Hogg and Ledolter

1.5-1 Control Charts

Here we explain the basic idea behind control charts and discuss how to construct and use such charts. The reasons these charts work so well will become clear when we know more about the variability of sample statistics; this discussion is taken up in Chapter 8.

All processes have some variation. When we manufacture a product, measurements on the final product will show inevitable variation from unintentional process changes as well as random variation. Many different factors enter into a production process, and a change in each will cause some variation in the final product. This variation may come from differences among machines, lot-to-lot variation, differences in suppliers and incoming raw materials, and so on. Despite the fact that considerable effort is generally directed toward controlling the variability in each of these factors, there will still be variability in the final product. In the end it is this variability that has to be controlled.

Statistical *control charts* or, more generally, statistical process control methods are procedures for monitoring the process variation and for generating information on the stability of the process. It is important to check the stability of processes, since unstable processes will result in lost production, defective products, poor quality, and, in general, loss of consumer confidence. For example, in the production of integrated-circuit boards, which involves several welding procedures, it may be the weld strength that is of importance. Selecting a small sample of such boards at regular intervals and measuring the weld strength by a certain pull test to destruction will provide valuable information on the stability of the welding process. In the production of concrete cylinders, it is the compressive strength that is of importance and that needs to be controlled. Measurements on a small number of concrete cylinders, say twice during each production shift, can give us valuable information on the variability of this process. In the production of thin wafers for integrated-circuit devices by high-temperature furnace oxide growth processes (see Exercise 1.5-2), it is the thickness of these very thin wafers that needs to be controlled. Measurements on the thickness of a few selected wafers from every other furnace run indicate whether the thickness of the product is stable. Here we have given only three examples. Many others can be found, and we encourage the reader to think of still others.

A control chart is a *plot* of a summary statistic from samples that are taken *sequentially in time*. Usually, it is the sample mean and a measure of the sample variability, such as the standard deviation or the range, that are plotted on such control charts. The charts in Figures 1.5-1 and 1.5-2 are two examples. The chart in Figure 1.5-1 shows the average compressive strengths of concrete blocks from samples of size 5. Twice during each shift, five concrete blocks were taken from the production line, their compressive strengths were determined, and the average was entered on the chart. Since we plot averages,

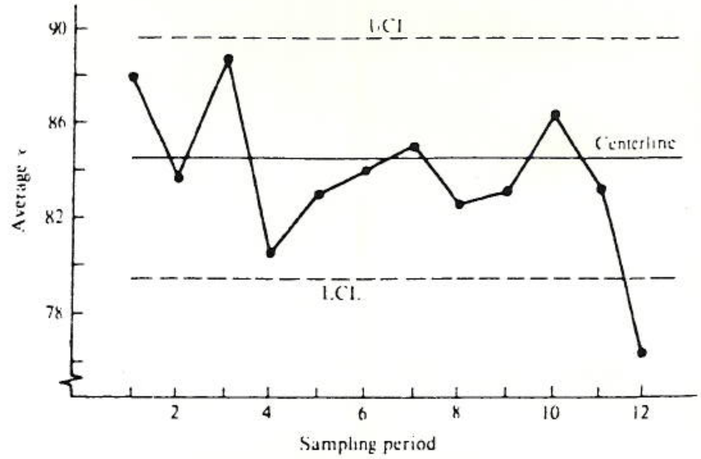

Figure 1.5-1 $\bar{x}$-chart for the sample means in Table 1.5-1.

we call this an *$\bar{x}$-chart.* In Figure 1.5-2 we display the variability within the samples over time, and plot the *ranges*; we call such a plot an *R-chart.*

Control charts also include bounds, or *control limits,* which help us determine whether a particular average (or range) is "within acceptable limits" of random variation. Through these limits, control charts try to distinguish between the variation that can normally be expected and the variation that is caused by unexpected changes. If one notices shifts in the process *mean* on the $\bar{x}$-chart and if the shifts are larger than those which can be expected under the usual pattern, we conclude that something significant has happened to the process. Similarly, if the process *variation* as measured on the *R*-chart changes by more than would be expected under usual circumstances, we conclude that the process variability is no longer stable.

The control limits are usually determined from past data that are collected when the process is actually in control and when no significant changes have

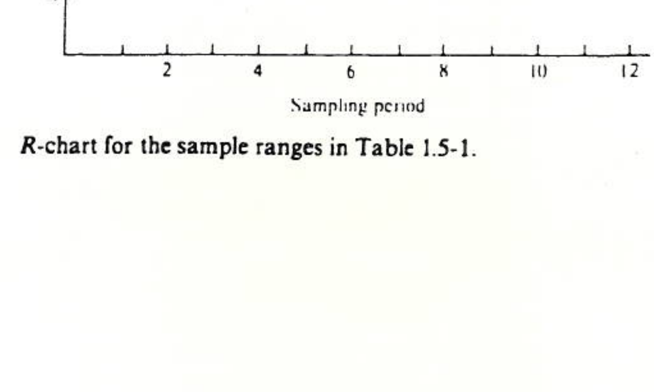

Figure 1.5-2 *R*-chart for the sample ranges in Table 1.5-1.

taken place. The construction of the control limits is very simple. We take samples of a few observations (usually, the sample size n is 4 or 5) at various times. It is usually recommended that $k = 10$ to 20 such samples be obtained before constructing the control limits. Depending on the application, these samples can be taken every 4 hours (see the weld strength example), twice a shift (compressive strength), from every other furnace run (wafer example), every hour, from every tenth batch, and so on. The frequency of the sampling depends on the stability of the process; the more stable the process, the longer the time between samples. It also depends on the potential loss that is caused when deteriorations of the process are not recognized on time, and of course on the cost of the sampling inspection. From each sample we calculate the average $\bar{x} = \sum_{i=1}^{n} x_i/n$ and the range $R = \max(x_1, \ldots, x_n) - \min(x_1, \ldots, x_n)$, and enter these quantities on the $\bar{x}$-chart and R-chart. From the k sample averages and ranges, we compute the grand average (average of averages),

$$\bar{\bar{x}} = \frac{1}{k} \sum_{j=1}^{k} \bar{x}_j,$$

and the average of the ranges,

$$\bar{R} = \frac{1}{k} \sum_{j=1}^{k} R_j.$$

These quantities form the respective centerlines in the $\bar{x}$-chart and the R-chart. The control limits (the lower control limit, LCL, and the upper control limit, UCL) in the $\bar{x}$-chart are given by

$$\text{LCL} = \bar{\bar{x}} - A_2 \bar{R} \qquad \text{and} \qquad \text{UCL} = \bar{\bar{x}} + A_2 \bar{R}.$$

The control limits in the R-chart are given by

$$\text{LCL} = D_3 \bar{R} \qquad \text{and} \qquad \text{UCL} = D_4 \bar{R}.$$

The constants A_2, D_3, and D_4 can be found in Table C.1 in Appendix C; they depend on the sample size n. These constants are chosen such that almost all future averages $\bar{x}$ and ranges R will fall within the respective control limits, provided that the process has stayed in control (which means that the level has not shifted and the variability has not changed). The precise meaning of "almost all" and the construction of the values in Table C.1 are explained later. For now it is sufficient to understand that the natural variability in the process leads to variability among the sample averages and ranges. If the process is stable, it is very rare that a sample average and range fall outside the control limits. On the other hand, if there are shifts and drifts in the process, the averages and/or the ranges will probably exceed the limits and generate an alarm.

Consider the data in Table 1.5-1, in which we list the compressive strength measurements from $k = 10$ samples of size $n = 5$. The process was sampled twice during each production shift, and the observations were taken while the

Table 1.5-1 Compressive Strength of Concrete (kg/cm^2)

	Sample	Compressive Strength					$\bar{x}$	R
Samples used to determine the control limits	1	91	88	88	90	83	88.0	8
	2	84	89	80	79	87	83.8	10
	3	93	90	87	89	85	88.8	8
	4	76	84	82	79	82	80.6	8
	5	83	85	81	80	86	83.0	6
	6	84	84	90	79	83	84.0	11
	7	83	89	80	82	91	85.0	11
	8	78	79	90	81	85	82.6	12
	9	82	81	87	86	79	83.0	8
	10	88	90	83	84	87	86.4	7
	Mean						$\bar{\bar{x}} = 84.52$	$\bar{R} = 8.9$
	11	79	87	82	85	83	83.2	8
	12	72	79	76	77	78	76.4	7

process was under control, or at least thought to be under control. With $n = 5$ observations in each sample, we find from Table C.1 that the constants are $A_2 = 0.577$, $D_3 = 0$, and $D_4 = 2.115$. Thus the control limits for the $\bar{x}$-chart are

$$\text{LCL} = \bar{\bar{x}} - A_2 \bar{R} = 84.52 - (0.577)(8.9) = 79.38$$

and

$$\text{UCL} = \bar{\bar{x}} + A_2 \bar{R} = 84.52 + (0.577)(8.9) = 89.66.$$

The limits on the R-chart are $\text{LCL} = D_3 \bar{R} = (0)(8.9) = 0$ and $\text{UCL} = D_4 \bar{R} = (2.115)(8.9) = 18.82$. These are the limits that are shown in Figures 1.5-1 and 1.5-2. We see that the averages and ranges of all 10 samples are within these limits. We could have expected this because we were told that the process was in control when these observations were taken. But let us plot the results of the next two samples, also given in Table 1.5-1, on these charts. There we find that the twelfth average $\bar{x} = 76.4$ is smaller than the lower control limit on the $\bar{x}$-chart. This fact should alert the user that this particular sample represents an unusual event. This should lead to an investigation (i.e., discussions with workers on the production line, checking whether there were changes in raw materials, looking for any other unusual condition) that will identify an assignable cause for this event. Finally, these causes should be eliminated.

Control charts are very useful methods that help us assess whether a process is stable. They alert the user to situations in which something has shifted. A point outside the control limits forces us to find an assignable cause for this unusual event and, more important, to make certain changes in the process that prevent such conditions from happening again. Control charts will uncover many external sources that lead to shifts in the mean level and in the variability of the process. Their graphical simplicity makes them a very valuable instrument for process control. The requirement to identify assignable causes and to eliminate them forces management and workers to take an aggressive attitude toward maintaining the quality of their work.

Control charts are not only useful for averages and ranges, but also for proportions, such as the proportion of defectives. Also, these charts are useful not just for manufacturing applications, but also in other areas, such as the service industry. In fact, they can be applied to virtually all situations in which data are taken sequentially in time.

Assume, for example, that we simply judge whether or not a manufactured item is satisfactory. That is, although we prefer to take more accurate measurements, here we just check an item on a pass–fail basis, whether it is within or outside specifications. Assume that an inspector at the production line checks a sample of n items at certain stated periods (hour, half-day, day, etc., depending on the numbers of items produced each day) and observes the number of defectives, say d, in these n items. If this is done for k periods, we obtain $d_1, d_2, \ldots, d_k$ defectives, respectively. The average fraction of defectives is

$$\bar{p} = \frac{d_1 + d_2 + \cdots + d_k}{nk}.$$

Statistical theory (as discussed in subsequent sections) assures us that if there are no changes in the process, almost all of the future fraction defectives, d/n, will be between the respective lower and upper control limits:

$$\text{LCL} = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}},$$

$$\text{UCL} = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}.$$

These control limits, together with the centerline at $\bar{p}$, are plotted on a chart; since we are plotting fraction defectives or percentages, we call it a *p-chart*. If future points fall outside these limits, this strongly suggests that the process has changed. In particular, a point exceeding the upper control limit indicates that the process has deteriorated. As a consequence, we should look for possible reasons for the sudden increase in the number of defectives.

Example 1.5-1 Each hour $n = 50$ fuses are tested. For the first $k = 20$ hours we find the following number of defectives:

1 1 3 0 2 4 0 0 1 2 3 2 0 1 1 1 3 0 0 2

Thus, since $nk = 1000$,

$$\bar{p} = \frac{27}{1000} = 0.027$$

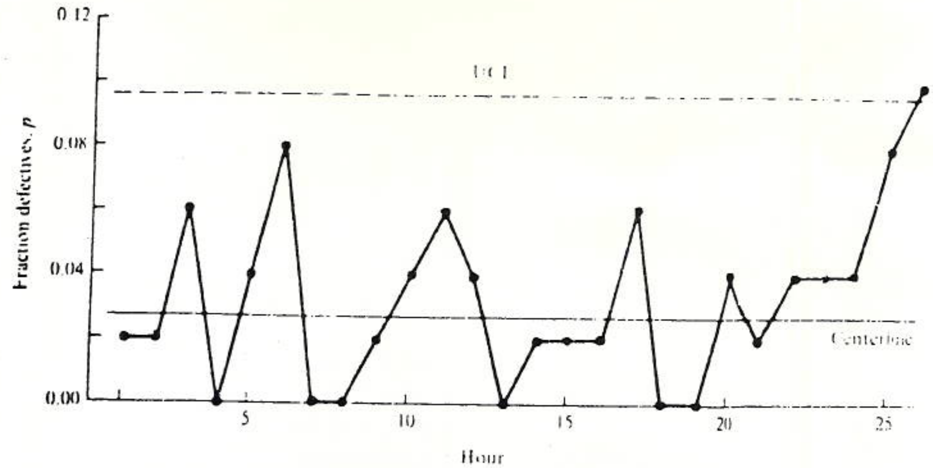

Figure 1.5-3 *p*-chart for the data in Example 1.5-1.

is the average fraction of defectives. We must first decide if this is an acceptable fraction for our particular process. If it is, then

$$\text{LCL} = 0.027 - 3\sqrt{\frac{(0.027)(0.973)}{50}} = -0.042,$$

$$\text{UCL} = 0.027 + 3\sqrt{\frac{(0.027)(0.973)}{50}} = 0.096.$$

Since LCL < 0, and since the fraction defective d/n can never be less than zero, in such cases we usually plot the LCL at zero, or omit it entirely. In Figure 1.5-3 we have plotted these 20 values of the fraction defective together with six more recent ones, those with d values of 1, 2, 2, 2, 4, and 5. Additional values would also be plotted as long as the process is "in control." However, we find that the sixth additional fraction defective is above the UCL. This suggests that the process has become unstable and that corrective action should be taken. In this example we have assumed that 2.7 percent defective is acceptable and that we are willing to produce at this level; this may not be the case for other items.

The *c-chart* is very similar to the *p*-chart except that here we count the number of flaws or defectives for a certain unit (bolt of fabric, length of wire, and so on) rather than the number of defectives in n items. Suppose that we determine the number, c, of blemishes in 50 feet of a continuous strip of tin plate. This is done each hour for k hours, resulting in $c_1, c_2, \ldots, c_k$ with an average of

$$\bar{c} = \frac{1}{k}\sum_{i=1}^{k} c_i.$$

The respective lower and upper control limits for the c-chart are

$$\text{LCL} = \bar{c} - 3\sqrt{\bar{c}},$$
$$\text{UCL} = \bar{c} + 3\sqrt{\bar{c}},$$

which are plotted together with the centerline, $\bar{c}$. If $\bar{c}$ is a satisfactory average for the process, the process is considered in control provided that future points plot within these control limits.

Example 1.5-2 We observe $k = 15$ 50-foot tin strips and obtain the following numbers of blemishes:

2 1 1 0 5 2 3 1 1 2 0 0 4 3 1

The average is $\bar{c} = 26/15 = 1.73$ and

$$\text{LCL} = 1.73 - 3\sqrt{1.73} = -2.22, \qquad \text{UCL} = 1.73 + 3\sqrt{1.73} = 5.68.$$

These 15 points, together with the 10 additional observations

3 1 1 0 2 2 5 0 1 2

are plotted on the c-chart in Figure 1.5-4. Of course, as long as the process is in control, as with these 10 additional points, future points are plotted on this c-chart. Occasionally, new control limits are calculated if the points continue to fall within the control limits; thus the control limits may change slightly. Points outside the control limits, however, indicate that the process has become unstable. Assignable causes for these unusual events should be found and eliminated.

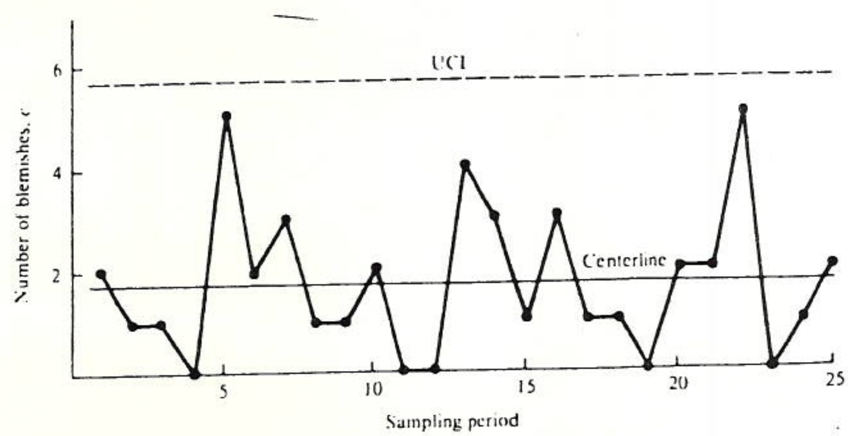

Figure 1.5-4 c-chart for the data in Example 1.5-2.

Table C.1 Factors for Determining the 3σ Control Limits in $\bar{x}$-Charts and R-Charts.

Number of Observations in Sample, n	Factors for $\bar{x}$-Charts		Factors for R-Chart	
	Using $\bar{s}$ A_3	Using $\bar{R}$ A_2	D_3	D_4
2	2.66	1.88	0	3.27
3	1.95	1.02	0	2.57
4	1.63	0.73	0	2.28
5	1.43	0.58	0	2.11
6	1.29	0.48	0	2.00
7	1.18	0.42	0.08	1.92
8	1.10	0.37	0.14	1.86
9	1.03	0.34	0.18	1.82
10	0.98	0.31	0.22	1.78
11	0.93	0.29	0.26	1.74
12	0.89	0.27	0.28	1.72
13	0.85	0.25	0.31	1.69
14	0.82	0.24	0.33	1.67
15	0.79	0.22	0.35	1.65
16	0.76	0.21	0.36	1.64
17	0.74	0.20	0.38	1.62
18	0.72	0.19	0.39	1.61
19	0.70	0.19	0.40	1.60
20	0.68	0.18	0.41	1.59

Source: Reproduced with permission from E. L. Grant, *Statistical Quality Control*, 2nd ed. (New York: McGraw-Hill, 1952), pp. 513 and 514.

Table C.

n	x
2	0
	1
	2
3	0
	1
	2
4	
	1
	2
	3
	4
5	0
	1
	2
	3
	4
	5
6	0
	1
	2
	3
	4
	5
	6
7	0
	1
	2
	3
	4