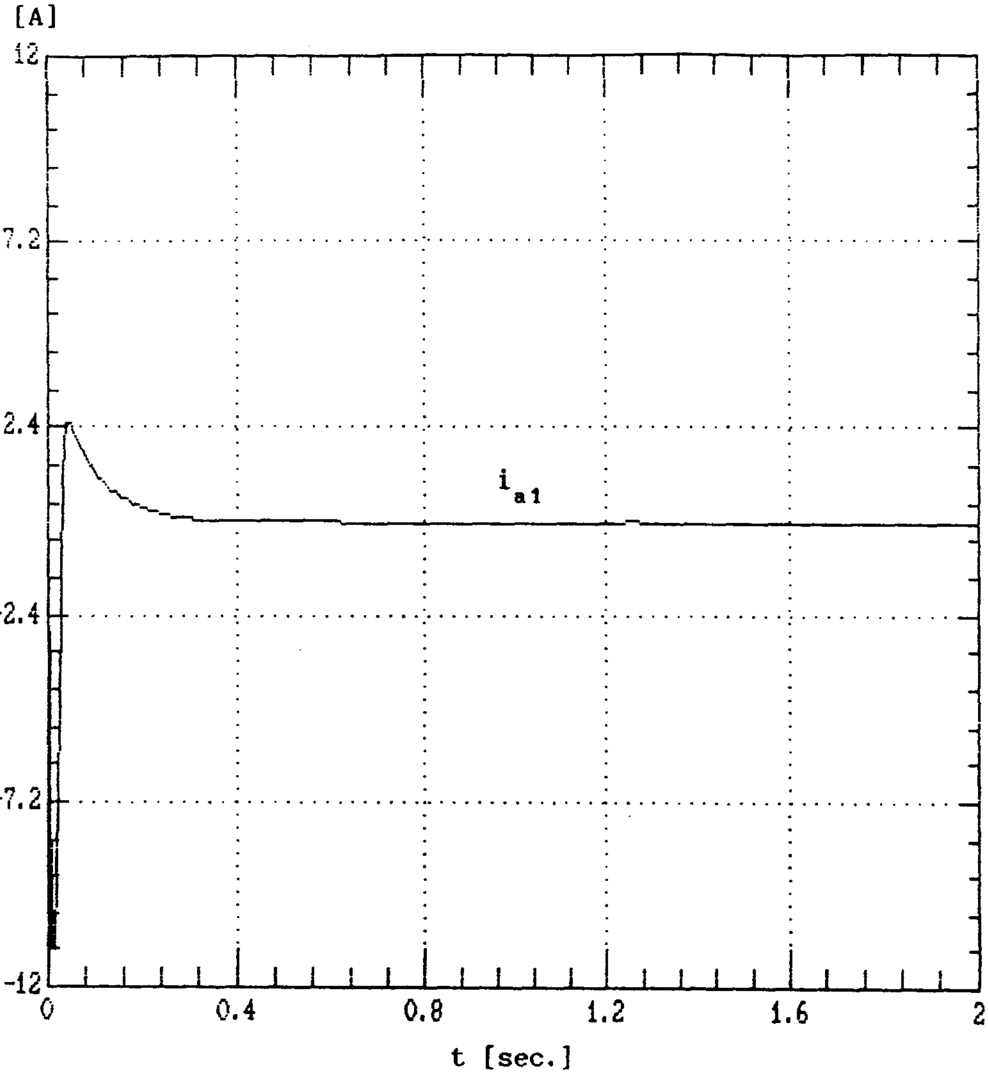

Fig. 3(a). $i_{a1}(t)$ armature current of first actuator.

into account, there is also viscous friction in IRM. Energy loss caused by this friction occurs, first of all, when noting elements move at high relative speed, e.g. in planetary gears. This kind of friction will be also considered in relation to actuator effectors. Therefore F_{aire} resultant described by formula (18) the F_{air} described in formula (17) decreased by viscous friction.

$$F_{aire} = F_{air} - k_{vai}\dot{q}_{ai}. \tag{18}$$

k_{vai} is a coefficient of transmission viscous friction. The $x_0 y_0 z_0$ base system associated with the IRM base has been assumed so far to be immobile, or, more precisely—inertial [21] or Galilean [22]. Only in such a coordinate system the formulas describing L Lagrangian function is useful. For IRM with a mobile base, the dynamics of its motion must be described in relation to another inertial (or Galilean) reference system. If a base system may be described in relation to inertial reference system by means of one coordinate q_0, then: an additional $\mathbb{T}_0 = \mathbb{A}_0$ matrix is created, $\mathbb{T}_{a0}$ and $\mathbb{J}_{a0}$ of an actuator driving the base is determined, IRM kinematics is modified ($\mathbb{T}_i = \mathbb{A}_0 \mathbb{A}_1 \mathbb{A}_2 \cdots \mathbb{A}_i$) and hitherto existing formulae can be used. Similarly, $\mathbb{T}_{\mathrm{Oil}}$ matrices in the formulas (15a–c) are modified. Summation indices will, naturally, change not from number one but from zero.

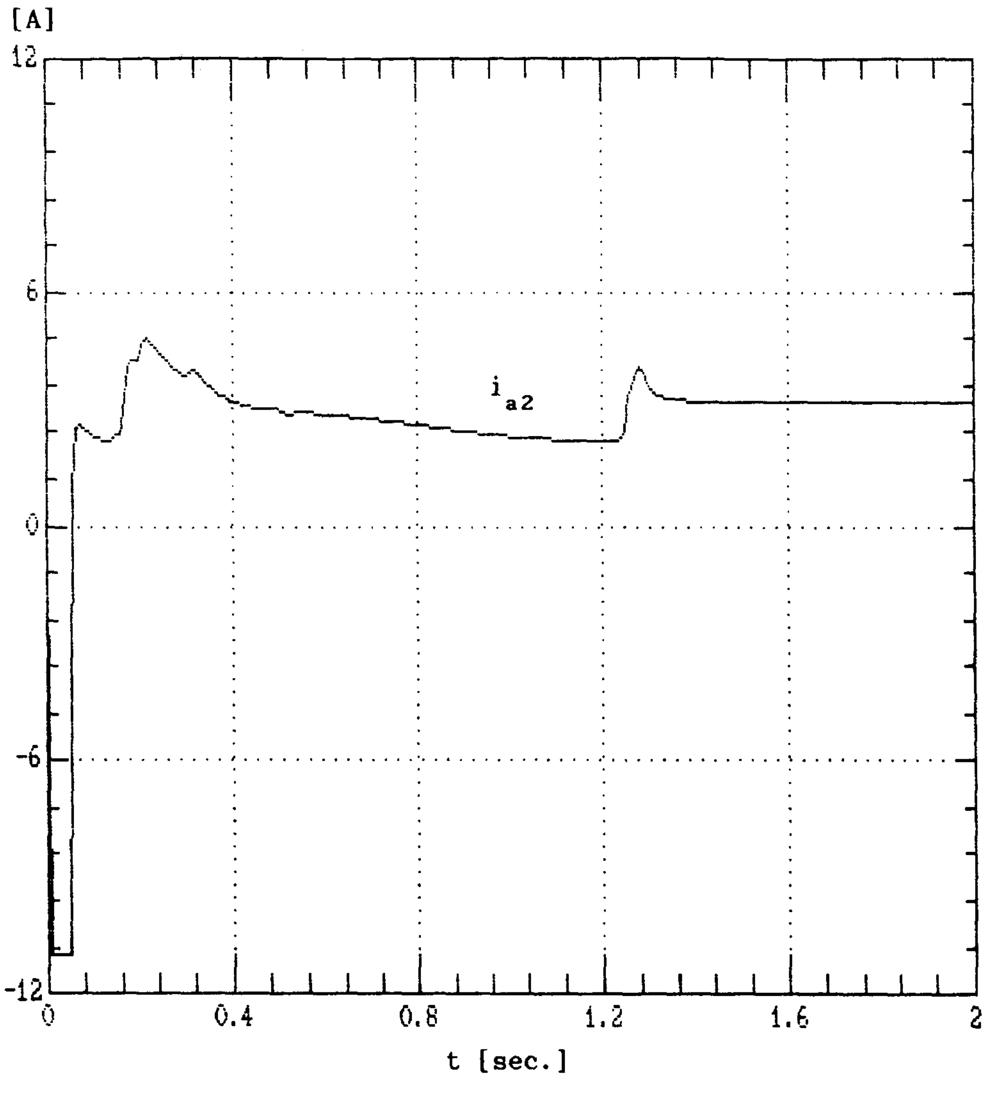

Fig. 3(b). $i_{a2}(t)$ armature current of second actuator.

3. EXAMPLE

Making use of the dynamics models described in the formulae (1)–(18), one can solve direct as well as inverse problems of IRM drives. An inverse problem of IRM drive dynamics consists in determining of F_{aid} driving forces of actuators for the known natural coordinates of the actuators. A direct problem of IRM drive dynamics consists in determining of actuator natural coordinates for the known F_{aid} forces.

The direct problem of IRM drive dynamics can be solved as follows:

(a) link driving forces are determined from the formula:

$$F_{id} = \sum_{j=1}^{N} \frac{\partial q_{ai}}{\partial q_j} F_{aire}; \tag{19}$$

(b) for reference initial values of natural coordinates and their first derivatives, link natural coordinates and their first derivatives, link natural coordinates are determined from the system differential equations (7);

(c) natural coordinates of actuators are determined.

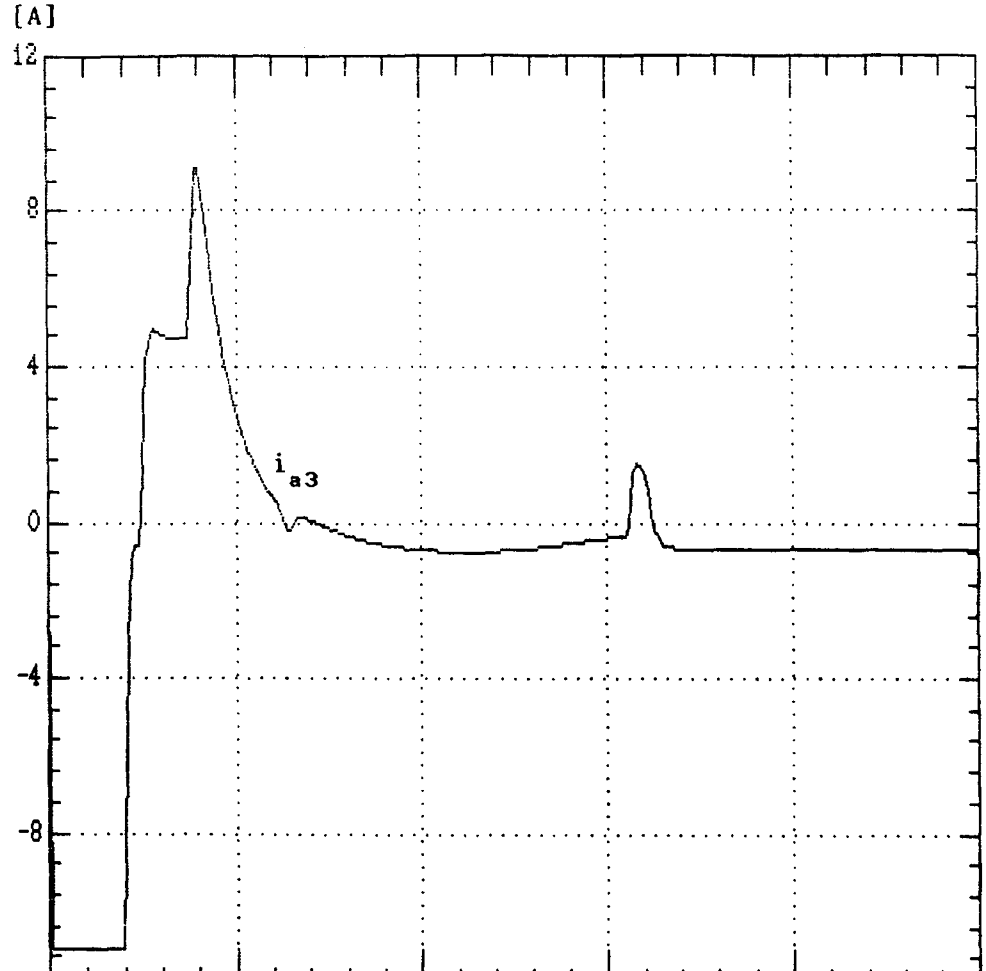

Fig. 3(c). $i_{a3}(t)$ armature current of third actuator.

The system of equations (7) is transformed into:

$$\sum_{j=1}^{N} D_{ij}\ddot{q}_j = C_i, \tag{20a}$$

$$C_i = F_{id} - \sum_{i=1}^{N}\sum_{k=1}^{N} D_{ijk}\dot{q}_j\dot{q}_k - D_i, \quad i = 1, 2, \ldots, N. \tag{20b}$$

For the known natural coordinates and for their first initial derivatives, D_{ijk} and D_i coefficients appearing in the equations (20b) can be determined, followed by D_{ij} and C_i coefficients which appear in the system of equations (20a). Solving the system equations of the second derivatives of link natural coordinates are obtained:

$$\ddot{q}_j = \ddot{q}_j(C_1, C_2, \ldots, C_N, D_1, D_2, \ldots, D_N, F_{1d}, F_{2d}, \ldots, F_{Nd}), \quad j = 1, 2, \ldots, N. \tag{21}$$

After the system of equations has been discretized (20a), natural coordinates and their first derivatives can be determined in the next discretization step, using the formulae

$$\Delta q_j = \dot{q}_j \Delta t, \quad \Delta \dot{q}_j = \ddot{q}_j \Delta t. \tag{22}$$

Δt is a time discretization step.

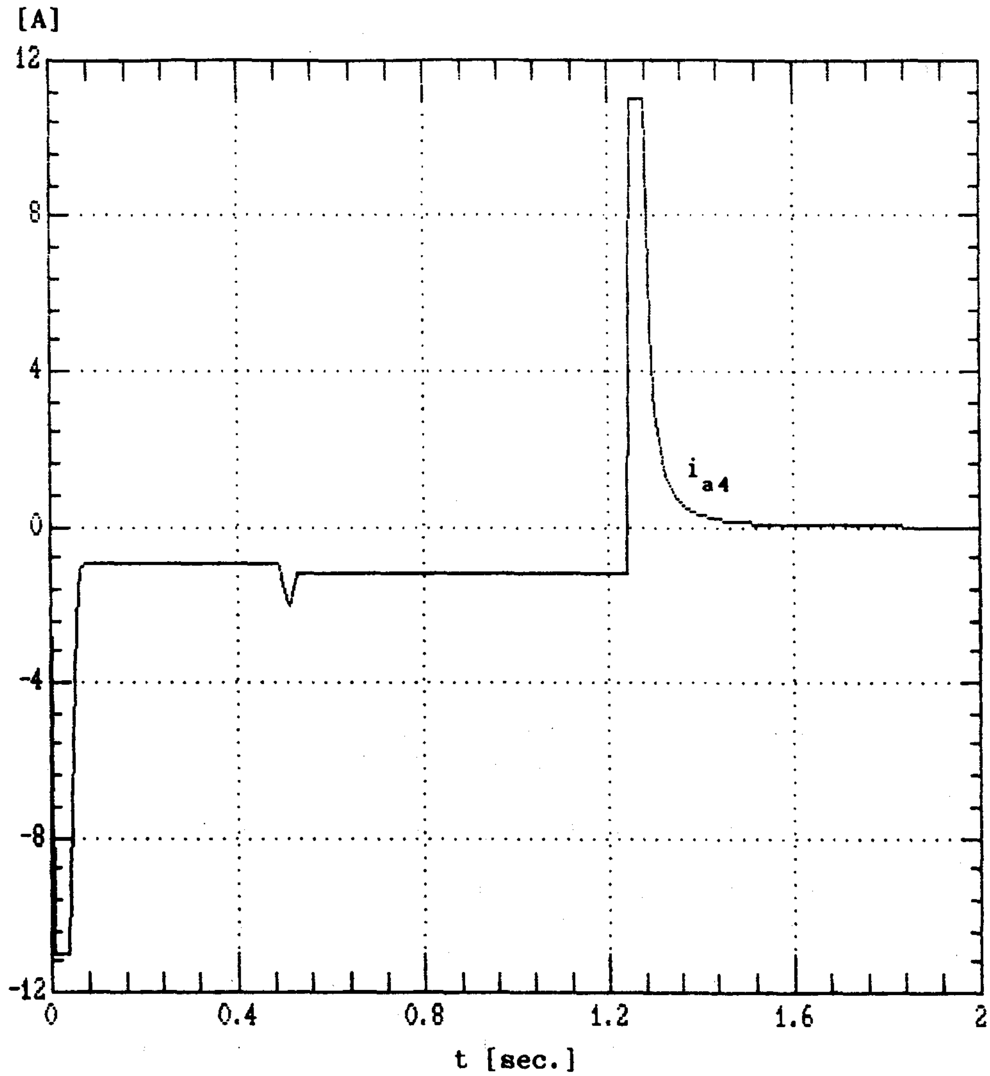

Fig. 3(d). $i_{a4}(t)$ armature current of fourth actuator.

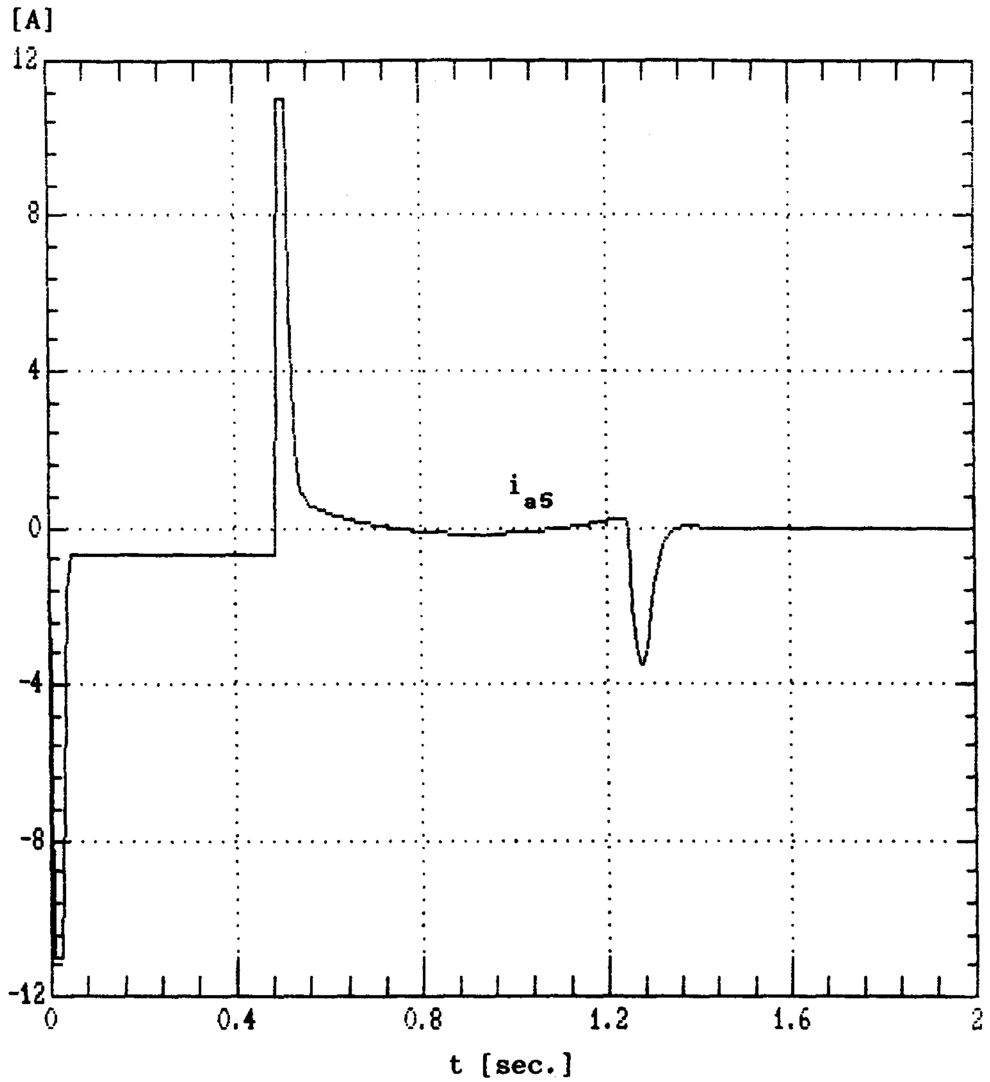

Fig. 3(e). $i_{a5}(t)$ armature current of fifth actuator.

The formulas (20)–(22) are the basis for computer simulating of servo controlled IRM motion [7]. They were used by the author to design the STER algorithm for computer simulation of IRM IRb-6.

In IRb-6 robot, d.c. motors generating driving force, which is a torque proportional to the armature current, are the robot's actuators.

Figure 3 shows exemplary currents of IRb-6 IRM motor armature. Use of STER algorithm allows to obtain natural coordinates of the actuators, shown in Fig. 4.

It follows from Fig. 4 that the armature currents from Fig. 3 have resulted in the static state of IRM IRb-6, where $q_{a1} = \cong -130.0$ rad, $q_{a2} \cong -66.7$ rad, $q_{a3} \cong -51.0$ rad, $q_{a4} \cong -199.0$ rad, $q_{a5} \cong -199$ rad. The state is maintained by currents $i_{a1} \cong 0$ A, $i_{a2} \cong 3.18$ A, $i_{a3} \cong -0.73$ A, $i_{a4} \cong 0$ A, $i_{a5} \cong 0$ A. Both Coulomb and viscous friction, described by formulae (17) and (18), have been taken into account in the computing. Time discretization step equalled 0.001 s.

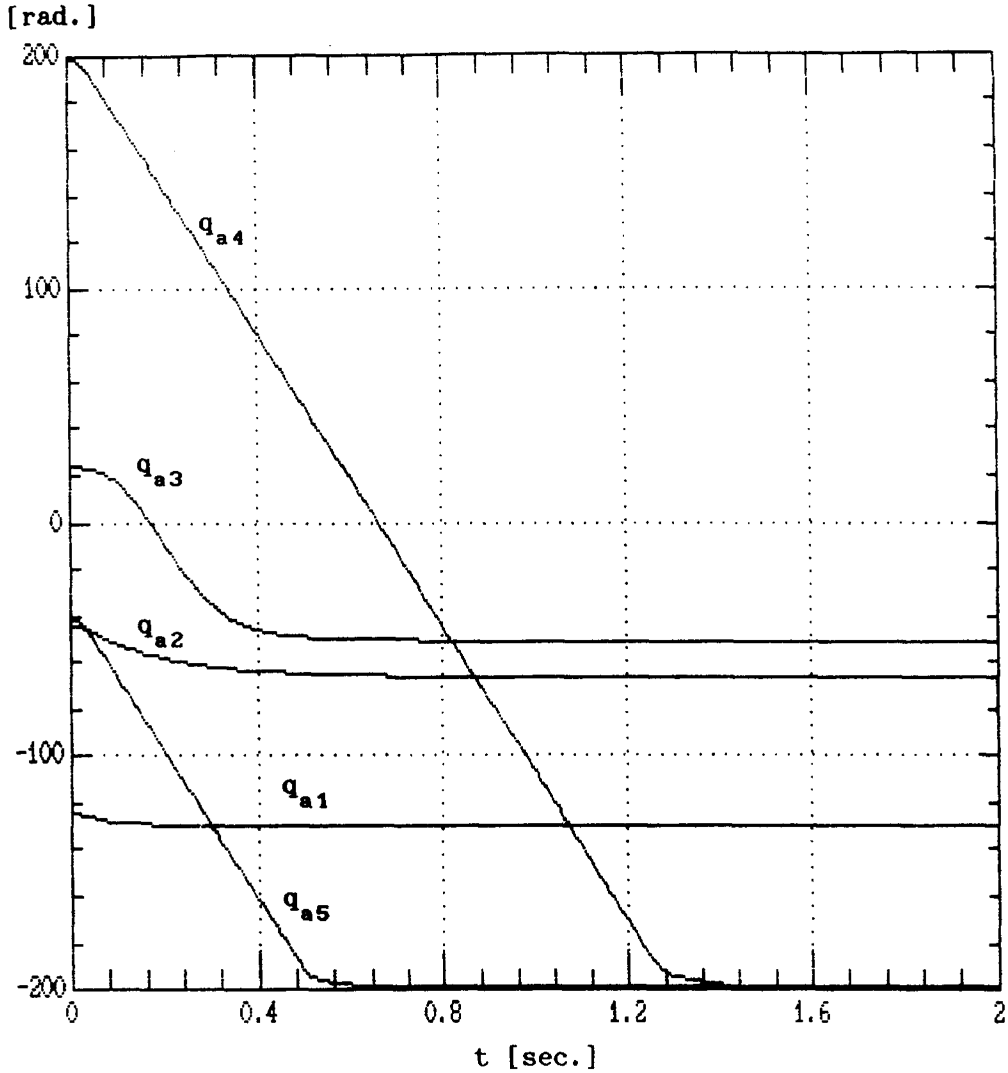

Fig. 4. $q_{a1}(t)$–$q_{a5}(t)$ natural coordinates of actuators.

4. CONCLUSION

The IRM models presented in the paper make it possible to solve direct and inverse problems of IRM dynamics, with viscous friction, starting and kinetic Coulomb friction, gravity forces, mass distribution of particular IRM links and actuator effectors being taken into account. The models are useful for IRM with drives placed beyond the axes of kinematic pairs joining links. Solution of these problems is a requisite of design motion computer simulation algorithms for IRM controlled by continuous and discrete servos of known structure. The present models of dynamics render it possible to design algorithms for computer analysis of changes of actuator effective inertia—a parameter due to which setting of servo-controllers can be determined [23, 24].

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APPENDIX

Computing of Lagrangian L [determined by formula (4)] derivatives, appearing in formula (6), and F_i reaction forces of links motion.

$$L=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_j}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)\dot q_j\dot q_k+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_j}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)\dot q_j\dot q_k$$
$$+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_{ai}}{\partial q_j}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\right)\dot q_j\dot q_k+\sum_{i=1}^{N}m_i\mathbf{g}^{\mathrm{T}}\mathbb{T}_i\bar{\mathbf{r}}_i.$$

Derivatives $\partial L/\partial\dot q_p$

$$\frac{\partial L}{\partial\dot q_p}=\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{T}_i\frac{\partial\mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)\dot q_k+\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_j}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm{T}}}{\partial q_p}\right)\dot q_j$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)\dot q_k+\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_j}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_p}\mathbb{T}_i^{\mathrm{T}}\right)\dot q_j$$
$$+\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_{ai}}{\partial q_p}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\right)\dot q_k+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_{ai}}{\partial q_j}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_p}\right)\dot q_j.$$

Because

$$\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_j}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm{T}}}{\partial q_p}\right)=\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_j}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm{T}}}{\partial q_p}\right)^{\mathrm{T}}=\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{J}_i^{\mathrm{T}}\frac{\partial\mathbb{T}_i^{\mathrm{T}}}{\partial q_j}\right)$$
$$=\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm{T}}}{\partial q_j}\right),$$ after j exchanged into k we obtain:

$$\frac{\partial L}{\partial\dot q_p}=\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)\dot q_k+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)\dot q_k$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial^{\mathrm{T}}_{ai}}{\partial q_p}\mathbb{T}_i^{\mathrm{T}}\right)\dot q_k+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_{ai}}{\partial q_p}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\right)\dot q_k$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot q_p}\right)=\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)\ddot q_k+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)\ddot q_k$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_p}\mathbb{T}_i^{\mathrm{T}}\right)\ddot q_k+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_{ai}}{\partial q_p}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\right)\ddot q_k$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial q_j}\left[\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)\right]\dot q_k\dot q_j+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}$$
$$\times\frac{\partial}{\partial q_j}\left[\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)\right]\dot q_k\dot q_j$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial q_j}\left[\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_p}\mathbb{T}_i^{\mathrm{T}}\right)\right]\dot q_k\dot q_j$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial q_j}\left[\mathrm{Trace}\left(\frac{\partial\mathbb{T}_{ai}}{\partial q_p}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\right)\right]\dot q_k\dot q_j.$$

$$\frac{\partial L}{\partial q_p}=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial}{\partial q_p}\left[\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_j}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)\right]\dot{q}_j\dot{q}_k+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial}{\partial q_p}\left[\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_j}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)\right]\dot{q}_j\dot{q}_k$$
$$+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial}{\partial q_p}\left[\mathrm{Trace}\left(\frac{\partial \mathbb{T}_{ai}}{\partial q_j}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\right)\right]\dot{q}_j\dot{q}_k+\sum_{i=1}^{N}m_i\mathbf{g}^{\mathrm{T}}\frac{\partial \mathbb{T}_i}{\partial q_p}\bar{\mathbf{r}}_i.$$

After j exchanged into k and k into j we obtain:

$$\frac{\partial L}{\partial q_p}=\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial q_p}\left[\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_j}\right)\right]\dot{q}_k\dot{q}_j+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial q_p}\left[\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_j}\mathbb{T}_i^{\mathrm{T}}\right)\right]\dot{q}_k\dot{q}_j$$
$$+\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial q_p}\left[\mathrm{Trace}\left(\frac{\partial \mathbb{T}_{ai}}{\partial q_k}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_j}\right)\right]\dot{q}_k\dot{q}_j+\sum_{i=1}^{N}m_i\mathbf{g}^{\mathrm{T}}\frac{\partial \mathbb{T}_i}{\partial q_p}\bar{\mathbf{r}}_i.$$

$$F_p=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_p}\right)-\frac{\partial L}{\partial q_p}=\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_p}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)\ddot{q}_k+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)\ddot{q}_k$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_p}\mathbb{T}_i^{\mathrm{T}}\right)\ddot{q}_k+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_{ai}}{\partial q_p}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\right)\ddot{q}_k$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\left[\frac{\partial}{\partial q_j}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_p}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)-\frac{1}{2}\frac{\partial}{\partial q_p}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_j}\right)\right]\dot{q}_k\dot{q}_j$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\left[\frac{\partial}{\partial q_j}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)-\frac{\partial}{\partial q_p}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_j}\mathbb{T}_i^{\mathrm{T}}\right)\right.$$
$$\left.+\frac{\partial}{\partial q_j}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_p}\mathbb{T}_i^{\mathrm{T}}\right)\right]\dot{q}_k\dot{q}_j+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\left[\frac{\partial}{\partial q_j}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_{ai}}{\partial q_p}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\right)\right.$$
$$\left.-\frac{1}{2}\frac{\partial}{\partial q_p}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_{ai}}{\partial q_k}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_j}\right)\right]\dot{q}_k\dot{q}_j-\sum_{i=1}^{N}m_i\mathbf{g}^{\mathrm{T}}\frac{\partial \mathbb{T}_i}{\partial q_p}\bar{\mathbf{r}}_i.$$

The expression will be reduced:

$$\sum_{k=1}^{N}\sum_{j=1}^{N}\left[\frac{\partial}{\partial q_j}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_p}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)-\frac{1}{2}\frac{\partial}{\partial q_p}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_j}\right)\right]$$
$$=\sum_{k=1}^{N}\sum_{j=1}^{N}\left[\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_j\,\partial q_p}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)+\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_p}\mathbb{J}_i\frac{\partial^2 \mathbb{T}_i^{\mathrm{T}}}{\partial q_k\,\partial q_j}\right)\right.$$
$$\left.-\frac{1}{2}\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_k\,\partial q_p}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_j}\right)-\frac{1}{2}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{J}_i\frac{\partial^2 \mathbb{T}_i^{\mathrm{T}}}{\partial q_p\,\partial q_j}\right)\right]$$
$$=\sum_{k=1}^{N}\sum_{j=1}^{N}\left[\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_j\,\partial q_p}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)+\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_j\,\partial q_k}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_p}\right)\right.$$
$$\left.-\frac{1}{2}\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_p\,\partial q_k}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_j}\right)-\frac{1}{2}\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_p\,\partial q_j}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)\right]$$
$$=\sum_{k=1}^{N}\sum_{j=1}^{N}\left[\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_j\,\partial q_p}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)+\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_j\,\partial q_k}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_p}\right)\right]$$
$$-\sum_{k=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_j\,\partial q_p}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_k}\right)=\sum_{k=1}^{i}\sum_{j=1}^{i}\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_j\,\partial q_k}\mathbb{J}_i\frac{\partial \mathbb{T}_i^{\mathrm{T}}}{\partial q_p}\right).$$

Similarly for the expressions in which $\mathbb{T}_i$ will be replaced by $\mathbb{T}_{ai}$

$$\sum_{k=1}^{N}\sum_{j=1}^{N}\left[\frac{\partial}{\partial q_j}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)-\frac{\partial}{\partial q_p}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_j}\mathbb{T}_i^{\mathrm{T}}\right)\right.$$
$$\left.+\frac{\partial}{\partial q_j}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_p}\mathbb{T}_i^{\mathrm{T}}\right)\right]=\sum_{k=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_j\,\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)$$
$$+\sum_{k=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left[\frac{\partial \mathbb{T}_i}{\partial q_p}\frac{\partial}{\partial q_j}\left(\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)\right]-\sum_{k=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_i}{\partial q_p\,\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_j}\mathbb{T}_i^{\mathrm{T}}\right)$$
$$-\sum_{k=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left[\frac{\partial \mathbb{T}_i}{\partial q_k}\frac{\partial}{\partial q_p}\left(\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_j}\mathbb{T}_i^{\mathrm{T}}\right)\right]+\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial q_j}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_p}\mathbb{T}_i^{\mathrm{T}}\right)$$
$$=\sum_{k=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left[\frac{\partial \mathbb{T}_i}{\partial q_p}\frac{\partial}{\partial q_j}\left(\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_k}\mathbb{T}_i^{\mathrm{T}}\right)\right]-\sum_{k=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left[\frac{\partial \mathbb{T}_i}{\partial q_k}\frac{\partial}{\partial q_p}\left(\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_j}\mathbb{T}_i^{\mathrm{T}}\right)\right]$$
$$+\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial q_j}\mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_p}\mathbb{T}_i^{\mathrm{T}}\right).$$

Consequently

$$F_p=\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm T}}{\partial q_k}\right)\ddot{q}_k+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_k}\mathbb{T}i\right)\ddot{q}_k$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_p}\mathbb{T}_i^{\mathrm T}\right)\ddot{q}_k+\sum_{i=1}^{N}\sum_{k=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_{ai}}{\partial q_p}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_k}\right)\ddot{q}_k$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left(\frac{\partial^2\mathbb{T}_i}{\partial q_j\,\partial q_k}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm T}}{\partial q_p}\right)\dot{q}_k\dot{q}_j+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\left\{\mathrm{Trace}\left[\frac{\partial\mathbb{T}_i}{\partial q_p}\frac{\partial}{\partial q_j}\left(\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_k}\mathbb{T}_i^{\mathrm T}\right)\right]\right.$$
$$\left.-\mathrm{Trace}\left[\frac{\partial\mathbb{T}_i}{\partial q_k}\frac{\partial}{\partial q_p}\left(\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_j}\mathbb{T}_i^{\mathrm T}\right)\right]+\frac{\partial}{\partial q_j}\left[\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_p}\mathbb{T}_i^{\mathrm T}\right)\right]\right\}\dot{q}_k\dot{q}_j$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\mathrm{Trace}\left(\frac{\partial^2\mathbb{T}_{ai}}{\partial q_j\,\partial q_k}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_p}\right)\dot{q}_k\dot{q}_j-\sum_{i=1}^{N}m_i\mathbf{g}^{\mathrm T}\frac{\partial\mathbb{T}_i}{\partial q_p}\bar{\mathbf{r}}_i.$$

The forces will be presented in the form:

$$F_p=\sum_{k=1}^{N}D_{pk}\ddot{q}_k+\sum_{k=1}^{N}\sum_{j=1}^{N}D_{pkj}\dot{q}_k\dot{q}_j+D_p$$

$$D_{pk}=\sum_{i=1}^{N}\left[\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm T}}{\partial q_k}\right)+\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_k}\mathbb{T}_i^{\mathrm T}\right)+\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_p}\mathbb{T}_i^{\mathrm T}\right)+\mathrm{Trace}\left(\frac{\partial\mathbb{T}_{ai}}{\partial q_p}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_k}\right)\right].$$

$$D_p=-\sum_{i=1}^{N}m_i\mathbf{g}^{\mathrm T}\frac{\partial\mathbb{T}_i}{\partial q_p}\bar{\mathbf{r}}_i.$$

$$D_{pkj}=\sum_{i=1}^{N}\left[\mathrm{Trace}\left(\frac{\partial^2\mathbb{T}_i}{\partial q_j\,\partial q_k}\mathbb{J}_i\frac{\partial\mathbb{T}_i^{\mathrm T}}{\partial q_p}\right)+\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\frac{\partial\mathbb{T}_{ai}}{\partial q_j}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_k}\mathbb{T}_i^{\mathrm T}\right)\right.$$
$$+\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial^2\mathbb{T}_{ai}^{\mathrm T}}{\partial q_j\,\partial q_k}\mathbb{T}_i^{\mathrm T}\right)+\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_p}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_k}\frac{\partial\mathbb{T}_i^{\mathrm T}}{\partial q_j}\right)$$
$$-\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\frac{\partial\mathbb{T}_{ai}}{\partial q_p}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_j}\mathbb{T}_i^{\mathrm T}\right)-\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial^2\mathbb{T}_{ai}^{\mathrm T}}{\partial q_j\,\partial q_p}\mathbb{T}_i^{\mathrm T}\right)$$
$$-\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_j}\frac{\partial\mathbb{T}_i^{\mathrm T}}{\partial q_p}\right)+\mathrm{Trace}\left(\frac{\partial^2\mathbb{T}_i}{\partial q_k\,\partial q_j}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_p}\mathbb{T}_i^{\mathrm T}\right)$$
$$+\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\frac{\partial\mathbb{T}_{ai}}{\partial q_j}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_p}\mathbb{T}_i^{\mathrm T}\right)+\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial^2\mathbb{T}_{ai}^{\mathrm T}}{\partial q_j\,\partial q_p}\mathbb{T}_i^{\mathrm T}\right)$$
$$\left.+\mathrm{Trace}\left(\frac{\partial\mathbb{T}_i}{\partial q_k}\mathbb{T}_{ai}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_p}\frac{\partial\mathbb{T}_i^{\mathrm T}}{\partial q_j}\right)+\mathrm{Trace}\left(\frac{\partial^2\mathbb{T}_{ai}^{\mathrm T}}{\partial q_j\,\partial q_k}\mathbb{J}_{ai}\frac{\partial\mathbb{T}_{ai}^{\mathrm T}}{\partial q_p}\right)\right].$$

After p changed to i, k into j, j into k, i into p as well as making use of the equation $\partial\mathbb{T}_i/\partial q_j\equiv 0$ for $i<j$ we obtain

$$F_i=\sum_{j=1}^{N}D_{ij}\ddot{q}_j\sum_{j=1}^{N}\sum_{k=1}^{N}D_{ijk}\dot{q}_j\dot{q}_k+D_i$$

where

$$D_{ij}=\sum_{p=\max(i,j)}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_p}{\partial q_i}\mathbb{J}_p\frac{\partial\mathbb{T}_p^{\mathrm T}}{\partial q_j}\right)+\sum_{p=i}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_p}{\partial q_i}\mathbb{T}_{ap}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_j}\mathbb{T}_p^{\mathrm T}\right)$$
$$+\sum_{p=j}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_p}{\partial q_j}\mathbb{T}_{ap}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_i}\mathbb{T}_p^{\mathrm T}\right)+\sum_{p=1}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_{ap}}{\partial q_i}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_j}\right),$$

$$D_i=-\sum_{p=i}^{N}m_p\mathbf{g}^{\mathrm T}\frac{d\mathbb{T}_p}{\partial q_i}\bar{\mathbf{r}}_p.$$

$$D_{ijk}=\sum_{p=\max(i,j,k)}^{N}\mathrm{Trace}\left(\frac{\partial^2\mathbb{T}_p}{\partial q_j\,\partial q_k}\mathbb{J}_p\frac{\partial\mathbb{T}_p^{\mathrm T}}{\partial q_i}\right)+\sum_{p=i}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_p}{\partial q_i}\frac{\partial\mathbb{T}_{ap}}{\partial q_k}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_j}\mathbb{T}_p^{\mathrm T}\right)$$
$$+\sum_{p=i}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_p}{\partial q_i}\mathbb{T}_{ap}\mathbb{J}_{ep}\frac{\partial^2\mathbb{T}_{ap}^{\mathrm T}}{\partial q_j\,\partial q_k}\mathbb{T}_p^{\mathrm T}\right)+\sum_{p=\max(i,k)}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_p}{\partial q_i}\mathbb{T}_{ap}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_j}\frac{\partial\mathbb{T}_p^{\mathrm T}}{\partial q_k}\right)$$
$$-\sum_{p=j}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_p}{\partial q_j}\frac{\partial\mathbb{T}_{ap}}{\partial q_i}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_k}\mathbb{T}_p^{\mathrm T}\right)-\sum_{p=\max(i,j)}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_p}{\partial q_j}\mathbb{T}_{ap}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_k}\frac{\partial\mathbb{T}_p^{\mathrm T}}{\partial q_i}\right)$$
$$+\sum_{p=\max(j,k)}^{N}\mathrm{Trace}\left(\frac{\partial^2\mathbb{T}_p}{\partial q_j\,\partial q_k}\mathbb{T}_{ap}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_i}\mathbb{T}_p^{\mathrm T}\right)+\sum_{p=j}^{N}\mathrm{Trace}\left(\frac{\partial\mathbb{T}_p}{\partial q_j}\frac{\partial\mathbb{T}_{ap}}{\partial q_k}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_i}\mathbb{T}_p^{\mathrm T}\right)$$
$$+\sum_{p=\max(j,k)}^{N}\left(\frac{\partial\mathbb{T}_p}{\partial q_j}\mathbb{T}_{ap}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_i}\frac{\partial\mathbb{T}_p^{\mathrm T}}{\partial q_k}\right)+\sum_{p=1}^{N}\mathrm{Trace}\left(\frac{\partial^2\mathbb{T}_{ap}^{\mathrm T}}{\partial q_j\,\partial q_k}\mathbb{J}_{ap}\frac{\partial\mathbb{T}_{ap}^{\mathrm T}}{\partial q_i}\right).$$

After regrouping we obtain

$$D_{ijk} = \sum_{p=\max(i,j,k)}^{N} \mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_p}{\partial q_j \partial q_k} \mathbb{J}_p \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_i}\right) - \sum_{p=\max(i,j)}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \mathbb{T}_{ap} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_k} \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_i}\right)$$
$$+ \sum_{p=\max(i,k)}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_i} \mathbb{T}_{ap} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_j} \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_k}\right)$$
$$+ \sum_{p=\max(j,k)}^{N} \left[\mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_p}{\partial q_j \partial q_k} \mathbb{T}_{ap} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_i} \mathbb{T}_p^{\mathrm{T}}\right) + \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \mathbb{T}_{ap} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_i} \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_k}\right)\right]$$
$$+ \sum_{p=i}^{N} \left[\mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_i} \mathbb{T}_{ap} \mathbb{J}_{ap} \frac{\partial^2 \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_j \partial q_k} \mathbb{T}_p^{\mathrm{T}}\right) + \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_i} \frac{\partial \mathbb{T}_{ap}}{\partial q_k} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_j} \mathbb{T}_p^{\mathrm{T}}\right)\right]$$
$$+ \sum_{p=j}^{N} \left[\mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \frac{\partial \mathbb{T}_{ap}}{\partial q_k} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_i} \mathbb{T}_p^{\mathrm{T}}\right) - \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \frac{\partial \mathbb{T}_{ap}}{\partial q_i} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_k} \mathbb{T}_p^{\mathrm{T}}\right)\right]$$
$$+ \sum_{p=1}^{N} \mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_{ap}}{\partial q_j \partial q_k} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_i}\right).$$

Generally, masses of actuator effectors, together with transmission elements fixed to them, are small, and that is why $\mathbb{J}_{ap}$ matrix elements are small as compared to the elements of $\mathbb{J}_p$ matrix. However, due to high transmission ratio, some elements of $\partial \mathbb{T}_{ap}/\partial q_j$ matrix are large. Therefore, in the above formulas, the expression with $\mathbb{J}_{ap}$ which do not contain two $\mathbb{T}_{ap}$ derivatives, will be ignored.

$$D_{ij} = \sum_{p=\max(i,j)}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_i} \mathbb{J}_p \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_j}\right) + \sum_{p=1}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_{ap}}{\partial q_i} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_j}\right),$$
$$D_{ijk} = \sum_{p=\max(i,j,k)}^{N} \mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_p}{\partial q_j \partial q_k} \mathbb{J}_p \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_i}\right) + \sum_{p=i}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_i} \frac{\partial \mathbb{T}_{ap}}{\partial q_k} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_j} \mathbb{T}_p^{\mathrm{T}}\right)$$
$$+ \sum_{p=j}^{N} \left[\mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \frac{\partial \mathbb{T}_{ap}}{\partial q_k} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_i} \mathbb{T}_p^{\mathrm{T}}\right) - \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \frac{\partial \mathbb{T}_{ap}}{\partial q_i} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_k} \mathbb{T}_p^{\mathrm{T}}\right)\right],$$
$$D_i = -\sum_{p=i}^{N} m_p \mathbf{g}^{\mathrm{T}} \frac{\partial \mathbb{T}_p}{\partial q_i} \bar{\mathbf{r}}_p.$$

If all actuators are installed in the axes of kinematic pairs so that $\partial \mathbb{T}_{ap}/\partial q_k = (\partial \mathbb{T}_{ap}/\partial q_k)\delta_{pk}$ (δ_{pk}—Kronecker delta), then

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{p=1}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \frac{\partial \mathbb{T}_{ap}}{\partial q_k} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_i} \mathbb{T}_p^{\mathrm{T}}\right) = \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{p=1}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \frac{\partial \mathbb{T}_{ap}}{\partial q_k} \delta_{pk} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_i} \delta_{pi} \mathbb{T}_p^{\mathrm{T}}\right)$$
$$= \sum_{j=1}^{N} \sum_{k=1}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_k}{\partial q_j} \frac{\partial \mathbb{T}_{ak}}{\partial q_k} \mathbb{J}_{ak} \frac{\partial \mathbb{T}_{ak}^{\mathrm{T}}}{\partial q_i} \delta_{ki} \mathbb{T}_k^{\mathrm{T}}\right) = \sum_{j=1}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_j} \frac{\partial \mathbb{T}_{ai}}{\partial q_i} \mathbb{J}_{ai} \frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_i} \mathbb{T}_i^{\mathrm{T}}\right).$$

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{p=1}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \frac{\partial \mathbb{T}_{ap}}{\partial q_i} \mathbb{J}_{ap} \frac{\partial_{ap}^{\mathrm{T}}}{\partial q_k} \mathbb{T}_p^{\mathrm{T}}\right) = \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{p=1}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_j} \frac{\partial \mathbb{T}_{ap}}{\partial q_i} \delta_{pi} \mathbb{J}_{ap} \frac{\partial \mathbb{T}_{ap}^{\mathrm{T}}}{\partial q_k} \delta_{pk} \mathbb{T}_p^{\mathrm{T}}\right)$$
$$= \sum_{j=1}^{N} \sum_{k=1}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_k}{\partial q_j} \frac{\partial \mathbb{T}_{ak}}{\partial q_i} \delta_{ki} \mathbb{J}_{ak} \frac{\partial \mathbb{T}_{ak}^{\mathrm{T}}}{\partial q_k} \mathbb{T}_k^{\mathrm{T}}\right) = \sum_{j=1}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_i}{\partial q_j} \frac{\partial \mathbb{T}_{ai}}{\partial q_i} \mathbb{J}_{ai} \frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_i} \mathbb{T}_i^{\mathrm{T}}\right).$$

Similar transformation having been performed, we obtain

$$D_{ij} = \sum_{p=\max(i,j)}^{N} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_p}{\partial q_i} \mathbb{J}_p \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_j}\right) + \delta_{ij} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_{ai}}{\partial q_i} \mathbb{J}_{ai} \frac{\partial \mathbb{T}_{ai}^{\mathrm{T}}}{\partial q_i}\right),$$
$$D_{ijk} = \sum_{p=\max(i,j,k)}^{N} \mathrm{Trace}\left(\frac{\partial^2 \mathbb{T}_p}{\partial q_j \partial q_k} \mathbb{J}_p \frac{\partial \mathbb{T}_p^{\mathrm{T}}}{\partial q_i}\right) + \delta_{jk} \mathrm{Trace}\left(\frac{\partial \mathbb{T}_j}{\partial q_i} \frac{\partial \mathbb{T}_{aj}}{\partial q_j} \mathbb{J}_{aj} \frac{\partial \mathbb{T}_{aj}^{\mathrm{T}}}{\partial q_j} \mathbb{T}_j^{\mathrm{T}}\right),$$
$$D_i = -\sum_{p=i}^{N} m_p \mathbf{g}^{\mathrm{T}} \frac{\partial \mathbb{T}_p}{\partial q_i} \bar{\mathbf{r}}_p.$$